RELEVANCE WITHOUT MINIMALITY

S YABLO

1. INTRODUCTION

[1]

A notion that comes up everywhere in philosophy is that of a circumstance "contributing" to, or "favoring," a result or outcome—or being "helpful" to, or a "factor" in, or "relevant" to, the result or outcome.

Causes for instance should bear positively on what they cause. They should be free of material to which the effect is not beholden. An argument's premises, or the assumptions employed in a proof, should help to make the case for its conclusion. If a premise can be dropped without invalidating the argument, it probably shouldn't have been there in the first place. Grounds should contribute to what they ground, both in toto and throughout. That it would be fair is a reason for φ-ing only if its fairness counts in favor of φ-ing. A data point does not confirm a hypothesis, or figure in the evidence for it, if it is irrelevant to whether the hypothesis is true.

This last example (of figuring in the evidence) helps us to clarify the kind of relevance at issue. Hempel distinguishes three progressively more complicated types of confirmation: absolute, comparative, and quantitative (Hempel [1945]). Quantitative confirmation theory tries to develop measures of the extent to which Q confirms P. Comparative confirmation theory tries to make sense of Q confirming P more than Q' confirms P. Confirmation

[1]Versions of this material were presented at Harvard (as the 2016 Whitehead Lectures), Rutgers (at the 2017 Unstructured Conference), Sydney, Monash, ANU, Otago, Toronto, Amsterdam, UNAM, and elsewhere. I owe thanks to Tim Williamson, Lloyd Humberstone, Nathaniel Baron-Schmitt, Daniel Muñoz, Brad Skow, Sally Haslanger, Kit Fine, Andy Egan, Dirk Kindermann, Sam Carter, Cian Dorr, Daniël Hoek, Friedericke Moltmann, Agustín Rayo, Melisa Vivanco, Axel Barceló, Kenny Easwaran, Roxanne Kurtz, Ned Hall, Christine Korsgaard, Laura Schroeter, Greg Restall, Frank Jackson, David Chalmers, Katrin Shulz, Robert van Rooy, and Francesco Berto.

of the absolute type is a "binary" affair, both in involving only two elements—Q and P—and allowing only two verdicts—Q confirms P or it fails to confirm P. Hempel mentions comparative and quantitative confirmation only to set them aside for a later stage of the investigation.

Relevance in the sense of this paper is a binary affair too. Z contributes to Y, or it does not, period.[2] Nothing will be said about comparative helpfulness, or degrees of helpfulness.

Various other subtleties will be set aside as well. Our focus will be on *actual* rather than potential helpfulness. Z is helpful to Y only if both obtain.[3] A factor that normally works against Y—Y holds despite this factor—may yet be helpful to it on a particular occasion, and vice versa. A normally neutral Z may join forces with Y's friends on some occasions and its enemies on others.

Plan of the paper: Relevance is standardly explained in terms of notions like *minimality*, *essentiality*, and *non-redundancy*. The standard explanation is reviewed in the next two sections, first from an analytic and then a quasi-historical perspective. We will argue that this explanation does not get to the heart of things: Z can contribute to Y even if Z is not essential to the case for Y and even if minimality considerations do not apply. The problem is seen to have hyperintensional aspects. A hyperintensional diagnosis is suggested and a possible solution sketched. The solution is in terms of "focussed" minimality, or minimality where a certain subject matter is concerned. We sum up at the end and sketch some further issues and possible applications.

2. DEPENDENCE

One imagines to begin with that Z contributes to Y just if Y counterfactually depends on Z, that is, Y would not have obtained if not for Z.

[C1] Z contributes to Y just if: $\overline{Z} \,\Box\!\!\rightarrow \overline{Y}$

[2]Helpfulness may of course be *contingent* on other facts.
[3]Much as both need to obtain for Y to hold *despite* Z. Helpfulness in our sense is roughly the opposite of despiteness.

This is for a couple of reasons not a a satisfactory answer. One is that Z and Y will in some applications (grounding, entailment,....) be necessary. $\overline{Z} \mathrel{\Box\!\!\rightarrow} \overline{Y}$ will in that case be a counterpossible conditional. Counterpossible conditionals are as theoretically elusive as helpfulness, and raise some of the same problems, e.g. both are hyperintensional.

Another reason not to rest too much on counterfactuals is familiar from the theory of causation. Z can contribute to Y even if Y would still have obtained (on an alternative basis) in Z's absence.

The mismatch is often explained as follows.[4] Y depends on Z if an X obtains with three properties: (i) it contains Z; (ii) $X \backslash Z$ does not suffice for Y; and (iii) Y is not overdetermined—it has no other bases (actual or counterfactual) that would do the job in X's absence.

But, granted that Y does not *depend* on Z if Y is overdetermined, Z contributes to Y regardless. Why should it affect Z's claim to be making a contribution that there are, or otherwise would be, other contributors about? This suggests our focus should not have been on dependence in the first place. We should be looking rather at a circumstance (defined by (i) and (ii)) that normally *makes* for dependence:

[$C2$] Z contributes to Y just if: an X obtains such that $X \Rightarrow Y$, but $(X \backslash Z) \not\Rightarrow Y$.

Merely counterfactual bases drop out of the picture on this approach. Alternative actual bases are freely acknowledged and seen as posing no threat. X need not be in any sense unique for Z to qualify as helpful to Y by figuring essentially in X.

The way Z achieves relevance on [$C2$] is by pulling an "almost" sufficient condition over the finish line: $A \not\Rightarrow Y$ but $(A+Z) \Rightarrow Y$. But, Z could do that even if it was largely *ir*relevant. Socrates did not die because he drank hemlock in a world where iPhones were later developed. But such a Z may well be helpful to is death by the lights of [$C2$]. This and certain related problems suggest that *everything* in X (not merely Z) should be essential to it qua basis for Y:

[4] Kment [2014], Strevens [2007].

[$C3$] Z contributes to Y just if: an X obtains such that $Z \leq X$, $X \Rightarrow Y$, and $\forall U < X$ ($U \not\Rightarrow Y$).

This, the *minimal sufficiency* model of relevance, is what I want to talk about in this paper. Of course it admits like any other philosophical model of various refinements. But we will not bother too much about these, since they don't affect the problem about to be developed. That problem runs deep and is not easily tweaked away.

The problem *formally* speaking is that not everything *has* a minimal basis. Surely we do not want to conclude from the fact that sufficient conditions for Y always contain smaller such conditions that none of these are wholly helpful to Y. Especially if nothing is even *relevant* to Y except by relation to an X that *is* wholly helpful.

The problem intuitively speaking is that X to be wholly helpful need only be wholly *welcome* from Y's perspective. So far is this from requiring X be wholly essential that X can be in a good sense wholly *inessential*. It can be composed of elements that would none of them be missed, though of course large enough combinations of them would be missed.

To put the intuitive problem another way, consider the idea of "extra help." This on the minimal sufficiency model looks to be a contradiction in terms, for if Z was extra—beyond what was needed—then it was no help at all. The model runs here completely contrary to intuition. Suppose the winning team in a tug of war was larger than necessary. Did the team win without any help from its members? Apparently so, if Z, to be helpful, must push an otherwise insufficient condition over the top.

3. HISTORY

Let us now re-approach the question "historically" (note the scare quotes). When do minimality pressures first begin to make themselves felt? When do we first encounter the problem just noted, that a development to be welcome from Y's perspective needn't be a sine qua non of a sufficient condition for Y?

Sufficiency had a long run in philosophy before anyone got worked up about irrelevant add-ons. There was the Principle of Sufficient Reason. Causes were events given which the effect was sure to follow. Classical validity was a matter of premise-truth sufficing for the truth of the conclusion. Grounds for such and such were and sometimes still are items or conditions prior to such and such and sufficient for it.

An obvious worry about these proposals is that they put a lower bound on, say, the cause, but not much of an upper bound, since $X \Rightarrow Y$ is monotonic in X (X^+ suffices if X does) while causation, grounding, and the like are more discerning. Socrates died not because he drank the hemlock in a toga, but because he drank the hemlock. The existence of even primes is grounded in 2 being an even prime, not 2 being an even prime while 5 is odd.

If sufficiency allows causes to get too big, we might think of asking Z also to be *necessary* for Y. Hume considers this in the *Treatise* and rejects it, on the theory that effects need not have been caused at all, let alone by their actual causes.

