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# Cause as an Implication *

**Abstract.** An appropriately unprejudiced logical investigation of causation as a type of implication relation is undertaken. The implication delineated is bounded syntactically. The developing argument then leads to a very natural process analysis, which demonstrably captures the established syntactical features. Next relevantly-based semantics for the resulting logical theory are adduced, and requisite adequacy results delivered. At the end of the tour, further improvements are pointed out, and the attractive terrain beyond present developments is glimpsed.

The notion of cause, having fallen from favour in the heydays of logical positivism, has enjoyed a contemporary resurgence. But despite its fashionability now, especially as a major foundational element in epistemology, the logical and structural properties of causation remain quite insufficiently examined. In this situation, who knows whether the foundations will carry the philosophical castles being built (they are never complete, and invariably ramshackle)? Our preliminary investigation of causal implication suggests they will not; like structurally and materially short-supplied high-rise buildings, they will come tumbling down.

In treating *cause* as like a conditional (of implicational type) there is a familiar problem of getting in a satisfactory way from natural language locutions, e.g. of the form "$\alpha$ causes $\beta$", to conditional forms which typically couple sentences. A variety of expressions plug into the causal form, not just event subjects but gerundives such as "smoking" – but not significantly sentences. One difficulty in working with "$\alpha$ causes $\beta$" as primitive is that not all the usual sentential connectives are particularly well-defined on the relevant substituents; e.g. negation becomes problematic with event clauses, though not as unintelligible as with proper names.

Still, it is not so difficult to make out what negation is doing applied to event clauses, to construe for instance $\sim$ (mowing the lawn), as not moving the lawn. Let us allow in *whatever* can be made to fill out variable places, and let variables correspondingly range over such makable-out values. The constraints will then be *imposed* by the formation rules, not fixed in advance. Connectives which combine variables $A$, $B$, $C$, ... delimit what can be made out. For

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comparison with familiar logics, let these connectives be, to start with, just &, $\vee$, $\sim$, giving the zero-degree wff. Then we add the causal connective $\ni$, read in context "(the) obtaining of... causes (the) obtaining of...". It can be left open for the present to what extent $\ni$ combines with extensional connectives &, $\vee$, $\sim$; for instance, whether an entire higher degree results or only a first degree part. This is a matter of what can be made out. Making out itself can eventually be taken up through significance.

In this way we arrive, hopefully then, at a sufficiently well-behaved form $A \ni B$, to begin logical investigations, a form where &, $\vee$, $\sim$ are well-defined on combinations of $A$ and $B$, and also on uniterated forms $A \ni B$. For example, $A \ni B$ may read $\ulcorner A$ causes $B\urcorner$, with $A$ and $B$ not being sentences, or $\ulcorner$should $A$ happen it would cause $B$ to happen$\urcorner$, or it may sometimes be read $\ulcorner A$'s happening causes $B$'s happening$\urcorner$, or it may even be read $\ulcorner$the truth of $A$ *causes* the truth of $B\urcorner$ (whatever that really means). The essential point is that we do not shift out of the causal idiom at the outset (as e.g. Burks [1], Zinovev [11], and others do), and so prejudge or prejudice several issues, such as whether causal idiom is modal, whether we are working with something like partial sufficiency or not, and so on.

In this way also we can wend through a minefield of research-paralysing objections (such as those laid by Davidson [2], e.g. p. 161) as to what the grammatical and logical forms of causal statements really are. In particular, we do not say, or require, that the relation of causation is represented through a sentential connective. It is enough that $\ni$ forms sentences on certain terms, which are sometimes propositional or fact-like or rendered such by happening or occurrence functions (so delivering a partial higher degree). Most conveniently it is much the same with *implies* as it is with *cause*, and as it is with bridge verbs such as *means* and *intends.* All these two-place connectors are sentence-forming on many linguistic items other than propositional or fact-like expressions. For instance "that flood caused a famine"; is parallelled by "that flood meant a famine"; "the failure of the sprinkling system caused that fire" is matched by similar sentences builtaround "meant" or "implied". What is this implication-like causal connection like?

The method of the reflexive equilibrium approach we then adopt is this: we try to reach, by working through a standard list of key logical principles for implication, a core logic and some important rejections for the notion of cause. We certainly leave open the option that various types of cause can be characterized through satisfaction of extra principles, e.g. reverse causation, proximate causation, logically extended causation, etc. In reaching a core we are much helped if we separate out various notions that have been confused with cause, such as conditionals. Having begun to move to a core causal logic, and associated typology, we begin to cast about for some logical explanation of the core, preferably of a semantical cast or semantically adaptable. In the presentation we borrow symbolism and terminology from [6].

