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MEETING OF THE ASSOCIATION FOR SYMBOLIC LOGIC

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The 1975–76 Annual Meeting of the Association for Symbolic Logic was held at the Statler Hilton Hotel in New York City on December 28–29, 1975, in conjunction with the Annual Meeting of the Eastern Division of the American Philosophical Association. Invited hour lectures were given by Harvey Friedman, *The logical strength of mathematical statements*, and by Jack Silver, *How to get rid of Jensen's fine structure*. A survey lecture was given by Stephen C. Kleene on *The work of Kurt Gödel*. In addition there was a joint symposium with the American Philosophical Association on *Sets, concepts and extensions*, by David Kaplan and Charles Parsons and moderated by Ruth Barcan Marcus. In addition twenty-five contributed papers were read and two were presented by title. Abstracts follow.

The Council met the evening of December 28.

PAUL BENACERRAF
SIMON KOCHEN
GERALD SACKS

BRUCE M. HOROWITZ. *Constructively nonpartial recursive functions and completely productive sets.*

Rose and Ullian [2] call a total function $f(x)$ constructively nonrecursive iff for some recursive $g(x)$, $f(g(n)) \neq \varphi_n(g(n))$ for all $n \in N$, where φ_n is the partial recursive function with index n. We define a partial function $f(x)$ to be constructively nonpartial recursive iff for some recursive $g(x)$, $f(g(n)) \not\simeq \varphi_n(g(n))$. We say $f(x)$ is constructively nonpartial recursive via $g(x)$.

THEOREM 1. *Sums and products do not necessarily preserve constructively nonpartial recursive functions.*

THEOREM 2. *For every* 1-1 *recursive function* $g(x)$, *there is a function* $f(x)$ *which is constructively nonrecursive via* $g(x)$.

THEOREM 3. *If* Domain $(f(x))$ *is productive, then* $f(x)$ *is constructively nonpartial recursive.*

THEOREM 4. *If* $f(x)$ *is constructively nonpartial recursive via an onto recursive function* $g(x)$, *then* $\{x/f(x)$ *is undefined*$\}$ *is recursively enumerable, provided* $\varphi_x(g(x))$ *defined implies* $f(g(x))$ *undefined.*

Myhill has shown productivity is equivalent to complete productivity. It is known that every productive set is productive via a recursive permutation. (See Rogers [1].) However, we have:

THEOREM 5. *There exists a set which is completely productive, but via no onto recursive function.*

More illuminating is:

THEOREM 6. *If* A *is completely productive via an onto recursive function, then* $\tilde{A}$ *is creative.*

Theorem 6 may also be proven directly, without reference to constructively nonpartial recursive functions. In fact, by modifying a suggestion of Rogers [1], we obtain the converse:

THEOREM 7. *If* A *is creative, then* $\tilde{A}$ *is completely productive via an onto recursive function.*

REFERENCES

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[2] G. ROSE and J. ULLIAN, *Approximations of functions on the integers*, ***Pacific Journal of Mathematics***, vol. 13 (1963), 693–701.

SY FRIEDMAN. *Recursion theory and Jensen's* $\diamond^+$*-principle.*

An effectivized version of Jensen's principle $\diamond^+$ is used to solve Post's Problem for many nonadmissible ordinals.

Let β be an ordinal such that L_β is primitive-recursively closed. β-RE $= \Sigma_1$ over L_β. $\beta^* = \Sigma_1$-projectum of $\beta = \mu\gamma$ ($\exists$ a 1-1 Σ_1 over L_β function $f: \beta \to \gamma$). If $\beta^* < \beta$, then $L_\beta \models$ "β^* is a cardinal". Let $\langle K_\delta \rangle_{\delta<\beta}$ be a canonical indexing of sets of ordinals in L_β (β-finite sets). For $A, B \subseteq \beta$, define $A \leq_\beta B$ iff there is a β-RE set R such that

$$K_\delta \subseteq A \leftrightarrow \exists\delta_1\exists\delta_2[\langle\delta, \delta_1, \delta_2, 0\rangle \in R \wedge K_{\delta_1} \subseteq B \wedge K_{\delta_2} \subseteq B^c],$$

$$K_\delta \subseteq A^c \leftrightarrow \exists\delta_1\exists\delta_2[\langle\delta, \delta_1, \delta_2, 1\rangle \in R \wedge K_{\delta_1} \subseteq B \wedge K_{\delta_2} \subseteq B^c].$$

$A <_\beta B$ iff $A \subseteq_\beta B$ and $B \not\leq_\beta A$. Let $C =$ the complete β-RE set. Post's Problem: Is there a β-RE set A such that $0 <_\beta A <_\beta C$?

THEOREM 1 (SACKS-SIMPSON). *If β is Σ_1-admissible, there are β-RE sets A, B such that $A \not\leq_\beta B$, $B \not\leq_\beta A$.*

THEOREM 2. *If $\beta^* < \beta$, and $L_\beta \models$ "β^* is a successor cardinal", then there exist β-RE sets A, $B \subseteq \beta^*$ such that $A \not\leq_\beta B$, $B \not\leq_\beta A$.*

Theorem 2 uses the following effective version of Jensen's $\diamond^+$-principle:

LEMMA. *(Same hypotheses as Theorem 2.) There exists a sequence $\langle S_\gamma \rangle_{\gamma<\beta^*} \in L_\beta$ such that*

(1) *$S_\gamma \subseteq 2^\gamma \cap L_\beta$.*

(2) *$L_\beta \models S_\gamma$ has cardinality $< \beta^*$.*

(3) *For any β-RE set A, $A \cap \gamma \in S_\gamma$ for unboundedly many γ.*

The proof of the Lemma is identical to the proof of ordinary $\diamond^+$ for a successor cardinal of L. Theorem 2 uses the Lemma, an effective version of Fodor's Theorem, and recursion-theoretic ideas similar to those used in the proof of Theorem 1.

GEORGE BOOLOS. *On deciding the truth of certain statements involving the notion of consistency.*

L is the normal axiomatic modal propositional logic whose sole special axiom schema is $\Box(\Box A \to A) \to \Box A$. (The 0-ary connectives $\perp$ and $\top$ count as sentences of L; $\Box p \to p$ is *not* one of its axioms.)

THEOREM 1 (KRIPKE, DE JONGH, SAMBIN). $\vdash_L \Box A \to \Box\Box A$, *all* A.

Let $\dagger$ be any mapping of sentences of L into those of Peano Arithmetic that commutes with connectives and is such that $(\Box A)^\dagger = \mathrm{Bew}_{PA}(\ulcorner A^\dagger \urcorner)$.