> If we define a 'cause' to be *An object precedent and contiguous to another, and where all the objects resembling the former are similarly precedent and contiguous to objects that resemble the latter*, we can easily grasp that there is no absolute or metaphysical necessity that every beginning of existence should be preceded by such an object (Hume [1740/2003], Bk I, section 14, "Of the Idea of Necessary Connexion.")

The issue for us concerns *natural* necessity, and Y's specific cause rather than its being caused at all. But causes are not naturally necessary either, for Hume. From *objects resembling X are always succeeded by objects resembling Y* it does not follow that *objects resembling Y are always preceded by objects resembling X*. [5]

Hume does seem to appreciate, even in the *Treatise*, that causes as he officially defines them are liable to be overloaded with extraneous detail. For we find him in the very next section (*I*, *15*, "Rules by which to judge of causes and effects") looking for ways to block this:

[5] Similarly a truth does not have only one possible truthmaker, and there is more than one possible reason for doing a thing.

> where several different objects produce the same effect, it must be by means of some quality, which we discover to be common amongst them...in order to arrive at the decisive point, we must carefully separate whatever is superfluous, and enquire by new experiments, if every particular circumstance of the first experiment was essential to it.

Hume suggests here a different way of keeping X within bounds. Rather than requiring causes to be necessary—so that Y no longer *holds* given just part of X—he asks them only to be non-redundant—so that Y is not *ensured* by just part of X. This becomes in the *Enquiry* (Hume [1740/2006]) a full-blown proportionality requirement:

> we must proportion the [cause] to the [effect] and can never be allowed to ascribe to the cause any qualities, but what are exactly sufficient to produce the effect.[6]

A proportional cause is an X such that X suffices for Y *and nothing less suffices*.[7]

[P] X is proportional to Y ($X \rightarrow Y$) iff

(i) X suffices for Y ($X \Rightarrow Y$)

(ii) for all $X' \leq X$, if $X' \Rightarrow Y$, then $X' = X$

Of course we are often interested in "contributory" causes which are not sufficient and hence not proportional. But Hume has an easy way to bring these on board. Z contributes to Y if it is *contained* in some proportional cause X of Y:

[H] Z is helpful to Y ($Z \rightsquigarrow Y$) iff an X obtains such that $Z \leq X$ and $X \rightarrow Y$.

By the *Humean Package* (*HP*), we'll mean these two ideas together. The first idea: X is proportional iff it is minimally sufficient. The second: Z is relevant iff it is contained in a proportional. X. What the two together offer is an account of relevance in terms of the prima facie much clearer notions of sufficiency and minimality.

[6] "A body of ten ounces raised in any scale may serve as a proof, that the counterbalancing weight exceeds ten ounces; but can never afford a reason that it exceeds a hundred" (Hume [1740/2003]).

[7] "Proportionality" in the sense of this paper is analogous but not identical to the notion at work in Yablo [1992b] and Yablo [1992a].

The Humean Package has a lot to be said for it. It is powerful and illuminating and deals correctly with a great many cases. Also it is adaptable. Since [P] and [H] do not contain the word "cause," they offer a general template that is potentially of very wide application. And indeed it is hard to think of an area of philosophical inquiry that hasn't employed it. The Hypothetico-Deductive model of confirmation is Humean in spirit; E confirms H just if H figures essentially in some suitable E-entailing body of knowledge. An action's good-making features, on one account, are those included in some condition that is minimally sufficient for its goodness. Recent theories of presupposition projection lay great weight on the relevance of the embedded sentence's truth-value to the truth-value of the whole.[8] A recent paper defines "P is a difference-making ground for Q" like this:

> for some scenario S which contains a full ground of Q, S minus {the fact that P} does not contain a full ground of Q (Krämer and Roski [2017]).

How can the fact P that 5 is odd (indivisible by 2) be relevant to the fact Q that there are primes, when that fact is fully assured independently? A scenario S consisting precisely of 5 and its indivisibility by 2, 3, and 4 contains thereby a full ground of Q. No lesser scenario contains a full ground, and in particular S minus the fact P of 5's indivisibility by 2 does not fully ground Q. 5's oddness contributes to the existence of primes because it makes the difference between a minimal ground for primes' existence and a near-ground.

4. EXTRA HELP

The problem as already indicated that a *non*-minimal condition X—one with elements that it doesn't need, to suffice for Y—can still be wholly, entirely helpful. Extra help is still help, and sometimes it is the only kind of help around.

One example of this sort comes from Zeno. A solid sphere takes up space. It has measure 1, say. The sphere's component points are helpful, surely? Certainly they are helpful en masse—en masse they just *are* the sphere. And it is hard to see how they

[8] Schlenker [2008]. See Schlenker [2009], p. 52-3 for a close cousin of the minimality problem.

could be helpful together if not individually. But since each point has measure 0, they would none of them be missed. None of the sphere's component points lies in a minimal subregion of the same measure, simply because there *are* no minimal subregions of the same measure.[9]

Hume would certainly have known of the Zeno puzzle. But he could not have known of the example to come, for it is based in events taking place in 1741, the year the *Enquiry* was published. God is pleased, let us stipulate, if and only if he is praised infinitely many days. Being praised *every* day should be pleasing, surely. But no, not if we go by the Humean Package. The reason was noted by John Newton, author of *Amazing Grace*:[10]

> When we've been here ten thousand years
> Bright shining like the sun
> We've no less days to sing His praise
> Than when we'd just begun.

Singing *every* day is out of proportion with the effect, since God would still be pleased if we waited 10000 years before beginning. And of course the same is true for any other set of days one might choose. There *is* no least infinite set of days. Every praise-day is helpful to the cause, but not because it figures in a minimal sufficer.

The moral so far is this. Minimality had better not be required for relevance, because you can't always get it. In fact minimality is not required even where you *can* get it.

The pope's crown was once supposedly made of three smaller crowns. Suleiman the Magnificent, not to be outdone, had *four* crowns in his crown. Suleiman's crown seems wholly relevant to *There are crowns*. But you could lop the upper sub-crowns off and still have a sufficient condition for the sentence's truth. Here we *can* point to a minimal sufficient basis for *There are crowns*. But there is no reason to do so. Suleiman's total crown is no less helpful for being four times larger than necessary.

[9] Skyrms [1983] emphasizes this version of the puzzle.

[10] A certain John Watson had a shipboard conversion that year off the coast of Ireland,....

The US Senate cannot conduct certain kinds of business unless 51 members are present (a quorum). Let us say the Senate is not "in order" without a quorum. Suppose that 52 senators are present on a given occasion. They all arrived at the same time and that the situation is in other ways symmetrical. The presence of these senators—the Gang of 52, let' s call them —seems wholly helpful to order obtaining. Of course there is a Gang of 51 present as well which also suffices, in fact there are 52 such gangs. Somehow though this does not detract from our initial judgment. The Gang of 52 is wholly relevant despite the fact that not all its members had to be there.

5. PREVIOUS PROPOSALS

That X can still be wholly relevant to Y, even if not all of it is needed, has not gone unnoticed. Fine has made the point in connection with truthmaking (proportional truthmakers in his parlance are *exact* rather than *inexact*) (Fine [2017]).[11] The Humean proposes in effect to

> to take the exact verifiers to be the minimal inexact verifiers, those that inexactly verify without properly containing an inexact verifier.

This of course precludes the possibility of extra help, but not if fusions of exact verifiers are allowed in as well. Consider Kratzer's theory of *exemplification*. The fact of two teapots exemplifies *There are teapots* despite its non-minimality. But the fact of two teapots and a dog does not. Why is the extra dog more of a problem than the extra teapot? Kratzer has an interesting idea about this: the parts of a p-exemplifying situation s must "earn their keep" by figuring essentially, not perhaps in s itself, but in a minimal p-verifying part of s.

> s exemplifies p iff for all s' such that $s' \leq s$ and p is not true in s', there is an s'' such that $s' \leq s'' \leq s$, and s'' is a minimal situation in which p is true. (A

[11] "With inexact verification, the state should be at least partially relevant to the statement; and with exact verification, it should be wholly relevant. Thus the presence of rain will be an exact verifier for the statement 'it is rainy'; the presence of wind and rain will be an inexact verifier for the statement 'it is rainy', though not an exact verifier (Fine [2017])

> minimal situation in which p is true is a situation that has no proper parts in which p is true.) (Kratzer [2002]: 660)

The fact of two teapots exemplifies *There are teapots* despite its non-minimality because everything in it is part of some minimal verifier or other. Here s and p are like our X and Y and exemplification is like being-sufficient-for-and-wholly-helpful-to.

