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INTUITIONIST LOGIC – SUBSYSTEM OF, EXTENSION OF, OR RIVAL TO, CLASSICAL LOGIC?

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Strictly speaking, intuitionistic logic is not a modal logic. There are, after all, no modal operators in the language. It is a subsystem of classical logic, not [like modal logic] an extension of it. But . . . (thus Fitting, p. 437, trying to justify inclusion of a large chapter on intuitionist logic 'in a book that is largely about modal logics').

The short well-known answer to the title question is, yes, *all* of those. It depends, in large measure, on how we formulate the systems, compare them, and apply them. Formulated as a system in connectives $\{\rightarrow, \vee, \&, \sim\}$, Lewis modal system S3 is a subsystem of classical logic S, similarly formulated, which results (for instance) by adjoining Peirce's implausible law. But with S recast as a system in connectives $\{\&, \sim\}$ (or $\{\vee, \&, \sim\}$), S3 reappears as extension of S, got by adding a modal connective $\rightarrow$ (interdefinable with $\Box$) and appropriate postulates.

With relevance logics, such as R and E, which are in many respects like modal logics (S3 is tantamount to E + Antilogism), it is similar: these logics are both subsystems of S and extensions of it. But there is a most significant difference from the modal situation, which concerns applications. The difference, important for applications to inconsistent and also incomplete theories, turns primarily on the scope of the rule of Material Detachment

MD. $A, \sim(A \mathbin{\&} \sim B) / B.$

While this rule, also called γ, is admissible for full relevance (quantificational) logics, it is not admissible for all relevant theories, by any means. Establishing its admissibility for a consistent theory, such as relevant arithmetic or analysis, is an important and apparently difficult problem, known as the γ problem. The problem is important because its positive solution would provide among other largesse, improved consistency arguments for corresponding classical theories.

Intuitionist logic is like relevant logic, only more so, in interesting respects. In the first place, MD is not admissible for the full intuitionist

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logic, but only for its classical sublogic, in connectives & and $\sim$. Unlike the admissibility of MD in relevance logic, it is easy to see the inadmissibility in full intuitionism. For in intuitionist logic, like classical and unlike E, $\sim A \leftrightarrow . A \rightarrow \Lambda$, where Λ is the absurd statement. Thus $\sim(A \ \& \ \sim B)$ is ($\leftrightarrow$) equivalent to $A \ \& \ (B \rightarrow \Lambda) \rightarrow . \Lambda$, which is equivalent in turn by intuitionistic suppression principles, given A is a theorem, to $(B \rightarrow \Lambda) \rightarrow \Lambda$, i.e. to $\sim \sim B$. But there are intuitionist theories which, as Heyting explains, counter the principle $\sim \sim B \rightarrow B$, rendering instances of the antecedent true but the corresponding consequent false. Hence γ is not not intuitionistically admissible. Indeed, the Double Negation rule $\sim \sim B/B$ is not an admissible rule of full intuitionist sentential logic. For $\sim \sim(\sim \sim A \rightarrow A)$ and $\sim \sim(A \vee \sim A)$, in particular, are intuitionist theorems (see Kleene, p. 119), though neither $\sim \sim A \rightarrow A$ or $A \vee \sim A$ are, on pain of classical collapse.

In the second place, whereas modal and relevance logics adjoin just one intensional, i.e. broadly modal, connective to classical connectives, intuitionist logic adds *two*, namely $\rightarrow$ and $\vee$. Neither of these connectives is extensional (in the basic sense of PM); for they do not satisfy the requisite condition, formulated for 2-place connective X,

Ext. Where $A \equiv B$ and $C \equiv D$ then $X(A, C)$ iff $X(B, D)$.

Thus they are intensional, or (broadly) modal in von Wright's unfortunate sense. For example, $A \equiv \ \sim \sim A$, but it is certainly not the case that $\sim \sim A \rightarrow A$, though $A \rightarrow A$; so $\rightarrow$ is broadly modal.

It *looks* of interest to ask, then, to what extent intuitionism admits of reformulation as a classically-based doubly-intensional logic. The main difficulty for any such 'new axiomatics' is the failure of MD beyond the $\{\sim, \&\}$ fragment. (That it holds for this fragment, i.e. for the classical base is known from an old result of Gödel: Kleene p. 493). Thus the classical $\{\&, \sim\}$ part requires formulation — if it is to admit of ready extension — without MD. The pattern of presentation, given that intuitionistically $\vee$ is stronger than $\rightarrow$, is clear.

A proposed 'new axiomatics' for intuitionistic logic, represented as an extension of classical logic, involves the following components:

1. An axiomatisation of classical logic, in connectives & and $\sim$. Ideally it will be formulated using rules that, unlike MD, admit of intuitionistic extension. But at worst it can be any $\{\&, \sim\}$ formulation of classical logic.