## 1. Apparent principles, and rejections, of causal implication

Clearer principles of a logic of cause, formulated in $\ni$, include these: *Transitivity* at least in rule form:

$$\frac{A \ni B \quad B \ni C}{A \ni C}$$

(In fact Conjuctive Syllogism, (CS) $A \ni B \;\&\; B \ni C \rightarrow . A \ni C$, looks alright.) It is a little theorem, given appropriate epilogical apparatus, that

$$\frac{A_1 \ni A_2 \quad A_1 \ni A_3 \ldots A_{n-1} \ni A_n}{A_1 \ni A_n}$$

*Irreflexivity*: $\sim (A \ni A)$. Here the logic of causes diverges, sharply and distinctly it seems, from the logic of implication (under usual construals and interpretations). It also departs from the logic of explanation. For one thing, explanation does not require existence. The scientific creation, the initial Big Bang, may lack a cause, because nothing existed before it, but it does not lack an explanation. For another, self-explanation is not ruled out logically. Explicit irreflexivity separates causal implication from what is now dubbed non-circular reasoning; for that, while rejecting Identity leaves its negation open[1].
*Modus Ponens* (MP):

$$\frac{A \quad A \ni B}{B}.$$

(Like substitutivity, e.g. where $\Gamma$ is null, this exposes $A$; for in the first premiss $A$ is not embedded in a context of form $\ni$). MP appears to settle against *mere* partiality of cause: the connection really is invariant. The principle also helps distinguish *cause* from *reason* (see further [8]). Similarly, where it makes sense (with $A$ and $B$ so exposed),
*Modus Tollens*:

$$\frac{\sim B \quad A \ni B}{\sim A}.$$

The principles arrived at so far are enough to justify the honorific title "implication" (though some would want to insist that it is an exceedingly weak or decidedly rum implication). The linguistic connections seem to be more or less in order. We can move from $A \ni B$, through ⌜if $A$ *then* $B$, as a matter of causation⌝, through ⌜$A$ causally implies $B$⌝, and back again, naturally enough, for many substitutions upon $A$ and $B$. The logical issues as to whether some

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[1] Marketed non-circular reasoning, as in Martin and Meyer's "*S* for Syllogism", diverges from causal implication, and perhaps from genuine non-circular reasoning, in its heavy endorsement of *Exported* Syllogistic principles (e.g. in $\rightarrow$ form, $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$ and its permuted mate).

sorts of containment connections (those of *im-plicatio*) can be induced we shall come to: but to anticipate, the answer is that they can.

From here on, however, principles get less and less clear. For consider *Contraposition* (which would give Modus Tollens from MP):

$$\frac{A \ni B}{\sim B \ni \sim A}$$

This seems wrong; for example, no lung cancer does not *cause* no heavy smoking, though it *implies* in some sense that there has been none. Cause induces an implication connection that *extends it* $\Bbbk$ say. Then in place of Contraposition the following appears to hold:

$$A \ni B/A \Bbbk B \qquad A \Bbbk B/\sim B \Bbbk \sim A.$$

The *direction* of cause also appears wrong for Contraposition. The matter of directionality of causation is of course an important, and not uncontroversial, one.

*Substitutivity*: in familiar form, $A \ni B, B \ni A/\Gamma(A) \ni \Gamma(B)$, where $\Gamma(A)$ is a compounding (at least truth-functional) of $A$, fails. Normally its success would be tied to that of Contraposition and analogous compounding principles. But in causal cases, joint satisfaction of the premisses is excluded. If both premisses were to obtain then by transitivity $A \ni A$, violating irreflexivity. So substitutivity fails by default.

*Connexivity*:

$$\frac{A_1 \ni A_2 \quad A_2 \ni A_3 \ldots A_{n-1} \ni A_n}{\sim (A_1 \ni \sim A_n)}$$

with special cases

*Aristotle*:

$$\sim (A \ni \sim A)$$

*Rule Strawson*:

$$A \ni B/\sim (A \ni \sim B).$$

Causation is, so to say, strongly consistent; for with causal consistency defined, $A \between B = \sim(A \ni \sim B)$, then $A \between A$, etc. Such connexive principles tend to make the theory strongly resistant to known logical technology.

*Composition*:

$$(\&\text{ form}) \quad A \ni B \quad A \ni C/A \ni B \,\&\, C$$

$$(\vee\text{ form}) \quad A \ni C \quad B \ni C/A \vee B \ni C.$$

But that is around about where familiar &-∨ behaviour terminates. By contrast, other lattice principles almost all fail. Thus, with $\dashv$ representing in the usual way rejection, $\dashv A \,\&\, B \ni A$; $\dashv A \ni A \,\&\, A$; $\dashv A \vee A \ni A$; $\dashv A \ni A \vee B$;

etc. Distribution, however, may hold where the same variables figure appropriately (?). Consider next
*Augmentation*:

$$\frac{A \ni B}{A \,\&\, C \ni B} \qquad \frac{A \ni B}{A \ni B \vee C}$$

Such principles are decidedly doubtful. They are in doubt for the same sorts of reasons that Augmentation is problematic, at least, for what is closely related to causation, counterfactual conditionals. For the tacked-on additions are doing no *causal* work, and more important may *interfere* with causal connections. Maybe instead what hold are the implicational extensions:

$$\frac{A \mathrel{|\!\!<} B}{A \,\&\, C \mathrel{|\!\!<} B} \qquad \frac{A \mathrel{|\!\!<} B}{A \mathrel{|\!\!<} B \vee C}$$

*Factor*, for instance in the &-form, $A \ni B/A \,\&\, C \ni B \,\&\, C$, is similarly dubious, because the tacked-on components typically do no causal work, and may indeed interfere.