THEOREM 2. *If $\vdash_L A$, then $\vdash_{PA} A^\dagger$.*

DEFINITION. An *atom* is a sentence $\Box^n \perp$, $n \geq 0$.

THEOREM 3. *For any variable-free sentence G there is a truth-functional combination H of atoms such that $\vdash_L G \leftrightarrow H$.*

Define # by: Con # $= -\Box\perp$; let # commute with connectives; Con(ϕ) # $= -\Box - \phi$ #. (Cf. problem 35 of H. Friedman, *One hundred and two problems in mathematical logic*, this JOURNAL, vol. 2 (1975), p. 117.) Then if ϕ is in Friedman's E, $\vdash_{PA} \phi^* \leftrightarrow \psi^\dagger$, where ψ is some truth-functional combination of atoms such that $\vdash_L \phi \# \leftrightarrow \psi$. Since $B^\dagger$ is false if B is an atom, the truth-value of any ϕ^* can always be effectively determined, and so Friedman's 35th problem has an affirmative solution.

DEFINITION. R is a *Rosser sentence* if

$$\vdash_{PA} R \leftrightarrow \forall x(\mathrm{Prf}(x, \ulcorner R \urcorner) \to \exists y < x\ \mathrm{Prf}(y, \mathrm{neg}(\ulcorner R \urcorner))).$$

THEOREM 4. *If G is variable-free, then $G^\dagger$ is equivalent to no Rosser sentence.*

DEFINITION. A sentence $A(p)$ with sole variable p is *fully modalized* if every occurrence of p lies within the scope of some occurrence of $\Box$.

THEOREM 5. *If $A(p)$ is fully modalized,* $\vdash_L \Box(p \leftrightarrow A(p)) \to (\Box p \leftrightarrow \Box A(\top))$.

Theorem 5 gives a decision procedure for provability for diagonalizations of predicates in a certain natural class containing Bew(x), $-$Bew(x), Bew(neg(x)), etc.

CHARLES LANDRAITIS. *Definability and well quasi-ordered classes of structures.*

Let K be a class of structures, R a well quasi-ordering on K (as in R. Laver, *On Fraisse's order type conjecture*, ***Annals of Mathematics***, 1971, pp. 89–111). Let L be a language for K closed under finite conjunction and containing a set S of sentences such that for $\mathfrak{A}$, $\mathfrak{B}$ in S, $\mathfrak{B} R \mathfrak{A}$ if and only if for every ϕ in S $\mathfrak{A} \models \phi$ implies $\mathfrak{B} \models \phi$.

THEOREM. *For any $\mathfrak{A}$ in K, there is ψ in L such that for any $\mathfrak{B}$ in K, $\mathfrak{B} \models \psi$ if and only if $\mathfrak{B} R \mathfrak{A}$.*

For an application, let C be the class of countable structures $(A, <, P_0, \cdots, P_{n-1})$ where $<$ linearly orders A, the P_i are unary predicates, the language is $L_{\omega_1\omega}$, S is the set of universal $L_{\omega_1\omega}$ sentences, and R is the relation of embeddability between structures.

COROLLARY. *For each $\mathfrak{A}$ in C there is a (universal) sentence ψ of $L_{\omega_1\omega}$ such that for $\mathfrak{B}$ in C, $\mathfrak{B} \models \psi$ if and only if $\mathfrak{B}$ is embeddable in $\mathfrak{A}$.*

The proof uses (what follows from) the main theorem of Laver's paper, cited above: C is well quasi-ordered under R.

D. FAUST, W. HANF, and D. MYERS. *The Boolean algebra of formulas.*

Let $F_\mu(T)$ be the Boolean algebra of formulas (modulo equivalence in the theory T) of first order logic with equality and nonlogical predicates specified by the similarity type μ. $F_\mu(T_1)$ and $F_\nu(T_2)$ are said to be recursively isomorphic if there is a recursive correspondence of the formulas of the two languages such that the implication of two formulas is a theorem of T_1 iff the implication of the corresponding formulas is a theorem of T_2. We make use of a dual isomorphism condition formulated by representing a pair of models as reducts of a combined model. In this way a set of ordered pairs of models, and in particular, a function between model spaces, can be regarded as a set of structures. Using this representation, it can be shown that two theories are recursively isomorphic iff there is a certain type of homeomorphism between their model spaces which, when regarded as a set of structures, is an axiomatizable class. The recursive saturation method of Barwise and Schlipf is used to replace a hard-to-verify condition, that the homeomorphism preserve elementary equivalence, by a simpler condition that it preserve isomorphism.

We use interpretations of a model $\mathfrak{A}$ in a model $\mathfrak{B}$ whose universe $B = A \times A$ (and thus a pair of free variables ranging over A corresponds to a single free variable ranging over B) to show that $F_\mu(>1)$ is recursively isomorphic to $F_{(2)}(>1)$ where μ is any undecidable finite similarity type (i.e. $\mu_i \geq 2$ for some i) and where >1 is the theory of all structures of the given similarity type which have at least two elements. The axiom >1 is used to insure that the Boolean algebras are atomless and therefore classically isomorphic; the full Boolean algebra F_μ has a finite number of atoms, the number depending on the number of relations specified by μ.

JOHN J. YOUNG. *A disputed thesis of the logic of subjunctive conditionals.*

Let subjunctive conditionals of the form

If p were the case, q would be the case,

and

If p had been the case, q would have been the case,

be represented as

(1) $$pS_1q$$

and

(2) $$pS_2q$$

respectively. Although there are exceptions there has been widespread acceptance of the view that a necessary condition of the truth of conditionals of the form (1) and (2) is the falsity of their antecedent: that is, (1) and (2) entail

(3) $$\sim p.$$

Let us call any such view a "Falsity of the Antecedent Thesis" (or "FA Thesis").

If some version of the FA thesis is correct, then it should be reflected in theories which purport to capture the logic of these conditionals. So, any such logic should contain theses like

(I_1) $$(pS_1q) \supset \sim p$$

and

$$(I_2) \qquad (pS_2q) \supset \sim p.$$

Although Burks [2] accepts a version of the FA Thesis, a number of other accounts of subjunctive conditionals [1], [3], [4] fail to acknowledge or include those like (I_1) and (I_2).