Kratzer's theory does indeed loosen the bonds between relevance and minimality. But minimality is still playing its same old role one level down; it's a problem, then, if "a statement may have inexact verifiers without having any minimal verifiers" (Fine [2017]). Kratzer's example is *There are infinitely many stars* (numbered (7) in her paper):

> If the proposition expressed by (7) is the proposition p that is true in any possible situation in which there are infinitely many stars, we are in trouble. [The] definition would predict that there couldn't be a fact that makes p true, ... Situations with five or six stars, for example, ... are not part of any minimal situation in which p is true.[12]

Kratzer's example shows that we also cannot "take the exact verifiers to be the quasi-minimal inexact verifiers, those that are minimal or do not contain any states that are not inexact verifiers" ((Fine [2017]); for situations with five or six stars do not inexactly verify p, either. Similarly a cause might still be wholly helpful to an effect, even if all its sufficient parts contain smaller such parts all the way down. Imagine a detector that buzzes when presented with a line segment of any positive length.[13]

---

[12]Ibid., 662. Kratzer's response, like ours below, invokes subject matter.

> As for (7), there is a reading that the German sentence (8) brings out more clearly.
>
> (8) Sterne gibt es unendlich viele.
> Stars are there infinitely many.
> As for stars, there are infinitely many of them.
>
> In (8), the common noun "Stern" has been topicalized. ..The proposition expressed by (8) might now be taken to be the proposition q that is true in a situation s iff (i) s contains all the stars in the world of s, and (ii) there are infinitely many stars in s. Consequently, if q is true in a world at all, there is always a minimal situation in which it is true, hence there is always a fact that exemplifies it (Ibid., 662).

[13]Yablo [2017a].

A couple of very different ideas were prompted by an observation of Williamson's (p.c., 2006) about logical containment explained in terms of minimal models.[14] Infinitary relevance can *sometimes* be dealt with as follows. Consider again the Zeno puzzle. How do the individual points in a sphere contribute to its positive measure, when each point is of measure zero? Well, the points are *collectively* relevant and none is more relevant than another. Perhaps X is wholly helpful to Y if

(1) X subdivides into the X_i's
(2) Y fails if *all* the X_i's fail
(3) one X_i is as relevant to Y as another

Or, looking back at Amazing Grace, we might reason as follows. The *number* of praise-days does not shrink if we add one more day, but the *set* does shrink. And cardinality considered as a measure on sets is a coarsening of membership; size in the how-many sense is monotonically grounded in size in the membership sense. Perhaps X is wholly helpful to Y if

(1) Y is to the effect that X is at least so big by a certain measure
(2) that measure is monotonically grounded in another, finer measure
(3) X_i bears on X's size by this finer measure

I don't want to pursue these ideas here for two reasons. They seem insufficiently general. And they miss another dimension of the problem, to which we're coming next.

6. HYPERINTENSIONALITY

One problem for the Humean package ([H] and [P]) is that minimality is not always available or even desirable. Another is that Humean proportionality looks to be "intensional":

> if X and Y are necessarily equivalent to X^* and Y^*, then X is proportional to Y only if X^* is proportional to Y^*.

[14] A model of a PC sentence S is an assignment of truth-values to atoms every classical extension of which is a valuation verifying S. A minimal model σ of S is a model with no proper sub-models. B is contained in A, according to the proposed explanation, if $\forall\alpha\exists\beta\ \alpha\subseteq\beta$ and $\forall\beta\exists\alpha\ \alpha\subseteq\beta$ (Angell [1989], Yablo [2014]:59, Fine [2015a]). Williamson pointed out that logical containment *appears* to make sense as well in infinitary settings where A and B's models are always non-minimal.

Because it is defined by [P] in terms of sufficiency and minimality. which themselves look to be intensional. (I say "look to be" because the less-than relation $\leq$, which defines minimality, remains to be discussed.)[15]

Is *relevance* intensional? It is not. An example on the Y side: *His praise is sung infinitely many days* (Y) is true in the same worlds as *His praise is sung infinitely many days after 12/31/12017* (Y^*). Singing every day starting now (X) is wholly helpful to Y but overkill when it comes to Y^*. Singing today is absolutely beside the point when it comes to singing infinitely often in the distant future.

An example on the X side:[16] In Alternative Eden, there are infinitely many apples on the tree of life but only one, $\mathfrak{Badapple}$, on the tree of knowledge of good and evil. Otherwise the story is much the same. Eve can't recall what it was that God had told her and decides to check it out with the serpent:

> EVE: What did God allow me to do again?
>
> SERPENT: I remember it was equivalent to this: *You take infinitely many apples*.
>
> [Eve eats all the apples and is banished.]
>
> EVE: Why did you say God had allowed me to take infinitely many apples?
>
> SERPENT: I said it was necessarily *equivalent* to that, and it was. *You take infinitely many apples v such that* $v \neq \mathfrak{Badapple}$ (X) holds in the same worlds as *You take infinitely many apples, period* (X^*). One apple cannot make the difference between an infinite set and a finite one.

Despite that they hold in the same worlds, it would seem that X's truth is wholly helpful to *Eve did what she was told*, whereas X^*'s truth is not.

[15] It could turn out that $U \leq X$ but $U \not\leq X^*$. See section 7.

[16] Fine [2017]

7. MEREOLOGY

What is it for X' to be $\leq X$ in $[P]$? You might think that $X' \leq X$ iff X implies X'. But although this is how content-parts are usually understood, the view quickly runs into problems.[17] For one thing it allows X to be knocked out of proportion with Y by $X \vee S$, provided that S too is sufficient for Y. Which is surely the wrong result.

Socrates' drinking the hemlock (X) suffices, let's assume, for his death (Y). X is proportional to the death only if nothing less suffices. Yet something less is bound to suffice, if less-than is just the converse of implication. For consider any other sufficient basis for death, say. falling off a high cliff (S). $X \vee S$ is a weaker sufficient condition for Y than X is. $X \vee S$ knocks X out of proportion with Y, if $\leq$ means is-true-in-more-worlds-than. So $\leq$ had better mean more than that.

The answer we *want* to give is that $X \vee S$, although *weaker* than X, is not *contained* in X. X to be proportional to Y should have no proper *parts* sufficient for Y. This notion of content-part is not available to the Humean, since it is pretty clearly hyperintensional. Our solution to hyperintensionality, looking ahead a bit, will be in terms of *ways*: statements true in the same worlds may not be true in the same ways in those worlds. Ways are the key as well to content-parts:

> $X' \leq X$ iff
>
> (i) every way for X to hold implies a way for X' to hold,
>
> (ii) every way for X' to hold is implied by a way for X to hold.[18]

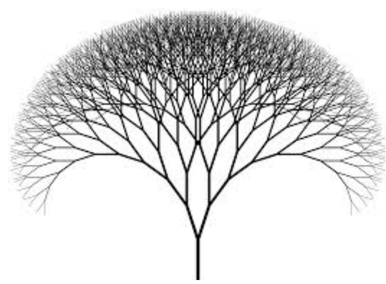
Ways bear also on the problem of minimality (the problem we are mainly concerned with in this paper) as we'll be seeing shortly. Suffice it for now to say that although the Humean Package faces multiple challenges, they all push in a similar theoretical direction.

[17] Gemes [1994, 1997], Fine [2013], Yablo [2014], Fine [2015a].

[18] Gemes [1994, 1997], Yablo [2014], Fine [2015a], Yablo [2016], Fine [2017],

8. BOTTOMLESS KINDS

A fractal is a geometrical figure containing isomorphic copies of itself; these will then have to contain isomorphic copies of *them*selves, and so on all the way down. Fractals are counterexamples par excellence to the minimality requirement. The fact that tree ***t*** exists ([***t*** *exists*], for short) is as helpful as it could be to *There are fractals*. You are not going to find a better candidate for a proportional, discerning, basis for the truth of *There are fractals* than the existence of ***t***.



Compare a fact that is clearly out of proportion with *There are fractals*: [***t*** *exists and Sparky exists*]. What is the difference exactly? You can throw the Sparky conjunct out, of course, and still be left with a fact sufficient for the existence of fractals. But one can also throw out part of the fact that ***t*** exists. For the immediate right subtree ***u*** of ***t*** is also a fractal, and ***t***'s existence consists in the joint existence of ***u*** and the rest of ***t***: ***v***. It is not clear as yet why [***u*** *exists* and ***v*** *exists*] would be more proportional to *There are fractals* than [***t*** *exists and Sparky exists*], or for that matter [***u*** *exists and Sparky exists*], given that the second conjunct is in each case dispensable.