The *higher degree*, insofar as it can be made out (or makes sense), is still more problematic. One principle that looked good to one of us at first is *DC*:

$$\frac{A \ni B \quad A \ni (B \ni C)}{A \ni B}.$$

But another principle that went along with DC is *much* less evident, namely *Contraction*:

$$\frac{A \ni (A \ni B)}{A \ni B},$$

whence

$$n\text{-place contraction: } \frac{A \ni \big(\ni \ldots (\ldots (A \ni B) \ldots)\big)}{A \ni B}.$$

Perhaps Contraction is a *defective* case of the DC principle; it ordinarily would be derived using $A \ni A$, no principle of causation.

As is by now evident, there are two directions of primarily first degree development feasible:

(1) embedding the logic in an entailment logic, as with principle (CS) of Conjunctive Syllogism above. Then principles like $\dfrac{A}{\sim\sim A}$, $\dfrac{\sim\sim A}{A}$ are automatically given in.
(2) elaborating the logic of its own. Then the pleasant weakness of inferential principles can become a serious initial difficulty.

## 2. Beginning on a typology – so postponing many complications

Various types or subvarieties of cause can be distinguished. As indicated, *augmentable* causation can be distinguished, either as that subclass also satisfying augmentation principles or, more satisfactorily, as the closure of the notion under augmentation rules; *a reversible* causation as the closure under contraposition; and so on. Different is actual (or realized) causation. The emphasis on what is actual is especially strong with notions like cause, but so far we have been operating, in a way resembling conditionals, with hypothetical or putative causes. *Actual* causes, ontologists want to insist, must happen. Let us distinguish actual causes then, for instance by the principle:

$$A \ni B \rightarrow . A \,\&\, B.$$

Then actual cause fails various principles that hold for (putative) cause, such as Modus Tollens; and would accordingly make a difficult starting point. In general, actuality requirements damage the smoothness and generality of a logic, here as elsewhere (e.g. with existential requirements and loading in quantification logic). However given (hypothetical) causation, actual causation can be defined as holding *where* the ancecedent is realized, i.e. in extensional approximation, $A \ni^E B =_{df} (A \ni B) \,\&\, A$. Then the distinguishing principle follows, where a suitable implication is available.

Quite different "types" again are those of *partial* cause – not really a cause – and *local* and *proximate* causes. A partial cause is a causal factor in Mill's sense, one significant component in a cause. At first shot it can be characterized enthymematically: $A$ is partial cause of $B$ iff $A$ is part of some cause $C$ which causes $B$. Local or immediate cause is a cause, delimited by a neighbourhood requirement (or even more narrowly, in the seventeenth century, by a juxtaposition requirement). Local cause contrasts with remote cause or action at a distance. Provided "neighbourhood" is not too narrowly construed, cause is a transitive closure of local cause. And local cause is of course cause restricted to a neighbourhood, e.g. $A \ni_L B$ iff $A \ni B \,\&\, (\mathrm{P}n)$ $(A \subseteq n \,\&\, B \subseteq n)$, or some such. Some topology can be pulled in to clarify neighbourhood behaviour. In physical theory action at a distance was made local by way of fields. Is there any holistic analogue (some sort of causal flux or "field") that can be profitably introduced into causal logic? These are all questions we can postpone until the more pressing logical work has advanced, until we have gained some semantical grip upon the core implicational notion.

## 3. An organizing process account

A process analysis both has much appeal and promises a neat semantical theory for causal implication – at least at the first degree, and presumably it admits of *some* higher degree extension. The basic connection, given in translation form [2], rather than in semantical form, is as follows:

P1. $A \ni B$ iff $A \to B \,\&.\, A < B$,

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|---|---|---|
| *process* | = *relation* | + *direction*: |
| causal | sufficiency | initiation, priority, e.g. |
| conditional | conditional | time order |

While there is nothing inevitably first-degree about this connection (unless priority is narrowly construed), suppose for the present that casual implication is first-degree, i.e. $\ni$ is not nested; this is after all the usual intuitive setting. Now the first-degree semantical analysis of implication or conditional, $\to$, is essentially a *constant conjunction* one; $A \to B$ is true iff for every situation where $A$ holds so does $B$. The situations involved are in no way restricted to those that actually occur, so there is no serious problem about accidental "conjunction". But the association may still not be causal, for instance $B$ could be a logical consequence of $A$ or $A$ an effect of $B$. The problem is resolved, it is supposed, by a priority ordering, which gives a direction. Commonly it is assumed that the priority ordering is a temporal one, but it strikes us as illegitimate to rule out time-reversing causation, such as is alleged to occur in some psychic phenomena, in this crude way. So we generalize time ordering to "action priority"; the time of $A$ may in principle be later than $B$, yet $A$ has initiation action or priority to $B$; casually $A$ is the *prior moment*. Those who find this idea paradoxical (e.g. because initiations or actions fix times) can however construe $c(A, a)$, the causal moment of initiation of $A$ at situation $a$, just as the time of $A$ at $a$.