Failure to resolve this disagreement rests in part on the fact that formulation of the FA Thesis varies. At least two patterns of analysis can be found: (i) "Implication" versions like (I_1) and (I_2), and (ii) "Conjunction" versions which take the general form

$$(C) \qquad (pSq) \equiv [(p > q) \,\&\, \sim p]$$

where "S" stands for either "S_1" or "S_2" and "$>$" is a nontruth-functional conditional connective. Once various versions of the FA Thesis are stated clearly, it is shown that serious difficulties beset *any* version of the FA Thesis which resembles (I_1), (I_2) or (C).

The role of the subjunctive mood is then briefly examined. The relation of the syntactic feature of mood to the semantics of these conditionals is discussed. It is suggested that the FA Thesis captures a pragmatic feature of subjunctive conditionals which has no bearing on their logical structure.

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[3] DAVID LEWIS, ***Counterfactuals***, Harvard University Press, 1973.

[4] ROBERT STALNAKER, *A theory of conditionals*, ***Studies in logical theory*** (Nicholas Rescher, Editor), Basil Blackwell, Oxford, 1968, pp. 98–112.

RAYMOND D. GUMB. *An extended joint consistency theorem for free logic with equality.*

A version of the Craig, Lyndon, and Robinson theorems is reported in free logic with equality, using techniques developed by Hintikka. Let $\mathbf{C}(S)$ be the set of all and only those individual, predicate, and sentence parameters occuring in S. The theorem is: $S_1 \cup S_2$ *is inconsistent if and only if there is a* wff B *such that*

(1) $S_1 \cup \{B\}$ *and* $S_2 \cup \{\sim B\}$ *are inconsistent*;

(2) $\mathbf{C}(B) \subseteq \mathbf{C}(S_1) \cap \mathbf{C}(S_2)$;

(3) *each predicate or sentence parameter* $f \in \mathbf{C}(B)$ *occurs* (a) *negatively in* S_1 *and positively in* S_2 *if it occurs positively in* B *and* (b) *positively in* S_1 *and negatively in* S_2 *if it occurs negatively in* B.

It is indicated that the following theorem of standard first order logic with equality does *not* hold in free logic with equality: if $S_1 \cup S_2$ is inconsistent, neither S_1 nor S_2 is inconsistent, and '$=$' occurs in neither S_1 nor S_2, '$=$' does not occur in F.

ROBERT F. BARNES and RAYMOND D. GUMB. *The completeness of presupposition-free tense logic.*

A novel construction is employed to provide a Henkin-style completeness proof for the system $QK_t^* =$, an axiomatic version of presupposition-free tense logic with equality in which neither the tensed versions of the Barcan formula nor their converses are theorems. The construction combines the truth-value semantics of Leblanc, Makinson's device of temporal attendants, and a modification of Henkin's construction to provide, for any deductively consistent set of wffs of $QK_t^* =$, an interpretation on which the wffs of the given set are all true. The construction proceeds iteratively, with three stages in each iteration:

(a) a "bookkeeping" stage, in which a set of index numbers is enlarged by adding new indices;

(b) a "generation/push-forward" stage, in which:

(i) for each newly added index, an "initial partial Makinson attendant" is created, and

(ii) for each of the previous indices, the corresponding partial attendant is enlarged by adding new wffs derived from its "ancestor";

(c) a"pull-back" stage, in which each partial attendant is enlarged by adding new wffs derived from all its attendants.

This provides a tree-ordered collection of maximally consistent, omega-complete (in suitably generalized senses) sets of wffs, which is then utilized to provide the desired interpretation. The completeness of $QK^*_t=$ and of QK^*_t, the comparable logic without equality, then follows.

An especially interesting feature of the construction, which contrasts with previous tableau-based proofs, is that the temporal-succession relation is not defined during, but after, the construction itself. This feature gives the present method a more uniform and general character, thus permitting its application to stronger logics as well. For this reason, the construction may also be applied to presupposition-free modal or combined tense-modal logics, especially those with a symmetrical possible-world relation, which have typically been recalcitrant to Henkin-style proofs.

R. BRADSHAW ANGELL. *Three systems of first degree entailment.*

In this paper I examine relationships between the sets of theorems derivable in three systems of logic so far as they claim to capture the notion of entailment between formulae of the sentential calculus. The three systems involved are: (1) Anderson and Belnap's system E (for "entailment"), (2) Parry's system AI (for "analytic implication") and (3) my own system AC (for "analytic containment"), which is as follows: *Primitives*: '.', '$-$' and '$\leftrightarrow$'; with other truth-functional connectives defined as usual and $\ulcorner(A \rightarrow B)\urcorner =_{df} \ulcorner(A \leftrightarrow (A.B))\urcorner$ and interpreted as 'A entails B' in the sense of 'A analytically contains B'; and *Axiom schemata*:

AC1. $(A \leftrightarrow --A)$,
AC2. $(A \leftrightarrow (A.A))$,
AC3. $((A.B) \leftrightarrow (B.A))$,
AC4. $((A.(B.C)) \leftrightarrow ((A.B).C))$,
AC5. $((A \vee (B.C)) \leftrightarrow ((A \vee B).(A \vee C)))$,
AC6. $((A \leftrightarrow B) \supset (-A \leftrightarrow -B))$,
AC7. $((A \leftrightarrow B) \supset ((A.C) \leftrightarrow (B.C)))$,
AC8. $((A \leftrightarrow B) \supset ((B \leftrightarrow C) \supset (A \leftrightarrow C)))$,
AC9. $((A \leftrightarrow B) \supset (B \leftrightarrow A))$,
AC10. $((A \rightarrow B) \supset (A \supset B))$.

Rule of inference: If $\vdash A$ and $\vdash \ulcorner(A \supset B)\urcorner$ then $\vdash B$.

Parry's AI contains AC, and the first degree entailment theorems of E include all first degree entailment theorems of AC. All three systems contain standard sentential logic and preserve the principle that if A entails B then A truth-functionally implies B while eliminating all of the "paradoxes of strict implication" which are theorems in C.I. Lewis's systems and in standard logic.