2. An axiomatisation of the further principles that the first intensional connective $\rightarrow$ conforms to. The form this takes depends on what happens at the first stage. But at worst it can follow standard axiomatisation of the $\{\rightarrow, \&, \sim\}$ part of intuitionism.
3. An axiomatisation of the further principles that the second stronger connective $\vee$ satisfies. Adequate postulates are known, namely the standard ones: $A \rightarrow .A \vee B$, $B \rightarrow .A \vee B$, $(A \rightarrow C)$ & $(B \rightarrow C) \rightarrow .A \vee B \rightarrow C$. (Indeed these are uniquely determined: see Gabbay).

There are some almost ready-made solutions to the problem so set: Gentzen formulations and closely related J systems (of Arruda-da Costa) both afford easy resolutions. Take, for instance, a formulation of intuitionist logic without Cut (e.g. Kleene's system G2, as below, or better a formulation that is like Gentzen's original formulation, at most singular in the right i.e. δ is null, γ singular). Restructure into three parts, first structural rules and negation and conjunction rules, second, implication rules, and third, disjunction rules. Structural rules can be reduced to thinning by taking a set formulation, i.e. where α, β, γ, δ are sets to: $\dfrac{\alpha \vdash \gamma}{C, \alpha \vdash \gamma}$ and $\dfrac{\alpha \vdash}{\alpha \vdash A}$. There is one axiom scheme: $A \vdash A$. Rules for & and $\sim$ are:

$$\&. \quad \frac{\alpha \vdash A, \delta \quad \beta \vdash B, \gamma}{\alpha, \beta \vdash A \,\&\, B, \delta, \gamma} \qquad \frac{\alpha, A \vdash C, \delta}{\alpha, A \,\&\, B \vdash C, \delta} \qquad \frac{\alpha, B \vdash C, \delta}{\alpha, A \,\&\, B \vdash C, \delta}$$

$$\sim. \quad \frac{\alpha \vdash A, \delta}{\alpha, \sim A \vdash \delta} \qquad \frac{\alpha, A \vdash}{\alpha \vdash \sim A}$$

Such a fragment enables the derivation of all classical tautologies, formulated in & and $\sim$, and indeed exactly that, where A is derivable iff $\vdash A$. Since the rules are all classically valid, it can deliver no more than that. Soundness is not so simple, calling for a restricted Cut theorem. But classical axioms are easily obtained, for instance

$$\cfrac{\cfrac{\cfrac{A \vdash A}{A, \sim A \vdash}}{(A \,\&\, B) \,\&\, \sim A \vdash}}{\vdash \sim((A \,\&\, B) \,\&\, \sim A)}$$

And in other respects the classical fragment behaves in the right sort of way.

For implication $\rightarrow$, add either appropriate Gentzen schemes, *or* some equivalent set of postulates, such as the following set

$$A \rightarrow .B \rightarrow A \qquad\qquad A, A \rightarrow B/B$$
$$A \rightarrow (B \rightarrow C) \rightarrow .A \rightarrow B \rightarrow .A \rightarrow C$$

(which correspond precisely to a two-way deduction theorem: $A_1, \ldots, A_{n-1}, A_n/B$ iff $A_1, \ldots, A_{n-1}/A_n \rightarrow B$). Finally, for disjunction, adjoin either the axiom schemes already indicated, or equivalently Gentzen schemes for $\vee$.

Whatever we think of the new axiomatisation that intuitionism supplies (or forces us into), it is evident on other grounds that there is a powerful case for some sort of new axiomatisation, which begins with classical $\{\&, \sim\}$ logic formulated with MD as an admissible rule only, and which then adds in turn intensional connectives of implication and disjunction, or better (though more difficult technically), disjunction and implication.

For it is a frequent observation, which received its classic contemporary elaboration in Strawson, that, of the standard supposedly truth-functional connectives, 'and', 'not', 'or' and 'only if', truth-functionality declines rapidly in the order exhibited. More exactly, so far as any English connective behaves truth-functionally, 'and' does; and 'not' and especially 'and' conform rather better to truth-functional preconceptions than do 'or' and the still more deviant 'only if'. It is a perennial theme, moreover, which can be also read out of Stoic logic and from certain medieval work.

It is of passing interest, then, that intuitionist logics and some of their neighbours can be construed as maintaining classical, truth-functional, behaviour for connectives 'and' and 'not' (or, to start with, absurdity), while diverging as regards behaviour of 'or' and 'only if'. From this perspective of course, intuitionism and minimalism become, like modal logics, extensions of classical logic, adding to it intensional connectives of disjunction and implication — not rivals to it, though not rule-preserving extensions. Thus are many claims in the standard literature upset, not only those of Fitting, but, for instance, scene-setting assumptions of Haack concerning intuitionistic deviance.

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