The priority ordering $A < B$ will be interpreted then, at $a$, through the precedence ordering $c(A, a) < c(B, a)$. Writing $c(A)$ for $c(A, T)$, i.e. the action of $A$ at actual (truth determining) situation $T$, then $A < B$ will be true when $c(A) < c(B)$, i.e. roughly when the time of $A$ precedes that of $B$. The conditional, $A \ni B$, is interpreted in the standard relevant and strict way in terms of formal (truth-) value preservation, i.e. as $I(A, a) = 1 \supset_a I(B, a) = 1$, where $I(A, c) = 1$ reads: $A$ holds at (situation, world) $c$. Thus, at the first-degree, we can arrive at

P2. $A \ni B$ is true iff $(a \in K)\big(I(A, a) = 1 \supset I(B, a) = 1\big) \,\&\, c(A) < c(B)$,

i.e. iff whatever apposite situation $a$, $B$ holds at $a$ when $A$ does and $A$ is initiated before $B$, i.e. iff $B$ is constantly conjoined with $A$ which is the prior. [3]

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[2] Connections of this sort appear in the literature; e.g. P1 is suggested in essentials by von Wright (who also introduces the notion of action, but unfortunately conflates it with manipulation). Indeed, any combination of a conditional with a tense-logical ordering will yield something like P1.

[3] As we found, on subsequently browsing relevant literature, we have been thoroughly anticipated — this is no deficiency here, but grist to our mill — in a semantical amount of general

Delineation of the class $K$ of *apposite situations* – perhaps certain situations neighbouring or similar to that where evaluation is taking place – is of course critical in a full theory and in many applications. In the further clean-up of P2, we may want to restrict situations also to those that conform to various constraints, for instance (at some risk of circularity) to natural laws. But those are important details that we can here comfortably leave in the background.

With but few assumptions, we can now substantially confirm initial intuitive principles. It will be assumed – what can be cogently argued – firstly that implication ($\rightarrow$) conforms at least to the principles of first degree relevant implication (thus it could be a strict implication, etc.), and secondly that the order $<$ is transitive and irreflexive (and so, antisymmetric).

PROPOSITION 1. *Causal conditional $\ni$ is an implication satisfying Rule Transivity, Irreflexivity, Modus Ponens, Modus Tollens but failing Identity.*

*Verification* uses P1 and properties given.
To proceed the further intuitive distance, it will be assumed, thirdly, that at least for extensional connectives in the same components priority is determined componentwise, that is, $c(\sim A) = c(A)$, $c(A \,\&\, A) = c(A)$, $c(A \vee A) = c(A)$. This leaves open, so far, what is said as to $c(A \,\&\, B)$, and so on, for different components.

PROPOSITION 2. *Causal conditional $\ni$ is a connexive implication satisfying Aristotle but failing Contraposition, Simplification, Addition (and elementary lattice conditions generally), and very important, failing* (SI) *Substitution.*

*Verification* is again elementary, but some examples are instructive. Consider Aristotle, which expands to $\sim (A \rightarrow \sim A) \vee \sim (c(A) < c(A))$, and is carried by the latter disjuinct. If Simplification held, then $A \,\&\, A \ni A$ would hold, implying $c(A) < c(A)$. Rule Contraposition fails because $<$ is antisymmetric. Etc. As the example of Aristotle reveals, some very curious negative statements are also verified. Let $\Delta_1(A)$, $\Delta_2(A)$ be any extensional functions just of $A$. Then $\sim (\Delta_1(A) \ni \Delta_2(A))$.

To fill out the first degree picture, it remains to impose appropriate connections between $c(A \,\&\, B)$ and the pair $c(A)$, $c(B)$. A natural approach to the issue of connections is to let actions (or times) pluralise, so the actions of $A \,\&\, B$ are those of $A$ and of $B$, i.e. $c(A \,\&\, B) = c(A) + c(B)$, where $+$ is some sort of union operation (a semi-lattice operation suffices). Then, since any truth-functional combination $\delta(A_1, \ldots, A_n)$ can be defined in terms of just

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form P2, in particular by Mackie, who proposed a conjunction of necessity-in-the-circumstances with causal priority (see also the discussion of Mackie in Leman [4], pp. 255–6). We agree with Mackie and Leman that a conditional or qualified-necessity analysis of causation needs to be supplemented by a further element, affording a foundation for causal assymetry. But we disagree with Leman that this foundation, provided by priority (or fixity), has to be epistemological (for many reasons, some of which Leman himself advances, p. 256).

connectives & and $\sim$, $c(\delta(A_1, A_2, \ldots, A_n)) = c(A_1)+c(A_2)+ \ldots +c(A_n)$. For instance, $c(A \vee B) = c(\sim(\sim A \,\&\, \sim B)) = c(\sim A \,\&\, \sim B) = c(\sim A)+c(\sim B) = c(A)+c(B)$ (given semi-lattice conditions and either suitable definitions of truth-functions or the principle $A \leftrightarrow B/c(A) = c(B)$ for $A$ and $B$ truth-functional). It follows, what may be a little surprising, that action reduces to that of sentential variables, at least in truth-functional cases. (Clearly, such distribution principles fail, or *should* fail, for such functors as the tense-logical $F$, it will be the case that: $c(A) \neq c(FA)$ for many $A$.)

PROPOSITION 3. *Causal implication satisfies various Composition principles, conforms to Augmentation only in restricted form, and fails Factorization principles unless also restricted.*

*Verification* and elaboration. Consider for instance &-Composition. $A \ni B$, $A \ni C/A \to B$, $A \to C$, $c(A) < c(B)$, $c(A) < c(C)$. Then, by the first pair $A \to (B \,\&\, C)$, while by the second $c(A) < c(B)+c(C) = c(B \,\&\, C)$, whence, assembling, $A \ni (B \,\&\, C)$. Factorization by contrast fails, for there is no guarantee that when $c(A) < c(B)$ that $c(A \,\&\, C) < c(B \,\&\, C)$, i.e. $c(A)+c(C) < c(B)+c(C)$. What happens with Augmentation, which similarly fails in general, is interesting. Consider the form: $A \ni B/A \,\&\, C \ni B$. The critical issue, given implication sustains augmentation, is when $c(A) < c(B)$ guarantees $c(A)+c(C) < c(B)$. Presumably generally only if $c(C) < c(B)$; roughly $C$ is casually relevant to $B$. In short, causally relevant additions can be tacked on, but not arbitrary ones.