Anderson and Belnap have shown that in E, $\ulcorner(A \rightarrow B)\urcorner$ is a theorem, where A and B are truth-functional, if and only if a certain effectively decidable relation holds between a disjunctive normal form of A and a conjunctive normal form of B. In Parry's system, $\ulcorner(A \leftrightarrow B)\urcorner$ is provable whenever A is any truth-functional formula and B is either its "full" disjunctive normal form or its "full" conjunctive normal form. In the system AC, I have shown elsewhere that $\ulcorner(A \leftrightarrow B)\urcorner$ will be a theorem if and only if the "maximum ordered conjunctive normal form" of A contains every conjunct which occurs in the "maximal ordered conjunctive normal form" of B. On the basis of these normal form theorems and derivations within these systems, we show that the following relationships hold (where the subscript 't' signifies the set of valid first degree formulae of the forms $(A \rightarrow B)$ or $(A \leftrightarrow B)$ In the given system):

$$(E_t \cap AI_t) = (AC + (A \rightarrow (A \vee -A)))_t,$$

$$E_t = (AC + (A \rightarrow (A \vee B)))_t,$$

$$AI_t = (AC + ((A \vee B) \rightarrow (-A \vee A)) + ((A \vee (B.-B)) \rightarrow A))_t,$$

$$(E_t \cup AI_t) = (AC + ((A \vee B) \rightarrow (-A \vee A)) + ((A \vee (B.-B)) \leftrightarrow A))_t.$$

ROBERT E. MAYDOLE. *On whether the general comprehension principle is consistent in the Łukasiewiczian logics.*

Let $Ł_\aleph$ be the nondenumerable-valued Lukasiewiczian first order logic with the binary predicate letter '$\in$' as the only predicate letter. Let C_2 be the two-valued classical first order logic with '$\in$' as the only predicate letter. Consider the Generalized Principle of Comprehension, GPC:

$$(\forall x_1)\cdots(\forall x_n)(\exists y)(\forall z)(z \in y \leftrightarrow F(z, y, x_1, \cdots, x_n)),$$

where $F(z, y, x_1, \cdots, x_n)$ is a wff with at most $z, y, x_1, \cdots, x_n$ as free. Is GPC model-theoretic consistent in $Ł_\aleph$?

GPC will be model-theoretic inconsistent in $Ł_{\aleph}$ if there is a transformation function g between the wffs of C_2 and $Ł_{\aleph}$ such that Thesis I and Thesis II are true. (We let 'con' stand for any wff which is contradictory in both C_2 and $Ł_{\aleph}$.)

THESIS I. *For every* wff A, *if* $(A \rightarrow \text{con})$ *is valid in* C_2, *then* $(g(A) \rightarrow \text{con})$ *is valid in* $Ł_{\aleph}$.

THESIS II. *There are* wff *instances* A *and* B *of GPC such that* (i) $(A \rightarrow \text{con})$ *is valid in* C_2; *and* (ii) *if there is a model* m *of* $Ł_{\aleph}$ *such that* $|B|^m = 1$, *then there is a model* n *of* $Ł_{\aleph}$ *such that* $|g(A)|^n = 1$.

In this paper we examine what appears to be a promising way of constructing a function g such that Thesis I and Thesis II are true. While we fail to construct such a function, and while the question of the model-theoretic consistency of GPC in $Ł_{\aleph}$ is left open, we believe that the methods and negative results of our investigation are worth discussing.

REFERENCES

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EMERSON C. MITCHELL. *A model of set theory with a universal set.*

Let T be the theory of the language of set theory saying that:

(1) Sets are extensional.

(2) Every set has a universal complement, i.e. given a set x there is a set y such that every set z is a member of y if and only if it is not a member of x.

(3) Every set x has a power set containing exactly the subsets of x.

(4) The result of replacing every member of a wellfounded set by some set is a set.

(5) The wellfounded sets form a model of Zermelo–Fraenkel set theory.

Then within the universe V of Zermelo–Fraenkel set theory there is a definable internal model of T. The members of the internal model are chosen by an inductive definition within V, and then a new membership relation is inductively defined such that the members of the internal model with the defined membership relation satisfy T. It happens that the members of the internal model which are wellfounded on the defined membership relation form an isomorphic copy of V. Thus one can regard the construction as an extension of V to a model of T.

COROLLARY. T *is consistent if and only if Zermelo–Fraenkel set theory is consistent.*

This paper constitutes the author's dissertation to be submitted to the University of Wisconsin-Madison. It was inspired by the author's hearing Alonzo Church give his paper *Set theory with a universal set*, ***Proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics*** XXV, 1974, pp. 297–308.

FREDERIC B. FITCH. *Excluded middle and the paradoxes.*

Whenever two-valued logic with an unrestricted abstraction rule (comprehension axiom) gives rise to a paradox, such as the Russell paradox, there is a circular dependence of a proposition on itself, and this is indeed true in all cases of impredicative reasoning. Two-valued logic can be slightly weakened, however, so that such self-dependent propositions fail to satisfy excluded middle (or at least fail to satisfy strong excluded middle in the sense described below), while propositions that are not self-dependent do satisfy strong excluded middle, since they depend ultimately on atomic propositions that satisfy strong excluded middle. By thus slightly weakening excluded middle (and so deviating slightly from two-valued logic), a system is obtained that has unrestricted abstraction and that is demonstrably consistent. This system, $C\Gamma$, is an extension of my system $C\Delta$ and is adequate as a foundation for the more essential parts of mathematics, including Riemann integration and the theory of continuous functions. The Skolem paradox does not arise since Cantor's theorems on nondenumerability fail because of their impredicative nature. Gödel's incompleteness result for elementary arithmetic also fails owing to the presence of a rule of ω-completeness, a rule which can be shown to fit into the overall consistency proof. A modal operator $\Box$ for necessity is definable in such a way that $\Box p = [[= = =] = p]$, where the equality symbol stands for a special equality relation. The property D of satisfying strong excluded middle is definable in such a way that $Dp = \Box[p \vee \sim p]$. Then $D(\Box p)$ and $D(Dp)$ are derivable as theorems. Propositions can be found which satisfy excluded middle but not strong excluded middle, e.g. ZZ where $Za = [\sim(aa) \& D(aa)]$. Here $\sim(D(ZZ))$ is provable, and so $\sim(ZZ)$ is also provable, as well as $ZZ \vee \sim(ZZ)$.

RUDOLF V.B. RUCKER. *Talking about the class of all sets: Large cardinals and Takeuti's nodal transfinite type theory.*

The class of all sets is not a set; and thus it is not the form of a possible thought. How then are we to interpret talk about such concepts as **C** (the class of all sets), or Ω (the class of all ordinals), or R_{Ω^+}, or (to be more extreme) $(\mu\alpha)$[for every ZF-term τ and for every $x \in C$, $\alpha \notin \tau[x, \Omega]$], etc.? Consider the following suggestive fact: If **C** is a model of ZF, then for any ZF-formula ψ and any $x \in \mathbf{C}$, $\mathbf{C} \models \psi[x]$ iff $\{\alpha : R_\alpha \models \psi[x]\}$ is stationary in Ω.