Call a kind K bottomless if to be a K is to contain smaller Ks. Clearly if K is bottomless then a minimal K is not to be expected. Are there other bottomless kinds, besides *fractal*?

A set is Dedekind infinite iff all of its members can be paired off 1-1 with its members other than x, for some x in the set. Suppose that S is equipotent in that sense with $S_1 = S\backslash\{x\}$, and let y be a member of $S\backslash\{x\}$. Then if $y \in S_1$, it follows on standard assumptions that S_1 is equipotent with $S_2 = S_1\backslash\{y\}$, and so on without limit. *Infinite set* is thus a bottomless kind.

A predicate P is *dissective* if a thing cannot instantiate it unless all its parts do.[19]. This does not ensure bottomlessness all by itself, but it does if we add that Ps always have proper parts. Sellars draws on this notion in his famous distinction between the "scientific" and "manifest" images of reality.

> Color expanses in the manifest world consist of regions which are themselves color expanses.[20]

The manifestly colored "expanses" form a bottomless kind then for Sellars. (Of an especially pure sort. Fractals can contain non- fractals, and infinite sets finite sets; but *no* part of a colored expanse can fail to be colored.) Aristotelian water is supposedly dissective, and a stretch of continuous motion subdivides into smaller stretches of continuous motion.

Now, these examples are not terribly recherche'. Why is minimality still insisted on when the problems are so obvious? Examples of a phenomenon should ideally be *visible*, so that people can see for themselves. But not if the phenomenon is invisibility. A visible example of that is impossible, and so the requirement seems silly.

A requirement can still retain its hold on us, to be sure, even after the problem is pointed out. A set of everything is impossible too, but that doesn't make it any less "what we wanted." Logicians tend to *regret* the unavailability of a universal set. They look for ways of approximating or simulating such a set.[21] Whereas there is nothing to regret in the fact that we can't lay our hands on a minimal fractal.

9. SCHEMATIZATION

The Humean Package is not a claim but a schema. Z is *causally* helpful, or relevant, to Y iff it figures in an X that *causally* suffices for Y, where nothing less causally suffices. Z is *ground*-relevant to Y (it is a difference-making ground, see section 3) iff it figures in a full

[19] Goodman [1966]
[20] Sellars [1963]
[21] This is part of the attraction of plural quantification.

ground X of Y such that nothing less than X fully grounds Y. Z is rationally relevant to Y iff it figures in an X that fully rationalizes Y, but ceases to do so when anything is deleted.

To bring this out into the open, we take the "simple" notions of sufficiency ($\Rightarrow$), proportionality ($\rightarrow$), and relevance ($\rightsquigarrow$) in [H] and [P] and ramify them, plugging in particular flavors $\Rightarrow^{\alpha}$, $\rightarrow^{\alpha}$, and $\rightsquigarrow^{\alpha}$ of these notions:

[$\mathbb{H}$] $Z \rightsquigarrow^{\alpha} Y$ iff

$Z \leq X$ for some (actual) X such that $X \rightarrow^{\alpha} Y$

[$\mathbb{P}$] $X \rightarrow^{\alpha} Y$ iff

(i) $X \Rightarrow^{\alpha} Y$,

(ii) for all $X' \leq X$, if $X' \Rightarrow^{\alpha} Y$, then $X' = X$.

The α-ized conditionals stand ambiguously for the modes of sufficiency and relevance with which philosophers have concerned themselves: causal, logical, modal, legal, nomological, explanatory, evidential, and so on.

None of this gets us closer to solving the problem, of course, rather it reminds us the problem's shape and size. Take again *He is praised infinitely many days*. It has an unending chain of progressively weaker sufficers: he is praised every day from today on (X_0), every day from tomorrow on (X_1),..., every day from 12/31/12019 on (X_n), and so on. The sufficiency in this case is ground-flavored. Writing $\Rightarrow^{\gamma}$ for "is sufficient in the manner characteristic of full grounds," we have

$X_0 \Rightarrow^{\gamma} Y$,

$X_1 \Rightarrow^{\gamma} Y$,

....

$X_n \Rightarrow^{\gamma} Y$,

.....

We're told by [$\mathbb{P}$] that $X \rightarrow^{\gamma} Y$ (X is proportional in the manner characteristic of full grounds for Y) iff X has no proper parts X' such that $X' \Rightarrow^{\gamma} Y$. But then it holds of no X_i on the above list that $X_i \rightarrow^{\gamma} Y$, since X_i has a proper part X_{i+1} such that $X_{i+1} \Rightarrow^{\gamma} Y$. Lacking a better candidate for proportional-full-ground-hood than these, we'd better give up as well on locating a difference-making ground Z for Y; these by [$\mathbb{H}$] will have to be parts of proportional grounds, a category we just declared to be empty. It is never true, therefore, that $Z \rightsquigarrow^{\gamma} Y$— Z is never relevant in the manner characteristic of difference-making grounds to Y—and in particular it is irrelevant to the sought-after outcome of infinitely many praise days that we sing on any occasions whatever.

This is in some sense old news, but it was run together with a different problem to do with causation. To get the different problem, we interpret Y as *God is pleased at the amount of praise he gets* and let the sufficiency be cause-and-effect-flavored (putting $\Rightarrow^{\epsilon}$ for $\Rightarrow^{\gamma}$). It holds of no X_i on the above list that $X_i \rightarrow^{\epsilon} Y$, since X_i has a proper part X_{i+1} such that $X_{i+1} \Rightarrow^{\epsilon} Y$. It is never true, by the same reasoning as before, that $Z \rightsquigarrow^{\epsilon} Y$. God does wind up pleased if praised every day, but not, it seems, on account of the singing on any particular day(s). An example with fewer distractions: A buzzer sounds at the weigh station when a truck enters weighing over 70,000 pounds. The buzzer goes off "for no particular reason," to go by [$\mathbb{P}$] and [$\mathbb{H}$]. Reasons—factors relevant to the effect—have to be drawn from conditions minimally causally sufficient for the effect, and there are no numbers minimally larger than 70,000.

Examples of the same general sort can be given for almost any mode of relevance: moral ($Z \rightsquigarrow^{\mu} Y$), evidential ($Z \rightsquigarrow^{\kappa} Y$), nomological ($Z \rightsquigarrow^{\nu} Y$), and so on. Which means that the Humean Package—a theory $\mathbb{H}$ of relevance built on the back of a theory $\mathbb{P}$ of proportionality—needs fixing. Our focus has been, and will continue to be, on $\mathbb{P}$. But $\mathbb{H}$ has issues of its own which will be mentioned briefly at the end.

10. CONCERN

Take again *He is praised infinitely many days*. It has an unending chain of progressively weaker sufficers: every day from today, every day from tomorrow, ..., every day from 12/31/12017, and so on. The weaker ones are *no better*, though. There *might* be something to regret in the sequence never terminating, if each new reduction brought a feeling of progress. But it doesn't.

Cantor was disappointed, it is said, with what we now call the infinite numbers. No $\aleph_\beta$ could satisfy him, because there was always a bigger one down the road, and bigger, in the infinity department, is better. A *truly* infinite number would be unsurpassable. This supposedly is why he called the $\aleph_\beta$s "transfinite," reserving "infinite" for a (putative) number too big for his system.[22]

Cantor had good reason to prefer higher $\aleph_\beta$s to lower ones. The larger had more of what he wanted: size. Reasoning in the other direction, if we perceive no advantage in X' = "singing every day from 12/31/12017," as the ground of God's pleasure, over X = "singing every day from today," then first does NOT have more of what we wanted. But the first is genuinely smaller.

Smaller is not necessarily better when it comes to proportionally causing Y = God's pleasure. X' may have an advantage over X with respect to causing some other outcome Y'. But they are "the same to Y," in symbols, $X' \equiv_Y X$. Or, more carefully, since they could conceivably be the same to Y *as causes*, but different as, say, *grounds*, $X' \equiv^\alpha_Y X$.