## 4. Running out initial semantics

The account given lends itself to ready semanticization. The semantics extends, or can be pushed, to the higher degree, i.e. for full sentential systems – given in our first attempt the implausible assumption that $c(A, a)$ is always determined through initial components (i.e. in pure calculi, variables). We shall offer semantics for full systems, though the significance of nested occurences of $\ni$ is often in doubt, to say the least, beyond the second degree. As we shall see, relevance itself provides some natural restrictions, and enables reduction of implausibility.

We take over semantics for relevant logic formulated in terms of connectives ($\to$, &, $\vee$, $\sim$). We view the semantics as fashioned for some deeper relevant system, for instance for a relevant theory of conditionals such as that of Hunter (not really for the excessively strong relevance systems of Anderson and Belnap or the analogous strict systems of Lewis, though they can be used for all of them). A relevant model structure m.s. $\langle T, K, O, R, {*}\rangle$ comprises a set $K$ of situations (or worlds), a subset $O \subseteq K$ of regular situations (where theorems hold), a base (actual) situation $T$ in $O$, a 3-place relation $R$ on $K$ (modelling implication) which collapses to a 2-place relation in irrelevant logics, and a reversal operation $*$ (which takes care semantically of negation).

At is happens, the regular situations will come to play a significant role in relevant causal implications, as in deeper relevant logics. We add to such relevant m.s. (further explained in [6]), a set $Ca$ of *moments* for each situation $a$ in $K$, and an operation $+$, on each $Ca$, which is commutative, associative, idempotent (i.e. $\langle Ca, +\rangle$ is a semi-lattice). The moments or actions of set $Ca$ (which will be constructed canonically just in terms of clauses of sentential variables, whatever these objects do) are open to a variety of interpretations, here as initiation or priority or fixing stages, but elsewhere — for different readings of "$\ni$" as grounds, information bits, contents. Order relations are defined for $\alpha$, $\beta$ in $Ca$ as follows:

$$\alpha \leqslant \beta \text{ iff } \alpha+\beta = \beta; \quad \alpha < \beta \text{ iff } \alpha \leqslant \beta \,\&\, \alpha \neq \beta.$$

The modelling itself takes weaker or stronger forms, depending upon whether causal implication is to be fully relevant or not. We present the stronger, more simplistic form first, and then give the variations demanded by relevance.

A CI m.s. $\langle T, K, O, R, {}^*, C, +\rangle$ adds to a relevant m.s. just a function $C$ from situations to sets of moments and a semilattice operation $+$. A CI (causal implication) *model* adds to a CI m.s. two valuation functions, $I$(or $\vee$) and $c$. Valuation $I$, from sentential parameters and situations to holding values 1(on) and 0(off), subject to a hereditariness requirement, is as in relevant semantics. Valuation $c$ (also subject to an hereditariness requirement, is also from sentential parameters and situations but maps to situationally-associated moments; i.e. $c(p, a)$, with $c$ representing the causal moment or initiation or time (or differently, information or content) of $p$ at $a$, belongs to $Ca$.

Both $I$ and $c$ are extended from initial sentential wff to all wff inductively. The evaluation clauses for $I$, as applied to connectives $\&$, $\vee$, $\sim$, $\rightarrow$, are just those of relevant semantics. The further clauses are as follows:-

$$c(A, a) = \sum_{j=1}^{n} c(p_j, a), \text{ i.e. } c(p_1, a)+ \ldots +c(p_n, a),$$

for $p_1, \ldots, p_n$ exactly the sentential parameters of $A$.

$$I(A < B, a) = 1 \quad \text{iff} \quad c(A, a) < c(B, a)$$

$$I(A \ni B, a) = 1 \quad \text{iff} \quad I(A \rightarrow B, a) = 1 \,\&\, I(A < B, a) = 1,$$

i.e. iff $(b, d)(Rabd \,\&\, I(A, b) = 1 \supset . I(B, d) = 1) \,\&\, c(A, a) < c(B, a)$. To make axiomatization and demonstration of its adequacy run smoothly further clauses for $\leqslant$ and $\approx$ we added:

$$I(A \leqslant B, a) = 1 \quad \text{iff} \quad c(A, a) \leqslant c(B, a);$$

$$I(A \approx B, a) = 1 \quad \text{iff} \quad c(A, a) = c(B, a).$$

Then

$$I(A < B, a) = 1 \quad \text{iff} \quad c(A, a) \leqslant c(B, a) \,\&\, c(A, a) \neq c(B, a),$$

i.e. iff $I(A \leqslant B, a) = 1 \neq I(A \approx B, a)$.[4]
Validity and other semantical notions for CI are defined as usual for relevant logics.