Some years ago, G. Takeuti presented [4] a way of extending the reflection result just cited to higher-order sentences in the context of a theory called NTT. My presentation of NTT will differ slightly from Takeuti's. NTT has an unusual language (the language of "transfinite type copy", which has variables of what would seem to be every nameable type,) a constant symbol Ω, and a new primitive unary predicate $\mathcal{N}$. The axioms groups are roughly: (i) $\mathcal{N}$ is a normal filter on Ω, (ii) For any ZF-formula ψ and any $x \in R_\Omega$, $\psi[x] \leftrightarrow \psi[x]^{(R_\Omega)}$, and (iii) For any ϕ in the language of transfinite type theory and any $x \in R_\Omega$, $\phi[x, \Omega] \leftrightarrow \{\alpha : \phi[x, \alpha]\} \in \mathcal{N}$. The idea behind (iii) is to use $\mathcal{N}$ to provide a semantic (set) interpretation of syntactic talk about higher types (concepts). Thus, e.g., $R_{\Omega+\Omega} \models T$ becomes $\{\alpha : R_{\alpha+\alpha} \models T\} \in \mathcal{N}$. If schema (iii) is only assumed for ZF-formulae one gets a weak theory, NTT^1. Solovay has shown in ZF that if there is a measurable cardinal, then NTT^1 is consistent (see [4, p. 102]).

I claim that a model of the full NTT has the form $\mathcal{U} = \langle U, \theta, \lambda_n, \mathcal{N}, \in\rangle_{n\in\omega}$, where (i) $\mathcal{N}$ is a normal filter on θ, (ii) $R_\theta < R_{\lambda_1} < \cdots < U$, and (iii) For any ZF-formula ψ and any $x \in R_\theta$, $\psi[x, \theta, \lambda_1, \lambda_2, \cdots, \lambda_n]^{(U)} \leftrightarrow \{\alpha : \psi[x, \alpha, \theta, \lambda_1, \cdots, \lambda_{n-1}]^{(U)}\} \in \mathcal{N}$. Note that U is a model of ZF, that we assume U is transitive, and that we do not require $\mathcal{N} \in U$. The intention here is that θ represents the class of all ordinals, and λ_1 represents the λ which Reinhardt [2, p. 200] calls "all *possible* Ω-classes", and λ_2 represents the ordinal of all the possible well-orderings beyond λ_1, etc.! Iterated applications of schema (iii) allow all talk about these increasingly imaginary concepts to be reduced to talk about $\mathcal{N}$ and the behaviour of the R_α's. E.g., if λ were the Reinhardt ordinal mentioned above, we could carry out the reduction:

$$(\Omega \text{ is } \lambda\text{-extendible}) \leftrightarrow (\{\alpha : \alpha \text{ is } \Omega\text{-extendible}\} \in \mathcal{N})$$

$$\leftrightarrow (\{\alpha : \{\beta : \alpha \text{ is } \beta\text{-extendible}\} \in \mathcal{N}\} \in \mathcal{N}).$$

THEOREM 1. *(ZF) If there is a measurable cardinal then there is a model of NTT.*

PROOF. Iterate the ultrapower ω times.

Takeuti showed in [4] that NTT is consistent with $V = L$, so we cannot expect a full converse to Theorem 1. However we do have the following partial converse.

THEOREM. 2. *(ZF) If there is a model $\mathcal{U} = \langle U, \theta, \lambda_n, \mathcal{N}, \in\rangle_{n\in\omega}$ of NTT, then there is a model $\mathcal{U}^* = \langle U^*, \theta, \lambda_n^*, \mathcal{N}, \in\rangle_{n\in\omega}$ of NTT for which $\mathcal{N}$ is a U^*-ultrafilter (in the sense of [1, p. 181]).*

PROOF. Let $U^* = L^{(U)}//(\theta + 1 \cup \{\lambda_n : n \in \omega\})$, and let $\lambda_n^* = l'\lambda_n$. (This notation is from [3].) If U happens to be a model of $V = OD$, then U can be used in place of $L^{(U)}$ in the definition of U^*.

THEOREM 3. *(ZF) If there is a model of NTT, then there is a model of ZF + "There is an ineffable cardinal which is ν-indescribable for arbitrarily large ν".*

PROOF. It can be shown that θ has the desired properties in the U^* of Theorem 2. The arguments depend on schema (iii), (iii)'s consequence that $\{\theta, \lambda_1, \lambda_2, \cdots\}$ is a set of indiscernibles, and the fact that the objects of rank $\geq \theta$ have names with set and indiscernible parameters in U^*. Given that concepts should have only a syntactic existence, U^* is a "natural" model of NTT.

THEOREM 4. *The U^* of Theorem 1 is a model of ZF which has a nontrivial elementary embedding j into itself such that θ is the first ordinal moved.*

PROOF. We let $j'\tau[x, \theta, \lambda_1, \cdots, \lambda_n] = \tau[x, \lambda_1, \lambda_2, \cdots, \lambda_{n+1}]$ for ZF-terms τ.

This j, of course, cannot be defined over U^* as one would then be able to see that the ordinal of

U^* has cofinality ω. Theorem 3 implies that θ is *a fortiori* "extendible" in U^*, although again the projection maps will not lie in U^*.

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STEPHEN G. SIMPSON. *First-order theory of the degrees of unsolvability.*

Let $(D, \cup)$ be the upper semilattice of degrees of unsolvability (cf. Kleene and Post, ***Annals of Mathematics***, vol. 59 (1954), pp. 379–407). It is shown that the first-order theory of $(D, \cup)$ is recursively isomorphic to the second-order theory of the natural numbers with plus and times. Also presented are results concerning first-order definability in $(D, \cup, j)$ where j is the jump operator.

JOEL FRIEDMAN. *The Universal Class has a Spinozistic partitioning.*

This paper is the sequel to a previous paper, in which it was shown that the set of finitely ranked sets has a Spinozistic partitioning, in a technically defined sense.

In this paper it is shown that the Universal Class, V, has a *strongly Spinozistic* partitioning, which may be defined as a partitioning with an absolutely infinite number of partition classes such that each partition class is ε-isomorphic to V (and hence to each other).

The proof[1] requires a Gödel-type exhaustion argument, in order to show the exhaustive property of the partitioning, as well as the following lemma, in order to show the disjointness property of the partitioning.

LEMMA. *There exists a proper class V' distinct from V such that V' is ε-isomorphic to V, $\phi \in V'$, and V' is pseudotransitive.*

A quasi-exhaustive sequence of elements z_β is then recursively defined, and the partition classes A_β are defined as follows:

DEFINITION. $A_0 = V'$, $A_\beta = V'(z_\beta/\phi)$ (where $X(z/\phi)$ is the result of replacing ϕ by z in X).