Of course, we don't know what that means as yet. If we did, we would take our existing account of proportionality, repeated (relabelled) from above,

[$\mathbb{P}$1] $X \rightarrow^\alpha Y$ iff

(i) $X \Rightarrow^\alpha Y$,

(ii) for all $X' \leq X$, if $X' \Rightarrow^\alpha Y$, then $X' = X$

and replace $X' = X$ in (ii) with $X' \equiv^\alpha_Y X$ to obtain

[$\mathbb{P}$2] $X \rightarrow^\alpha Y$ iff

(i) $X \Rightarrow^\alpha Y$,

(ii) for all $X' \leq X$, if $X' \Rightarrow^\alpha Y$, then $X' \equiv^\alpha_Y X$

[22] Cantor was hospitalized for depression in 1899. *That Obscure Object of Desire*, or rather the novel it is based on, appeared in 1898. This is probably a coincidence.

There are infinitely many so and so's is about size in the *how-many* sense, not the *membership* sense. That is why there is nothing to be gained proportionality-wise by knocking out one of the so and so's. Y doesn't *care* about, it's not *concerned* with, the kind of size where subsets are smaller.

11. WAYS AND WORLDS

Parthood, proportionality, concem and the like are hyperintensional notions. So we will have to expand our toolkit, for, "The possible worlds apparatus can only draw intensional, not hyperintensional, distinctions."[23] Fortunately, as we began to see in section 7, the role traditionally played by worlds is better played anyway by *ways*.[24] And ways are hyperintensional right out of the box. P is true in the same worlds as $(P{\equiv}Q)\vee(P{\equiv}\neg Q)$, but the latter has different ways of being true.[25] A solid figure occupies most of the open sphere $\{<x, y, z> \mid x^2+y^2+z^2 < 1\}$ in a world w just if it occupies most of the closed sphere $\{<x, y, z> \mid x^2+y^2+z^2 \leq 1\}$. But while to occupy all of the closed sphere is a way of occupying most of the closed sphere, it is not a way of occupying most of the open sphere.[26]

What is meant by a "way for P to hold"? I do not know how to define the notion and will not even try. This may seem like obscurantism, but really it is standard practice in semantics. Lewis for instance does not attempt to define "world in which P is true." He divides the problem into two parts:

(1) what is a "world in which a proposition is true"? and
(2) what is the "proposition expressed by P"?

The first sub-problem is trivial, if propositions are sets of worlds; $\mathcal{P}$ is true in w iff $w\epsilon\mathcal{P}$. The second, associating propositions with sentences, is a problem for everyone. It

[23] Berto [2017]
[24] See also Yablo [2017b].
[25] Assuming anyway that the second can be true by way of P's truth and Q's falsity. This will be so on some accounts but not others.
[26] Continuous motion occurs in the same worlds as discrete continuous motion*s*. But if a particle moves continuously from noon to one and then three to four, that is more of a way for continuous motion*s* to occur than continuous motion. (Thanks here to Kit Fine.)

is nothing special to do with worlds, and it is not made more difficult by the worldly conception. One approach is to let the intensions of atomic expressions be given outright; the intensions of complex expressions are then determined compositionally. Or we could determine atomic intensions using some kind of covariational metasemantics. Or we could approach the matter holistically, as Lewis does himself.[27]

If propositions are sets of truth-supporting circumstances, then S is true-in-certain-circumstances just if those circumstances belong to the proposition that S. One can work with the first notion today while leaving until tomorrow the problem of explaining how a sentence comes to express this proposition rather than that. This is again a common strategy in semantics. Propositions for Kratzer are sets of situations. Propositions for Humberstone are sets of possibilities. Propositions for Yalcin are sets of probability-measures. Propositions for truthmaker semanticists are sets of ways.

12. WHICH WAYS

Suppose I am right that way-for-it-to-be-that-P is on a par methodologically speaking to world-in-which-P. There still remains the question of *which* unanalyzed relation is intended. Here my job is in one respect easier than Lewis's: "ways for something to be the case" is a more familiar and commonsensical notion than "worlds where P is true." But it's in another respect harder, for ways are a miscellaneous lot and I need to direct your attention to a particular instance of the genre. Our target in the end is ways for it to be that P, but it helps to look more generally at ways for a thing x to φ. (Ways for it to be that P then fall out as the case where x is a world w and to φ is to be a P-world.)

So, let's try it. To begin it would be good to have some *paradigms* of ways for x to be thus and so, and some anti-paradigms (or foils) as well.

> DISJUNCTS: For x to sing is a way for x to sing or dance.
>
> INSTANCES: For x to sing is a way for something to sing.
>
> DETERMINATES: For x to yodel is a way for x to sing.

[27] Lewis [1974], Lewis [1983]

Here by contrast are some paradigms of *failure* to be a way.

CONJUNCTIONS: To sing and dance is a way of singing.

GENERALIZATIONS: For everything to sing is a way for x to sing.

MANNERS: To yodel badly is a way of singing.

PREQUELS: To dance is a (good) way of failing the course.[28]

PRECONDITIONS: To dance is a way of persisting over time.[29]

A few simple principles will help us to sort these cases out, and to see to a first approximation why the line is drawn where it is.

BY-WAY	For x to ψ is a way for x to φ only if: x φ's *by* ψ-ing.

So, for instance, x sings by yodeling, and sings or dances by singing; and *something* sings, by way of x singing. But x does not sing by singing and dancing, or persist over time by dancing.

This is only part of the story, however, for one can fail the course by dancing (when one ought to be paying attention); and to dance is not in the relevant sense a way of failing the course. If, as the story goes, one can get to Carnegie Hall by practicing, still practicing is not for us a way of getting to Carnegie Hall. Rather the practicing and singing are causes, or prequels. Here is a second principle aimed at prequels:

WAY-IN	For x to ψ is a way for x to φ only if: x φ's *in* ψ-ing.

Hank Williams sang sometimes by yodeling, and, what is more, *in the act of* yodeling. But one does not fail the course in the act of dancing, or get in the act of practicing to Carnegie Hall.

One of the foils (MANNERS) remains to be dealt with. A beginning yodeler does on the face of it sing by yodeling badly, and also *in* yodeling badly. Why then is yodeling badly not a way of singing? One is tempted to say that, as long as Bert is yodeling, how well

[28]The instructor disapproves of dancing; or, to dance cuts into your study time.

[29]Assuming here that an instaneous entity cannot dance.

he does it is *irrelevant* to whether he sings. But relevance is what we are attempting to explain, so this does not get us very far.

Luckily this particular type of irrelevance can be identified independently, without getting into grander issues about relevance as such. For ways are intimately related to *parts*, and relevance has a counterpart virtue on the side of parts that is easier to get a grip on. This is illustrated in (1)-(6).[30]

$$\text{(1) to be } \left\{\begin{array}{c} R \\ \overline{R} \end{array}\right\} \text{ is } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} R\&S \\ \overline{R}\vee\overline{S} \end{array}\right\}$$

$$\text{(2) to be } \left\{\begin{array}{c} R\vee S \\ \overline{R}\&\overline{S} \end{array}\right\} \text{ is NOT } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} S \\ \overline{S} \end{array}\right\}$$

$$\text{(3) to be } \left\{\begin{array}{c} R\vee S \\ \overline{R}\&\overline{S} \end{array}\right\} \text{ is } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} R\dot{\vee}S \\ R\equiv S \end{array}\right\}$$

$$\text{(4) to be } \left\{\begin{array}{c} R\supset S \\ R\&\overline{S} \end{array}\right\} \text{ is NOT } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} R\&S \\ \overline{R}\vee\overline{S} \end{array}\right\}$$

$$\text{(5) to be } \left\{\begin{array}{c} R\supset S \\ R\&\overline{S} \end{array}\right\} \text{ is } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} R\equiv S \\ R\not\equiv S \end{array}\right\}$$

$$\text{(6) to be } \left\{\begin{array}{c} R\equiv S \\ R\not\equiv S \end{array}\right\} \text{ is NOT } \left\{\begin{array}{c} \text{part} \\ \text{a way} \end{array}\right\} \text{ of being } \left\{\begin{array}{c} R\&S \\ \overline{R}\vee\overline{S} \end{array}\right\}$$

We see from (1)-(6) is that parts and ways are in a certain sense *duals* of one another: to ψ is part of φ-ing just if to $\overline{\psi}$ is a way of $\overline{\varphi}$-ing. Turning this around and focussing on the direction of interest,

PART-WAY	To ψ is a way of φ-ing only if: to $\overline{\psi}$ is part of what is involved in $\overline{\varphi}$-ing.

[30] R and S could be, for instance, *red* and *square*. $\dot{\vee}$ is exclusive disjunction.