For a matching stronger axiomatization of causal implication, CI, we add to any relevant-based affixing logic in connectives ($\rightarrow$, $\&$, $\vee$, $\sim$) – the *carrier* logic – the following postulates for further connectives $\ni$ and $\leqslant$ of CI, where $A \approx B =_{df} A \leqslant B \,\&\, B \leqslant A$ and $A < B =_{df} A \leqslant B \,\&\, \sim (A \approx B)$:

1. $A \ni B \leftrightarrow . A \rightarrow B \,\&\, A < B$ (i.e. P1)
2. $A \leqslant C \,\&\, B \leqslant C \rightarrow . A \leqslant C$ (i.e. $\leqslant$ is transitive)
3. $A \leqslant C \,\&\, B \leqslant C \rightarrow . A \,\&\, B \leqslant C$ (i.e. $\leqslant$ composes)

4($I$). $D \rightarrow . A \leqslant B$, where the sentential parameters of $A$ are a subset of those of $B$, i.e. in symbols $V(A) \subseteq V(B)$.

The irrelevant form 4($I$) immediately yields

4. $A \leqslant B$, where $V(A) \subseteq V(B)$,

as well as much irrelevant junk, such as $q \rightarrow . p \leqslant p$. In irrelevant logics which offer some control (over classical licence), $D$ of 4($I$) can be restricted in form, for instance to $\Box D$ in modal logics and $D \ni D$ in intuitionism (and in $I$-systems); and those logics of course enable the derivation of 4($I$) thus restricted from 4. It is the quest for relevance, and thus for replacement of 4($I$) and its variations by 4, that leads to weaker and deeper axiomatizations of causal implication.

It is a simple matter to reaxiomatize in terms of $<$ (and $\approx$) as primitive (or in terms of a single union operation). Replace 2, 3 and 4 by

$2^1$. $A < B \,\&\, B < C \rightarrow . A < C$

$3^1$. $A < B \,\&\, B < C \rightarrow . A \,\&\, B < C \vee . A \,\&\, B \approx C$.

$4^1$. $A < B$, where $V(A) \subset V(B)$.

A main reason for some indirection is that $4^1$ complicates semantical validation, excluding (correctly) some finite representations of $c(A, T)$. To work directly with $<$ it is enough to require that for $p_i$ distinct from $p_j$, $c(p_i, a) \neq c(p_j, a)$. Introduction of $\leqslant$ is however required for the type of completeness argument adopted.

Although the carrier logic is relevant-*based* in the sense of admitting a relevant logic, it may be decidedly irrelevant, e.g. in classical or intuitionistic

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[4] Part of the apparent circuity in argument here involved can be traced back to the relevant behaviour of negation (see the final stages of the completeness proof). In irrelevant settings, such as strict implication, such circuity is easily avoided.

directions. Thus any of a huge range of carrier logics may be chosen (including indeed, with minor variations in the argument, many not based on relevant systems).

Some theorems used in showing completeness are recorded

T1. $A \leqslant A$; by 4.
T2. $A \leqslant A \& B$; $B \leqslant A \& B$, by 4.
T3. Where $A_1 \leqslant A_2$ then $A_1 \& B \leqslant A_2 \& B$.

For, as $A_1 \leqslant A_2$ and $A_2 \leqslant A_2 \& B$, by T2, $A_1 \leqslant A_2 \& B$. But, by T2, $B \leqslant A_2 \& B$ also; whence by 3, $A_1 \& B \leqslant A_2 \& B$.

T4. Where $B_1 \leqslant B_2$ then $A \& B_1 \leqslant A \& B_2$. Similar to T3.
T5. Where $A_1 \leqslant A_2$ and $B_1 \leqslant B_2$ then $A_1 \& B_1 \leqslant A_1 \& B_2$. By T3, T4.
T6. $D \rightarrow .{:}B \leqslant B$, $D \rightarrow .B \leqslant A \& B$, $D \rightarrow .A \leqslant A \& B$; applying 4($I$).

*Stronger adequacy result.* A wff $A$ is a theorem of CI iff $A$ is CI-valid. (In fact the argument will yield strong completeness.)

*Argument*: (I) *Soundness.* The following properties of relevant semantics much simplify verification of postulates:

$I(C \rightarrow D, T) = 1$ iff for every situation $a$, where $I(C, a) = 1$ then $I(D, a) = 1$. Similarly $I(C \leftrightarrow D, T) = 1$ iff for every situation $a$, $I(C, a) = I(D, a)$. Then new postulate 1 is immediate from the evaluation clause for $\rightarrow$.

*Ad* 2. $\leqslant$ is transitive on $Ca$, for each $a$.

*Ad* 3. $c(A \& B, a) = c(A, a) + c(B, a)$; but where $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$ then $\alpha + \beta \leqslant \gamma$, by semi-lattice properties,

*Ad* 4($I$). Suppose $V(A) \subseteq V(B)$. Then $c(A, a) = \sum c(p_A, a) \subseteq \sum c(p_B, a) = c(B, a)$, where $p_A$ ranges over variables of $A$ and $p_B$ over those of $B$. Hence $I(A \leqslant B, a) = 1$, and thus where $I(D, a) = 1$ for arbitrary wff, so classically does $I(A \leqslant B, a) = 1$.