From this we see that a nontrivial settheoretical theorem may be directly suggested by a metaphysical system (nontrivial in the nonimmodest sense).

Moreover, this Spinozistic partitioning can, I believe, be built up to a full-fledged settheoretical model of my formalization of Spinoza's metaphysical system, (Part I of his ***Ethics***). This would then amount to a consistency proof for that formalized system (to appear).

[1] I am indebted to Professors R. Solovay and J. Silver for certain key ideas in the proof, as well as helpful suggestions required to complete the proof.

JOHN RICHARDS. *On making 'sense' of Frege's functions.*

Alonzo Church's latest revision of his logic of sense and denotation is based on several changes in Frege's theory. Church makes the following claims:

I. A function is saturated.

II. A name of a function is an object-name.

III. A name of a function has both sense and denotation.

IV. Although Frege would not accept any of the above, he would if we substitute "value-range of a function" for "function".

Church's formulation is an explicit rejection of the Fregean theory of functions. The formulation, which depends essentially on the *ungerade* occurrence of function-names, is a theory of objects, not functions.

In this paper I argue against each of these proposed changes. Most significantly, in IV there is an equivocation on "value-range of a function". For Church, a *function* is a *set* and the *sense of a function-name* is a *property*. Church is lead to reject Frege's claim that the form of the expression for a function is essential to the function. If a function is just a set of objects, then there is no sense in which the form of the expression is essential to it.

In the final section I suggest an expansion of Church's formulation in light of Frege's absolute dichotomy between a function and an object. Church's Δ function must be limited to object-names. Alternative (2), which makes the sense of A and B the same whenever $A = B$ is logically valid, is appropriate for identity of senses of object-names. It is necessary, however, to establish a separate category for function-names, and a new symbol η to represent the relation between the sense of a function-name and a function. Alternative (0) (synonymous isomorphism) is a necessary and sufficient condition for identity of the senses of function-names.

MICHAEL D. RESNIK. *A remark on Frege's class theory.*

Frege's original class theory contains the following generalization of the Russell paradox. Let f satisfy

(1) $(x)(y)(f(x) = f(y) \supset x = y)$

and define 'R' and 'r' by

(2) $Rx \equiv (\exists F)(x = f(\hat{y}Fy) \cdot - Fx)$,

(3) $r = \hat{x}Rx$.

Then by second order logic we have

(4) $Rf(r)$,

while from Frege's axiom (Vb) we obtain

(5) $- Rf(r)$.

Frege blocked the derivation of (5) by modifying (Vb) to

(Vb′) $\hat{x}Fx = \hat{y}Gy \supset (z)(z \neq \hat{x}Fx \supset (Fz \equiv Gz))$.

However, if f satisfies

(6) $(x)(f(x) \neq x)$

in addition to (1) the derivation of (5) can be easily reinstated.

By adding to Frege's system the axiom

(7) $\wedge \neq \vee$

one can define f by $f(\wedge) = \vee$, $f(\vee) = \wedge$, $f(x) = \{x\}$ if $x \neq \vee$, $x \neq \wedge$ and $x \neq \{x\}$; and $f(x) = \{x, \vee, \wedge\}$ otherwise. It can then be shown that f satisfies (1) and (6) with the result that Frege's "repaired" system has no model in domains of more than one object.

ROBERT S. TRAGESSER. *Numbers.*

C. Parson's criterion for the identity and existence of the natural numbers presented in *Frege's theory of number* is explicated. The explication of the identity criterion is "schematic" and so is neutral among the mutually deviating concepts of mathematical existence discussed by Parsons in his ***Ontology and mathematics.***

It is argued that the identity criterion alone is sufficient to found number theory. The existence criterion is superfluous. This supports and refines in a certain direction Benacerraf's thesis that "numbers are not objects".

However, using a phenomenological criterion explicit in Peirce and Husserl, and implicit in Gödel (as a foundation for his mathematical realism), it is argued that "the natural numbers" form an "*objective domain*", and in this sense have "objective reality". Some of the curious logical properties of objective domains are elaborated, e.g., that objects in a "domain" have no life independent of that domain (this makes further sense of certain points connected with Benacerraf's thesis), that the objects in an objective domain do not form a set (this is just as true of the domain of the natural numbers as it is of the domain of set theory!). These properties have foundational importance. What is their mathematical importance?

The "strict finitism" of Ésénine–Volpine and Rashevskii is briefly discussed, as well as the views of Wittgenstein and N. P. White.

JOHN GRANT. *Confirmation of empirical theories by observation sets.*

A model-theoretic framework is described for the study of empirical theories. An empirical theory is assumed to be formalized in first-order logic with equality, and an observation set consists of a set of observation reports. The relationship of various theories to a fixed observation set as well as the relationship of various observation sets to a fixed theory are investigated.

A criterion is proposed for the notion of an empirical theory being confirmed or not confirmed by an observation set. This criterion is more generally applicable than Hempel's criterion; it gives the intuitively correct result in some cases where Hempel's criterion gives counterintuitive results; it gives the same results as Hempel's criterion for the paradoxes of confirmation; and it does not satisfy Hempel's conditions of adequacy for any definition of confirmation. A criterion is also given for the comparison of observation sets with regard to how highly they confirm a theory. Finally it is shown that a historical aspect can be introduced into the logical theory of confirmation.

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DONALD NUTE. *The logic of causal conditionals of universal strength.*

We explore the relationship between causation and the logic of counterfactual conditionals. At least one author, David Lewis, has attempted a partial analysis of causal statements within a formal system intended to represent the counterfactual conditional of ordinary discourse. Causal conditionals of universal strength are a species of counterfactual conditional, but any attempt to define causal conditionals solely in terms of counterfactuals of ordinary discourse does not appear promising. Instead we undertake to investigate the logical structure of causal conditionals and the relationship between these and counterfactual conditionals without presupposing that one may be defined in terms of the other. First we outline a system of counterfactual logic which we have already developed in [3]. We then proceed to extend this calculus by the addition of axioms for a causal conditional of universal strength. This is not a simple extension of the counterfactual logic, so it is necessary to provide axioms which relate the new causal connective and the original counterfactual connective. Completeness results are provided using model theory adapted from Lewis [1] and Nute [2].