The thought is that irrelevancies that prevent ψ from being a way of φ-ing will show up under negation as "extrusions" that prevent $\overline{\psi}$ from being a part of $\overline{\varphi}$.

How does yodeling poorly fare by this test, as a way of singing? It is a way of singing only if part of what is involved in *not* singing is to either not yodel at all, or else yodel well. To avoid yodeling may indeed be part of what it takes not to sing. But to yodel well *conflicts* with not-singing! It cannot be part of H-ing to F or G, if to G prevents one from H-ing. Hence to yodel poorly is not, by our test, a way of singing.

The test detects, of ψ-ing in a certain fashion or manner, that it is not a way of φ-ing, when ψ-ing full stop is a way of φ-ing. A test attuned to so fine a distinction might seem in danger of being *too* sensitive. But it accepts all our paradigms. To sing remains a way of singing or dancing, since to do neither is in part not to sing. To yodel remains a way of singing, since not to sing is in part not to yodel. For Al to sing remains a way for someone to sing, since part of what is involved in noone's singing is for Al in particular not to sing.

Three principles have been roughly indicated: BY-WAY, WAY-IN, and PART-WAY. They correctly classify between them all of the cases presented above (Disjuncts, Instances, Determinates, Conjunctions, Generalizations, Manners. Prequels, Preconditions), and all the examples I know of. A number of questions suggest themselves about the principles, and there are other principles that might be considered. But our topic is relevance and it is time to get back to it.

13. ABOUTNESS

Here is where we left things: Z contributes$^{\alpha}$ (or is relevant$^{\alpha}$) to Y if Z is part of an X that is proportional$^{\alpha}$ to Y. X is proportional$^{\alpha}$ to Y if Y does not *care about* the difference between X and those of its proper parts X' that also suffice$^{\alpha}$ for Y—X is minimal in the respects that matter. (Our notation for this was $X' \equiv^{\alpha}_{Y} X$.)

The "caring" is of course metaphorical. But the "about" and the "mattering" can be cashed out to some extent in terms of *ways of being true*. How is it that $P \vee \neg P$ is on a different topic than $Q \vee \neg Q$, when they are true in the same worlds? Well, they are true in different ways in those worlds. Why does the subject matter of $P\&Q$ include the subject matter of P, but not that of $(P\&Q)\vee R$, when (writing $|S|$ for the set of S-worlds), $|P\&Q|$ is a subset both of $|P|$ and $|(P\&Q)\vee R|$? Well, $(P\&Q)\vee R$ holds, in certain worlds, in ways not implied by any way for $P\&Q$ to hold; while the same cannot be said of P in comparison to $P\&Q$.

This section attempts to make the notion of subject matter precise enough for the proposed application (which comes later) to "minimality in the respects that matter" and to relevance. We ask, first, what are subject matters considered as entities in their own right, and second, what is *the* subject matter of a particular sentence?

A subject matter M, THE NUMBER OF STARS, for instance, is given by specifying the ways matters can stand where M is concerned: the ways, in this case, are for there to be no stars, or one star, or etc. Formally M is a collection of set-of-worlds propositions. If M = $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, ...\}$ then $\mathcal{A}, \mathcal{B}, \mathcal{C},...$ constitute between them the ways matters can stand M-wise. Subject matters can be more or less fine-grained, for instance, THE NUMBER OF STARS is coarser-grained than WHICH STARS EXIST (aka THE STARS) and finer-grained than WHETHER THE NUMBER OF STARS IS PRIME:

THE STARS =

{|*Nothing is a star*|, |*The only star is Sol*|,... |*The stars are Sol, Polaris, Vega,..*|,...}

THE # OF STARS =

$$\{|\exists_0 x\ star(x)|, |\exists_1 x\ star(x)|, ...|\exists_k x\ star(x)|,...\}$$

WHETHER THE # OF STARS IS PRIME =

$$\{\cup_{prime(k)} |\exists_k x\ star(x)|, \cup_{\overline{prime(k)}} |\exists_k x\ \ star(x)|\}$$

The subject matter S of a particular sentence S is made up of S's ways of being true; it is the set of all set-of-worlds propositions $\mathcal{S}$ such that S is true in way $\mathcal{S}$ in some world w. *Stars exist* has a way of being true for each possible nonempty ensemble of stars; so its subject matter is what above we called THE STARS, except the first, star-less, cell must be dropped since *Stars exist* is false in that cell.

14. "EVERY BIT AS SUFFICIENT"

Let's review. Z is relevant$^{\alpha}$ to Y just if Z is part of an X that is proportional to Y—an X no proper part X' of which "undercuts" X by sufficing$^{\alpha}$ for Y on a more economical basis. When *does* a still-sufficient proper part X' of X undercut X in this way? This may seem a strange question. How can X NOT be undercut by X', if X' is every bit as sufficient for Y?

Not so fast, though. Normally, for X' to be "every bit as sufficient" as X is for Y would mean that Y holds in as many X'-worlds as X-worlds, viz. all of them. But another reading is possible if statements hold, not only in worlds, but in *ways* in worlds.

Suppose that Alice has three children and Bert has two. Is Alice any more of a parent than Bert? One is pulled both ways on this. Certainly it is no more *true* of Alice that she is a parent. But she is *some* sort of advantage parental-status-wise; for the truth of *Alice is a parent* is more often *witnessed* than that of *Bert is a parent*. The advantage is compounded if the witnesses to the one truth form a proper subset of the witnesses to the other; Alice has two children with Bert, and one not with Bert. Now she is more of a parent than Bert in a further sense. She is, in addition to being more often a parent, more *richly* or *comprehensively* a parent than Bert is, since her status as a witnessed by those who witness Bert's status and another besides.

Now we begin to see how X can avoid being undercut by X', though Y is no less definitely true in X'-worlds than X-worlds: Y is not as *richly provided for* in X'-worlds as in X-worlds.

Consider again *God is praised every day from now on*. It is better proportioned to *God is pleased* than *God is praised and dogs bark every day from now on*, because it doesn't matter to God whether dogs bark. Why would *God is praised every day from tomorrow on* not be better proportioned to *God is pleased* than *God is praised every day from now on*? *God is pleased* is still guaranteed if the praise starts tomorrow, but it is not as richly guaranteed as in worlds where the praise starts today. In worlds with less barking but as much praise, *God is pleased* is every bit as richly guaranteed.

Given two statements P and Q both of which are true in w, let us say that P is as richly true in w as Q if every way Q holds in w is a way as well for P to hold. So for instance $A \vee B \vee C$ is at least as richly true as $A \vee B$ in all worlds, since it is true when $A \vee B$ is true and in a (possibly proper) superset of the ways in which $A \vee B$ is true.

Writing $\|P\|^u$ for P's ways of being true in v, we can say that P is as richly true in u as Q is in v just if $\|Q\|^v \subseteq \|P\|^u$. For X' to suffice as fully as X does for Y is a matter of how comprehensively true $\exists W(W \Rightarrow^\alpha Y)$ is in X-worlds, in comparison with X'-worlds. Specifically

[$\mathbb{S}$] X' suffices as fully as X for Y — $X'/X \Rightarrow^\alpha Y$ — iff

(i) $X \Rightarrow^\alpha Y$,

(ii) $X' \Rightarrow^\alpha Y$, and

(iii) ($\forall X'$-worlds u) ($\exists X$-world v) [$\exists W(W \Rightarrow^\alpha Y)$ is as richly true in u as in v]

15. SUPER-HUMEANISM

Return now to the idea of proportionality as minimal sufficiency. Perhaps there is something right about it after all, *if* we are careful about what is being minimized subject to which constraints. X' is in a position to undercut X if (but only if) it suffices for Y as fully as X does.

[$\mathbb{P}$3] X is proportional$^\alpha$ to Y ($X \rightarrow^\alpha Y$) iff

(i) X suffices for Y ($X \Rightarrow^\alpha Y$),

(ii) ($\forall X' \leq X$) if X' suffices as fully as X does for Y (if $X'/X \Rightarrow^\alpha Y$), then $X' = X$.[31]

The super-Humean package ([$\mathbb{P}$3] and [$\mathbb{H}$]) — recall that

[$\mathbb{H}$] Z is helpful$^\alpha$ to Y ($Z \rightsquigarrow^\alpha Y$) iff an X obtains such that $Z \leq X$ and $X \rightarrow^\alpha Y$

— deals rather easily with cases of the kind considered so far in this paper. Let us work through this with an example.