(II) *Completeness.* To the canonical m.s. for the carrier logic, defined as usual but on *non-null* situations (see [6]), are added further details for $C$ and $+$. The requisite canonical semi-lattice is arrived at by standard methods for Lindenbaum algebras. An equivalence on wff, $\tilde{a}$ is defined for each situation $a$ thus: $A\tilde{a}B$ iff $A \leqslant B \in a \& B \leqslant A \in a$. Then $\tilde{a}$ is reflexive and transitive by postulates 4 and 2 and symmetric by the definition. Let $|A|_a$ be the equivalence class of $A$ under $\tilde{a}$, i.e. $(B{:}\ B\tilde{a}A)$. Then $Ca$ is the class of all these classes, i.e. $Ca = (|A|_a{:}\ A$ is *a* wff); and correspondingly $|A|_a + |B|_a = |A \& B|_a$. The latter definition is admissible, because, by virtue of T5, $|A \& B|_a$ is suitably independent of the choice of $A$ and $B$.

Canonical valuations are defined as usual: $I(p, a) = 1$ iff $p \in a$ and $c(p, a) = |p|_a$, for every sentential parameter $p$ and every situation $a$. These interconnections are extended inductively to every non-definitionally supplied wff $A$. Connectives $\approx$, $<$ and $\ni$ can be construed as definitionally supplied. The details beyond those for the carrier logic are these:

o $\quad c(A, a) = |A|_a$, for every wff $A$ and situation $a$.

The generalizing step is as follows, where $p_1, \ldots, p_n$ are all the sentential parameters of $A$:

$$\begin{aligned} c(A, a) &= \textstyle\sum c(p_j, a), && \text{by its evaluation rule} \\ &= \textstyle\sum |p_j|_a, && \text{by given basis} \\ &= |p_1 \,\&\, \ldots \,\&\, p_n|_a, && \text{by iteration of } |p \,\&\, q|_a = |p|_a + |q|_a \\ &= |A|_a. \end{aligned}$$

It is in this final step that use of non-null situations and of 4($I$) (or a modal-style variant upon it) is crucial. By 4($I$), $D \rightarrow . p_1 \,\&\, \ldots \,\&\, p_n \leqslant A$ and $D \rightarrow . A \leqslant p_1 \,\&\, \ldots \,\&\, p_n$. As $a$ is non-null choose some wff $D$ in $a$. Then by $\rightarrow$-closure of situations, $p_1 \,\&\, \ldots \,\&\, p_n \leqslant A \in a$ and $A \leqslant p_1 \,\&\, \ldots \,\&\, p_n \in a$, whence $p_1 \,\&\, \ldots \,\&\, p_n \tilde{a} A$, vindicating the final step.

o $\quad I(A \leqslant B, a) = 1 \quad$ iff $\quad A \leqslant B \in a.$

In view of the evaluation rule for $\leqslant$ it suffices to show $c(A, a) \leqslant c(B, a)$ iff $A \leqslant B \in a$. Now

$c(A, a) \leqslant c(B, a)$ iff $|A|_a \leqslant |B|_a$, as above
iff $|B|_a + |A|_a = |B|_a$, by a semi-lattice definition of $\leqslant$
iff $|B|_a = |A \,\&\, B|_a$, by definition of $+$
iff $B \tilde{a} A \,\&\, B$, by canonical definitions
iff $B \leqslant A \,\&\, B \in a \,\&\, A \,\&\, B \leqslant B \in a$, by canonical definitions
iff $A \leqslant B \in a$, as is now shown.

Suppose $A \leqslant B \in a$. Since $D \rightarrow . B \leqslant B$ by T6 and $a$ is non-null, $B \leqslant B \in a$. Hence, by 3, $A \,\&\, B \leqslant B \in a$. Also since $D \rightarrow . B \leqslant A \,\&\, B$ by T6, $B \leqslant A \,\&\, B \in a$. Suppose the converse; the operative conjunct is $A \,\&\, B \leqslant B \in a$. By T6, $D \rightarrow . A \leqslant A \,\&\, B$, so $A \leqslant A \,\&\, B \in a$, whence $A \leqslant B \in a$ by 2.

Further steps for positively-defined connectives are straightforward. For example,

o $\quad I(A \approx B, a) = 1$ iff $A \approx B \in a.$

It suffices to show $c(A, a) \approx c(B, a)$ iff $A \approx B \in a$, i.e. $c(A, a) \leqslant c(B, a) \,\&\, c(B, a) \leqslant c(A, a)$ iff $A \leqslant B \in a \,\&\, B \leqslant A \in a$. But this follows from the previous case. Similarly a derived step for $A \ni B$ is straightforward, given that for $A < B$. But the latter, negatively-defined wff provides problems in relevant settings. For while $I(A < B, a) = 1$ iff $A \leqslant B \in a \,\&\, A \approx B \notin a$ and $A < B \in a$ iff $A \leqslant B \in a \,\&\, \sim (A \approx B) \in a$, nothing guarantees $A \approx B \notin a$ i.e. $\sim (A \approx B) \in a^*$, is equivalent to $\sim (A \approx B) \in a$ unless modal collapse ensures $a = a^*$. So we avoid $<$ in the canonicalness argument, as (given its definitional eliminability) we are entitled to do. The remainder of the completeness argument is standard (see e.g. [6]).