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GARREL POTTINGER. *Normalization as a homomorphic image of Cut-elimination.*

We define the systems $H_{\lambda L}$ and H_λ of intuitionist propositional logic and show essentially that the Cut-elimination theorem for $H_{\lambda L}$ is equivalent to the normalization theorem for H_λ and that the strong Cut-elimination theorem for $H_{\lambda L}$ is equivalent to the strong normalization theorem for H_λ. These results are obtained by (1) defining (a) reduction relations $\gtrsim$ and $\geq$ appropriate to $H_{\lambda L}$ and H_λ, respectively, and (b) a many-one mapping $\mathcal{N}$ from the class of $H_{\lambda L}$ derivations onto the class of H_λ derivations and (2) proving that, for all derivations $\mathbf{D}$ and $\mathbf{E}$ of $H_{\lambda L}$, $\boldsymbol{D} \gtrsim \mathbf{E}$ if, and only if, $\mathcal{N}(\mathbf{D}) \geq \mathcal{N}(\boldsymbol{E})$.

Our results are like those proved by a similar method for systems of intuitionist predicate logic and arithmetic in Zucker [1974], but they are superior to Zucker's results in the following two respects. (1) They apply to systems including disjunction, whereas Zucker's correspondence between derivations fails to yield the desired theorems for systems including disjunction. (2) $\mathcal{N}$ is shown to be a *natural* mapping according to the usual way of explaining the intuitive sense of derivations in intuitionist propositional logic, whereas Zucker does not provide an intuitive reason for using the particular mapping he employs.

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JONATHAN P. SELDIN. *The predicate calculus in $\mathscr{F}_{22}$.*

The system $\mathscr{F}_{22}$, which is essentially a type-free intuitionistic predicate calculus without conjunction, alternation, and negation but with quantification over propositional functions, was introduced and proved consistent in a weak sense by Curry in [CSC]. A proof that the system is consistent in a stronger sense (there called Q-consistency) appears in [QCF]. A proof that this stronger form of consistency holds when a restriction of Leibnitz' rule for equality is adjoined appears in [EFTT].

In this paper, the system $\mathscr{F}_{22}$ is extended to include the other connectives and quantifiers. Systems are formed which correspond to the systems TA*, TM*, TJ*, TD*, TC*, TE*, and TK* of Curry [FML, Chapter 7]. The definition of canob is extended as in [CLg. II, §15D1], and from this definition the consistency result of [EFTT] is obtained for all of the systems. (This result is a proof normalization theorem, and for the systems TD*, TC*, TE*, and TK*, proof normalization steps of the kind considered in [PNG] are used.) Then a theorem corresponding to [CLg. II, Theorem 15D4] (which says that if $M \vdash X$ is provable in one of the systems of [FML], then $M, K \vdash X$, where K is a set of grammatical conditions as defined in [CLg. II, p. 424] but where we must have $\alpha \equiv \mathsf{E}$ as indicated in Curry [CSC, p. 491], holds in the corresponding $\mathscr{F}_{22}$ system) is proved.

If Rule $\mathsf{H}\Xi$, which is $\mathsf{L}X, \mathsf{L}Y \vdash \mathsf{H}(\Xi XY)$, is replaced by $\mathsf{L}X, \mathsf{F}X\mathsf{H}Y \vdash \mathsf{H}(\Xi XY)$ (so that the system is one of relative canonicalness as defined in [CLg. II, §15C5]), then the resulting systems are very similar to the description-free parts of the systems of Stenlund; the $\mathscr{F}_{22}$ system corresponding to TK* corresponds to the system of Stenlund [LDE] and the one corresponding to TJ* corresponds to the system of Stenlund [DIL] (provided in the latter case that Stenlund's rule $\perp \vdash A$ is replaced by $\perp, A \in F \vdash A$; but I think this replacement is needed anyway to prove the theorem of §2.4).

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THOMAS M. LESCHINE. *Propositional logics determined by topological matrices: Logics with restricted substitution.*

Let X be a topological space and let B be any open base for the pseudo-Boolean algebra $O(X)$ of open subsets of X such that $X, \varnothing \in B$. The pair $M(X, B) = \langle [B], \{X\} \rangle$ forms a *topological matrix*, where $[B]$ is the subalgebra of $O(X)$ generated from B. If $\boldsymbol{M}$ is any class of topological matrices, let $\boldsymbol{M}^*$ be the set of propositional formulas valid in each matrix of class $\boldsymbol{M}$, and let $\boldsymbol{M}^*b$ be the set of formulas *b-valid* in each matrix of $\boldsymbol{M}$ in the following sense: f is b-valid in $M(X, B)$ if those homomorphisms into $[B]$ which map propositional variables to members of B satisfy f. Thus $\boldsymbol{M}^* \subset \boldsymbol{M}^*b$. $\boldsymbol{I}$ and $\boldsymbol{C}$ are the sets of intuitionistically and classically provable formulas respectively.

THEOREM 1. *If $\boldsymbol{V}$ is the class of all topological matrices, $\boldsymbol{V}^* = \boldsymbol{V}^*b = \boldsymbol{I}$.*

THEOREM 2. *Given any class $\boldsymbol{M}$ of matrices and any formula f, $f \in \boldsymbol{M}^*$ iff $\varepsilon f \in \boldsymbol{M}^*b$ for all substitutions ε.*

This leads to intermediate logics definable by restricted substitution when appropriate topological conditions on base B are not inherited by [B]. Thus let $pV\neg p$ and $\neg\neg p \rightarrow p$ be instances of axiom schemes $fV\neg f, \neg\neg f \rightarrow f$ respectively, in which p is a propositional variable. Then if $M(X, B)$ is any matrix, THEOREM 3. *$\{M(X, B)\}^* = C$ iff $\{M(X, B)\}^*b = C$ iff B is a closed-open base. In particular, $p \vee \neg p \in \{M(X, B)\}^*b$ iff B is closed-open.*

THEOREM 4. *$\neg\neg p \rightarrow p \in \{M(X, B)\}^*b$ iff B is regular open. If B is regular open but not closed-open, $\{M(X, B)\}^*b \neq \mathbf{M}^*$ for any matrix class $\mathbf{M}$.*

Let Cn be the matrix consequence operator defined on matrix $M(X, B)$ and let the *b-consequence operator* Cnb be defined be relativizing the definition of Cn to consideration of only those mappings used in defining b-validity above.

THEOREM 5. *Consequence operator* Cnb *is structural iff* Cnb = Cn. Cnb satisfies the cancellation property [1].

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PETER EGGENBERGER. *Remarks on Brouwer's notions of time, mind and mathematics.*

When Brouwer talks about time he means the succession of moments of existence. Consciousness or *mind* starts with the perception of time. All objects *including mathematical objects* are mental and temporal creations in the sense that the mind creates them during some moment or series of moments of existence. Brouwer's mind should be conceived, perhaps, as *mental life*. The mind, mental life, is not itself an object. Brouwer thinks eternal or atemporal objects do not exist. The external world is a creation of the mind and it is neither eternal nor temporal.