[31] This is obtainable from [$\mathbb{P}$2] by unpacking "$X' \equiv^\alpha_Y X$" as "$X' \neq X \supset X'/X \not\Rightarrow^\alpha Y$."

Alice wins a prize (Y) if she moves for at least one hour between noon and two. Let X be her moving from 12.30 to 1.30 inclusive — her moving through the closed interval [12.30, 1.30]. How can X be proportional to Y, when X' — her moving through the open interval (12.30, 1.30) — is a proper part of X that also suffices?

Well, X suffices *better*. For there are X'-worlds, like the world w where she moves precisely at times later than 12.30 and before 1.30, where Y is not as well provided for as it is in X-worlds. Here are some of the ways it is true in w that Alice moves so as to win the prize: she moves at all times t such that

... $12.30<t<1.30$

... $12.30<t<1.30$ & $t\neq 1.00$

... $12.30<t<1.30$ & $t\neq 12.45$ & $t\neq 1.00$ & $t\neq 1.15$

... $12.30<t<1.30$ & $t\notin\mathcal{S}$ (for some countable set $\mathcal{S}$ of times)

What is nice about X-worlds is that Alice moves, not only in *these* prize-winning ways in them, but a bunch of additional prize-winning ways: among them moving at all times t such that

... $12.30\leq t\leq 1.30$

... $12.30\leq t\leq 1.30$ & $t\neq 1.00$

... $12.30\leq t\leq 1.30$ & $t\neq 12.45$ & $t\neq 1.00$ & $t\neq 1.15$

... $12.30\leq t\leq 1.30$ & $t\notin\mathcal{S}$ (for some countable set $\mathcal{S}$ of times)

Similar reasoning shows that $\mathcal{T}$-motion — moving at all times in $\mathcal{T}$ — is proportional to winning the prize just if $\mathcal{T}$ is a subset of [12.00, 2.00] of measure ≥ 1. It follows by [$\mathbb{H}$] that $\mathcal{T}$-motion *contributes* to winning the prize just if $\mathcal{T}$ is a non-empty (possibly measure 0) subset of [12.00, 2.00].

16. FORMS OF RELEVANCE

Our primary question has been, what can proportionality mean if every X has a proper part that is still sufficient? That question has now been addressed. X is proportional to Y if X, in addition to being sufficient for Y, lacks proper parts that are *as* sufficient. Equivalently X is the least $X' \leq X$ such that X' is no less sufficient for Y than is X. [32]

Mention was made at the beginning of certain *further* questions that were going to be set aside, about pro tanto relevance for instance. These were to do not with proportionality, but the explanation of helpfulness in *terms* of proportionality:

[$\mathbb{H}$] $Z \rightsquigarrow Y$ iff a fact X obtains such that $Z \leq X$ and $X \rightarrow Y$.

Helpfulness so understood is *in-situ*: Z is helpful to Y only if Z is the case, and its helpfulness depends very sensitively on what else is the case. It is holistic and situational in a way by moral or epistemological particularists. A Z that is helpful to Y qua part of X_1 may be helpful to $\overline{Y}$ qua part of X_2. If one thinks, for instance, that the party was good because it was fun— fun is by nature a good-maker—rather than fun being a good-maker derivatively, on this occasion—-because of featuring essentially in a sufficient condition for goodness that happens to obtain— then [$\mathbb{H}$] is not going to satisfy you.

That is the first subtlety. A distinction obviously exists between in-situ helpfulness, on the one hand, and per se or presumptive helpfulness on the other. And we have not seen how to draw it. Second, helpfulness of the sort defined by [$\mathbb{H}$] is *factive*: Z is helpful to Y only if Y is really the case. But then, what of the idea of U holding *despite* Z? Z in some sense favors $\overline{U}$ here—otherwise why say "despite"?— but the favoring cannot be factive, since $\overline{U}$ must fail, if U holds despite Z. That Z can fight in a losing cause is the second subtlety. Our focus has been on factive helpfulness; nothing has been said about would-be, defeated helpfulness.

These are subtle distinctions, on which a lot has been written, especially in moral and legal philosophy. I have no theory to offer. but a suggestion: people working on them should be talking to people working on subject matter and ways (and vice versa). For a number of lines in this area can be drawn in quantificational terms with pretty much

[32]If it can happen that every X that is sufficient$^\alpha$ for Y has a proper part that is still sufficient$^\alpha$ for Y, perhaps it can happen too that every X that is sufficient$^\alpha$ for Y has a proper part that is *as* sufficient$^\alpha$ for Y as X is. See the Appendix.

our existing machinery. Z is helpful *in situ* to Y iff an X holds that (i) contains Z and (ii) is proportional to Y—that is, letting $\overrightarrow{Y}$ range over conditions proportional to Y, Z is helpful (*in situ*) to Y iff a $\overrightarrow{Y}$ holds that contains Z. $\overrightarrow{Y}$ is still available to be quantified over, however, whether it holds in this world or not:

- Z is *always* helpful to Y iff in every Z-world, a $\overrightarrow{Y}$ holds that contains Z
- Z is *sine qua non* helpful to Y iff all $\overrightarrow{Y}$s contain Z
- Z is *intrinsically* helpful to Y iff some $\overrightarrow{Y}$s contain Z but no $\overrightarrow{Y}$ contains $\overline{Z}$
- Z is always *pertinent* to Y iff in each Y ($\overline{Y}$)-world, a $\overrightarrow{Y}$ ($\overrightarrow{\overline{Y}}$) holds that contains Z or $\overline{Z}$
- Z is more *often* helpful to Y than V iff more $\overrightarrow{Y}$s contain Z than U
- Z is more *likely* helpful to Y iff $\overrightarrow{Y}$s containing Z are likelier than $\overrightarrow{Y}$s containing V
- Z helps *more* than V (for a given $\overrightarrow{Y}$) iff Z forms a larger part of $\overrightarrow{Y}$ than V does
- Z helps more *simpliciter* iff it forms a larger part on average of its containing $\overrightarrow{Y}$s

These analyses are sketchy and schematic. A lot of work would be needed before one would want to rest anything on them. But we have enough to construct a toy logical model that illustrates how the distinctions work. Outcomes Y are sentences of propositional logic. Sufficiency is logical implication. Ways for things to be are conjunctions of literals (negated and unnegated atoms). A way $\overrightarrow{Y}$ for Y to be true is a conjunction of literals that implies Y and has no subconjunctions that imply Y.[33] One can easily check that

- p is always helpful to $p \vee q$, but not always helpful to $p \& q$ or $p \equiv q$
- p is sine qua non helpful to $p \& q$, but not sine qua non helpful to $p \vee q$ or $p \equiv q$
- p is intrinsically helpful to $p \vee q$ and $p \& q$, but not intrinsically helpful to $p \equiv q$
- p is always pertinent to $p \equiv q$, but not always pertinent to $p \vee q$ or $p \& q$
- p is more often helpful than its negation to $p \& q \& r \vee \overline{p} \& \overline{q} \vee p \& \overline{q} \& s$[34]
- neither of p, $\overline{p}$ is more likely helpful to $p \& q \& r \vee \overline{p} \& \overline{q} \vee p \& \overline{q} \& s$ than the other[35]

[33] $\overrightarrow{Y}$ is in Quine's terms a *prime implicant* of Y (Quine [1955]). Compare the notion of a minimal model in note 14. (Minimal models aren't always available, of course, but they are here. Also we are bracketing hyperintensionality issues for the moment.)
[34] Since p figures in two ways for $p \& q \& r \vee \overline{p} \& \overline{q} \vee p \& \overline{q} \& s$ to hold, while $\overline{p}$ is involved just in $\overline{p} \& \overline{q}$.
[35] Since $p \& q \& r \vee p \& \overline{q} \& s$ and $\overline{p} \& \overline{q}$ are both 25% likely, if probabilities are assigned in the obvious way.

- $\overline{p}$ helps $p\&q\&r \vee \overline{p}\&\overline{q} \vee p\&\overline{q}\&s$ more on balance than p does[36]

One can do at least *some* justice, then, I suggest, to presumptive helpfulness in the present framework. Indeed one can pull apart different flavors of the notion. But there are other subtleties that are still out of reach.