## 5. More fully relevant causal logics

While stronger causal implication systems do not enable the derivation of any irrelevant causal implication claims – because the implication conjunct serves to filter out any irrelevance priority may introduce – still the logics are irrelevant in containing theses of the form $C \rightarrow D$ where $C$ and $D$ share no variables. Amusingly we could have avoided this problem, and other hassles by basing the theory of causal implication on an irrelevant strict implication or associated irrelevant conditional.[5]

To obtain a more fully relevant logic including both $\ni$ and $\rightarrow$, and so on the present approach $\leqslant$, involves reducing postulate 4($I$) to 4. But this reduction, like felling a rainforest tree, takes some other things with it. It seems to require a certain withdrawal to $O$, and restructuring as follows: Model structures are unchanged, but the interpretation functions concerning $c$ are varied. In particular, for $a \notin O$, $I(A \leqslant B, a)$ is as before; for $a \notin O$ however $I(A \leqslant B, a)$ can be determined otherwise, or arbitrarily, by the modelling. So we take this opportunity to remove an obvious deficiency, and also qualify the assessment of $c(A, a)$ in term of $c(p_j, a)$ to $a \in O$. Otherwise, for $a \in O$, $c(A, a) \in Ca$, as for initial assessments. As a result, intensional wff are no longer assessed, priority-wise, through their sentential parameters. This is a major improvement. With $c(A, a)$ so decoupled, $I(A \leqslant B, a)$, etc., can most conveniently be left intact, as $c(A, a)$ gives requisite arbitrariness.

The effect of the decoupling of $c(A, a)$ gives the right results; validation of 4($I$) breaks down. Call the relevant logics with 4($I$) weakened to 4, but postulates 1-3 intact and semi-lattice principles of implication strength[6], DCI systems, deeper causal implication. The modified semantics with $c(A, a)$ decoupled supplies DCI-validity.

*Deeper adequacy result.* $A$ is a theorem of DCI iff $A$ is DCI-valid.

*Argument* varies that for CI. In soundness, only the details for 4 are different. What hold generally in 4($I$) now holds only for $a \in O$, which serves to validate 4 but not 4($I$). Completeness calls for more changes. For $a \in O$, $c(A, a)$ is coupled to its components; for $a \notin O$ define $c(A, a) = |A|_a$. Then $c|A, a| \in Ca$ as required. The proof given for CI that $c(A, a) = |A|_a$ works only for $a \in O$ in the absence of 4($I$): no matter, since the rest is obtained (not by hard work but) by stipulation. Given $c(A, a) = |A|_a$ generally, proof of $I(A \leqslant B, a) = 1$ iff $A \leqslant B \in a$ is as before.

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[5] Such an irrelevant approach to deeper relevant logics themselves is not entirely excluded.

[6] This introduces residual irrelevance, which it would be pleasing to eliminate, in the shape of the principle

5. $B \leqslant A \leftrightarrow . A \leqslant A \& B \& A \& B \leqslant A.$

In [9] it is shown how to eliminate even this residual irrelevance.

There remain sharp limits to the extent to which this style of approach, however relevant, can be improved. The reason is, of course, that a semantical elucidation of causal priority through atomic variables is ultimately unsatisfactory.

## 6. Corollaries, problems, further ado

An apparent corollary is decidability in those cases where the carrier logic succumbs to filtration methods. Another apparent corollary is that several philosophical castles *will* tumble down because their foundations are not adequate. A main reason in that they do not admit a sufficiently broad class of situations to support proper analyses of causal implication and duly distinguish it from other notions, such as mere time-ordered association (see [7]).

Among the problems we have left open are some we started with, the extent and shape of the higher degree of causal implication, and the purer theory. The semantical theory we have presented is thoroughly entwined with that of some (sufficiency) implication. But it would be interesting also to know what the pure theory of looks like, and what the ($\ni$, &, $\vee$, $\sim$) part comprises. We can present a small amount of information[7], some suggestions, and some conjectures. First, where the carrier relevant logic is near to or at basic affixing relevant logic, the pure $\ni$ logic contains no theorems, but only a rule structure (with rules such as transitivity). For the carrier logic contains only implicational theorems of the form $A \rightarrow A$ (see [6]) and these causal implication strips off. The ($\ni$, &, $\vee$ $\sim$) logic is a richer connexive logic, as we know. Determining its extent is a more difficult enterprise, one to which present semantical technology (which works better applied to stronger systems) is still not well adapted. (One reason is that Affixing rules certainly appear to fail for causal implication.) We conjecture that algebraic methods however will yield both an axiomatization, and a conservative extension result. The status of many higher degree principles, which could be added to CI and DCI, remains open and, so far as we know, largely investigated. Such business is part of the more immediate further ado.

There is always further ado. We have not looked at the quantificational theory; we have not considered the [false] principle of universal causation which obtains a nice nontrivial formulation within the theory, simply as

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[7] One important thing we *do* know is that there are several notions of logical significance of a similar doppelganger, double-banger, type to causal implication, permitting analysis in terms of a pair of more tractable notions. Containment logics, which have a rather similar semantical theory, are one sort, relational logic another, connexive logics another, logics for reason itself yet another. The connections will be made in a companion paper on reasoning (i.e. Sylvan and Goddard [8]), where some of the history of these connected developments will be duly acknowledged.

$(B)(\mathrm{PA})(A \ni B)$; we have not investigated the foundations we suspect the theory affords for "the logic of conditions"; nor have we glanced at the whole great ocean beyond this pleasant pebbly shore.

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