Time has three roles in Brouwer's notion of mathematics. Ontologically, as just stated, mathematical objects are temporal. Epistemologically, the fundamental concept of mathematics, one-twoity or succession, is abstracted from time. Constructively, mathematical objects can be constructed *over* time, i.e. moments of existence.

Mind also has three roles in Brouwer's notion of mathematics. Ontologically, again as stated, mathematical objects are mental creations. Epistemologically, the properties of mathematical objects must be discernible by the mind. Constructively, mathematical objects are constructed by *free* acts of the mind.

Many criticisms and elucidations of Brouwer's notion of mathematics are mistaken or misleading because they ignore the role of time and the mind in Brouwer's mathematics. Griss's criticisms of Brouwer are based upon a misunderstanding of Brouwer's fundamental concept of mathematics: Griss ignores the role of time. Discussions of Brouwer's creating subject tend to treat it as an ideal mathematician. For Brouwer, it is the real mathematician and "empirical mathematicians" are "ideal objects".

An adequate metamathematical interpretation of Brouwer's intuitionistic mathematics must start from a faithful analysis of the notions of time and mind as they appear in Brouwer's writings. Until we possess such an analysis we cannot be said to have an accurate intuitive or informal conception of intuitionistic mathematics.

MAŁGORZATA ASKANAS. *On truth and provability in Peano arithmetic.*

We consider the following proof of Gödel's Incompleteness Theorem: Assume we have a Gödel numbering for which the function assigning to the Gödel number n of a formula $F_n(x)$ the Gödel number of $F_n(n)$ is expressible. For $A \subset N$, let A^* be the set of all n for which the Gödel number of $F_n(n)$ is in A. If H expresses A, some formula H^* will express A^*. Let h be the Gödel number of $\neg H^*(x)$ and let p be the Gödel number of $\neg H^*(h)$. Then p is the number of a true sentence of arithmetic iff p is not in A. Thus an expressible set—in particular, the set of Gödel numbers of all theorems—cannot coincide with the set of Gödel numbers of all true sentences.

We formalize this argument within Peano Arithmetic as follows. With every formula $\psi(x)$ we associate a sentence Sat ψ whose meaning, intuitively, is that the set Ψ of sentences whose Gödel numbers lie in the set expressed by ψ is the truth set. More concretely, Sat ψ expresses the fact that Ψ contains all true atomic sentences, for every sentence contains either it or its negation but not both, contains a disjunction (conjunction) iff it contains at least one of the disjuncts (both conjuncts), and contains an existential (universal) quantification iff it contains at least one of the instanciations (all the instanciations). The following results can then be established:

THEOREM. *For every formula $F(x)$, the sentence* $\mathrm{Sat}\,\psi \to (\exists x)(F(x) \equiv \psi(x))$ *is a theorem of Peano Arithmetic.*

MAIN THEOREM. *For every formula F,* $\neg$Sat *F is a theorem of Peano Arithmetic. In particular, if* Prov *expresses the set of Gödel numbers of theorems,* $\neg$Sat Prov *is a theorem.*

We further construct two sentences Compl and ω-Consist to express, respectively, completeness and ω-consistency of arithmetic, and show

LEMMA. *The formula* Compl & ω-Consist $\to$ Sat Prov *is a theorem of Peano Arithmetic.*

THEOREM. $\neg$(ω-Consist & Compl) *is a theorem of Peano Arithmetic.*

ALBERT A. MULLIN. *On the computability of spline functions.*

This note deals with the interface between mathematical logic and numerical analysis.

LEMMA 1. *The decision problem as to whether or not the graph of an arbitrary spline function in more than one variable contains a lattice point (Gitterpunkt in the sense of Minkowski) is recursively unsolvable.*

On the other hand,

LEMMA 2. *There exists a Turing machine for determining all of the lattice points, if any, in the graph of an arbitrary convex polygon in* E^2.

LEMMA 3. *There exists a Turing machine for determining all of the lattice points, if any, in the graph of an arbitrary conic section in* E^2.

Proofs of Lemmas 2 and 3 use the Church–Turing Thesis.

Problem. Does there exist a Turing machine for determining all the lattice points, if any, in the graph of an arbitrary spline function in *one* variable?

Finally, questions of the computational complexity of B-splines are discussed informally.

LARRY W. MILLER. *The significance of the ordinals* $(n;1)$ *and* $(n;\Omega_{n+1})$.

In [1], I showed how to use techniques of Veblen, Bachmann, and Isles to form hierarchies of normal functions and get constructive ordinal notations for ordinals $< (n;1)$ and $(n;\Omega_{n+1})$ for each n, where

$$(n;x) = \boldsymbol{\omega}(\boldsymbol{\Omega}(\cdots(\boldsymbol{\Omega}_n(x,0),0)\cdots,0),0), \quad \boldsymbol{\Omega}_n : \Omega_{n+1} \to \Omega_{n+1}$$

is the normal function Ω_n^x, $\Omega_0 = \omega$, and $\Omega_1 = \Omega$. E.g., $(0;1) = \varepsilon_0$, $(0;\Omega) = \Gamma_0$, $(1;1) = \omega(\varepsilon_{\Omega+1}, 0)$, and $(1;\Omega) =$ Bachmann's $H(1)$. In this paper, I show the significance of the ordinals $(n;1)$ and $(n;\Omega_{n+1})$ by establishing the following conjectures of [1].

1. $(n;1)$ and $(n;\Omega_{n+1})$ are the ordinals of Takeuti's systems of ordinal diagrams $O(n+1,1)$ and $Od(n+1,1)$ under $<_0$. Generally, $(n;t)$ and $(n-1;\boldsymbol{\Omega}_n(\Omega_{n+1}, -1+t))$ are the ordinals of $O(n+1,t)$ and $Od(n+1,t)$ under $<_0$.

2. $(n;1)$ and $(n;\Omega)$ are the least ordinals inaccessible with respect to Feferman–Aczel iteration functionals of diagonalization over Ω_n through finite and transfinite types $(n;\Omega)$ is the least fixed point of $(n;x)$.

3. $(n;1)$ and $(n;\Omega_{n+1})$ are the proof theoretic ordinals of the theories ID_n of n-fold iterated generalized inductive definitions and ID_n^* which is to ID_n as predicative analysis ID_0^* is to first order arithmetic ID_0, i.e., predicative construction given the inductively defined set.

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