The elements of a proportional X must be in some broad sense *relevant* to Y. Helpfulness is a kind of relevance, but it is not the only kind. X may need to contain also facts which ensure of the helpful bits that they *are* helpful.[37] It may need to contain facts to ensure that the helpful bits *suffice*.[38] It may need to specify of potential spoilers that they did not materialize. (That my promise was not coerced is not a reason for lending you my car. Rather it is part of why it is not the case that the reason — my promise — was defeated.[39]) It may need to contain also threats to Y, if the events that blocked them were triggered by those very threats.[40] X may need to contain "intensifiers," trumps, switches, and other spin-control devices. All of this is fascinating and important but bracketed in the present paper. Extremely useful here are Hawthorne [2002], Dancy [2004], Sartorio [2005], Sartorio [2006], Sartorio [2008], Leuenberger [2014], Skiles [2015], Skow [2016], Baron-Schmitt [2017], Munoz [2017].

17. SUMMING UP

Relevance is important. Hume thought he'd explained it with minimality, but the explanation doesn't work. A variant using *focussed* minimality—minimality where a certain subject matter is concerned— seems to do better.

Subject matter relies on *ways of holding*, a notion with some of the features we were trying to explain, for instance, worlds being *relevantly* alike: alike on the score of how something is true in them. But we can in good conscience treat ways as primitive, by

[36]Since p takes up a third of $p\&q\&r$ and $p\&\overline{q}\&s$, while $\overline{p}$ takes up a half of $\overline{p}\&\overline{q}$.
[37]Say, by clarifying that another potential route to Y was blocked (Yablo [2002]).
[38]Say, by clarifying the nature of Y's demands: that Elsie is not a raven helps to make the case for *All ravens are black* by clarifying that it does not require Elsie to be black (Yablo [2014]:61ff, Skiles [2015]).
[39]Dancy [1983]
[40]Paul and Hall [2013]

analogy with worlds (section 12), especially since much f the work traditionally assigned to worlds is better done by ways (Fine [2015b,c], Yablo [2017b]). Also a certain amount can be done to indicate *which* primitive is intended.

Where we come out is, Z is relevant to Y iff it figures in a X that is proportional to Y. X is proportional iff its sufficient parts X' are not in the relevant sense *as* sufficient.

18. APPENDIX

If proportionality is explained as in [$\mathbb{P}$3], then we see in principle how X can defend its claim to proportionality with Y against a still-sufficient proper part X'. But have we really laid the problem to rest? If it can happen that every X that is sufficient$^\alpha$ for Y has a proper part that is still sufficient$^\alpha$ for Y, perhaps it can happen too that every X that is sufficient$^\alpha$ Y has a proper part that is *as* sufficient$^\alpha$ for Y as X is.

How would that work exactly? A proper part X' of X that was as sufficient$^\alpha$ for Y as X was would have to hold in a certain kind of $\overline{X}$-world u: one in which $\exists W(W{\Rightarrow}^\alpha Y)$ was as richly true as in v, for some X-world v. It is not obvious that a world like u must always exist. Why should it not be possible to load X up in advance with all worlds of the relevant type—all worlds where Y is richly enough provided for?

Zorn's Lemma tells us the conditions under which this is doable. Suppose we are given a partially ordered set ($\mathbf{K}$, $\preccurlyeq$). A subset $\mathbf{C}$ of $\mathbf{K}$ is a chain iff $\mathbf{C}$ is totally ordered by $\preccurlyeq$, and bounded below iff $\exists \mathbf{x}\epsilon\mathbf{K}\ \forall \mathbf{y}\epsilon\mathbf{C}\ \mathbf{y}{\preccurlyeq}\mathbf{x}$. From Zorn we know that

$\mathbf{K}$ has a minimal element ($\exists \mathbf{k}\epsilon\mathbf{K}\ \forall \mathbf{j}\epsilon\mathbf{K}\ (\mathbf{j}{\preccurlyeq}\mathbf{k} \supset \mathbf{j}{=}\mathbf{k}$) if all $\mathbf{K}$-chains are bounded below.

An X exists that is minimal among proper parts of X_0 no less sufficient (than X_0 is) for Y, provided that (*) holds:

> (*) if $X_0{\geq}X_1{\geq}X_2$...., and X_i is no less sufficient$^\alpha$ for Y than X_0, then the X_is have a common part X_ω such that X_ω is no less sufficient$^\alpha$ for Y than X_0.

This follows from two not unnatural assumptions.

1 if $\mathcal{X}_0{\subseteq}\mathcal{X}_1{\subseteq}$... is a chain of ways for things to be, $\cup_k\mathcal{X}_k$ is a way for things to be

2 if $\mathcal{A}, \mathcal{B}, \mathcal{C}$,... are ways for things to be, $\exists S$ ($\mathcal{A}, \mathcal{B}, \mathcal{C}$,... are S's ways of being true)

Proof of (*) from [1] and [2]. Let $X_0 \geq X_1 \geq X_2$..... and let $\mathcal{X}_\omega$ be $\cup_k |X_k|$. By definition of $\geq$, $\mathcal{X}_\omega$ is the union of all unions $\mathcal{X}_\omega^{\mathsf{C}}$ of chains of truthmakers $\mathcal{X}_0, \mathcal{X}_1$,... for X_0, X_1,.... respectively. $\mathcal{X}_\omega^{\mathsf{C}}$ is a way for things to be by [1]; and an X_ω exists with the $\mathcal{X}_\omega^{\mathsf{C}}$s as its truthmakers by [2]. X_ω is part of each X_k since each truthmaker $\mathcal{X}_\omega^{\mathsf{C}}$ for X_ω contains an $\mathcal{X}_i$. All we need to show now is that X_ω is as sufficient$^\alpha$ for Y as X_0. This is immediate from the facts that (i) each X_ω-world is an X_k-world for some k, and (ii) $\exists W(W \Rightarrow^\alpha Y)$ is for each X_k-world u as richly true in u as in some X_0-world v. □

Since from [1] and [2] it follows that (*), X_0 has by Zorn's Lemma (again assuming [1] and [2]) a part that cannot be further reduced while remaining as sufficient$^\alpha$ for Y.

References

Richard B Angell. Deducibility, entailment and analytic containment. *Directions in Relevant Logic, Dordrecht: Kluwer Academic Publishers*, 1989.

Nathaniel Baron-Schmitt. Contingent grounding. manuscript, 2017.

Francesco Berto. Impossible worlds and the logic of imagination. *Erkenntnis*, 82(6):1277–1297, 2017.

Jonathan Dancy. Ethical Particularism and Morally Relevant Properties. *Mind*, XCII(368): 530–547, 1983.

Jonathan Dancy. *Ethics without principles*. Oxford University Press on Demand, 2004.

Kit Fine. A note on partial content. *Analysis*, 73(3):413–419, 2013.

Kit Fine. Angellic content. *Journal of Philosophical Logic*, pages 1–28, 2015a.

Kit Fine. I. a theory of truth-conditional content: Conjunction, disjunction and negation. *unpublished manuscript*, 2015b.

Kit Fine. Ii. a theory of truth-conditional content: Subject-matter, common content, remainder, and ground. *Unpublished manuscript*, 2015c.

Kit Fine. Truthmaker semantics. *A companion to the philosophy of language*, pages 556–577, 2017.

Ken Gemes. A new theory of content I: Basic content. *Journal of Philosophical Logic*, 23(6): 595–620, 1994.

Ken Gemes. A new theory of content II: Model theory and some alternatives. *Journal of Philosophical Logic*, 26(4):449–476, 1997.

Nelson Goodman. *The structure of appearance*. Vrin, 1966.

John P Hawthorne. Blocking definitions of materialism. *Philosophical Studies*, 110(2): 103–113, 2002.

Carl G. Hempel. Studies in the Logic of Confirmation (I.). *Mind*, 54(213):1–26, 1945.

David Hume. *A Treatise of Human Nature*. Oxford University Press, Oxford, 1740/2003.

David Hume. *An Enquiry Concerening Human Understanding*. Clarendon Press, Oxford, 1740/2006.

Boris Kment. *Modality and explanatory reasoning*. OUP Oxford, 2014.

Stephan Krämer and Stefan Roski. Difference-making grounds. *Philosophical Studies*, 174 (5):1191–1215, 2017.

Angelika Kratzer. Facts: Particulars or information units. *Linguistics and Philosophy*, 25 (5-6):655–670, December 2002.

Stephan Leuenberger. Grounding and necessity. *Inquiry*, 57(2):151–174, 2014.

David Lewis. Radical interpretation. *Synthese*, 27(3-4):331–344, 1974.

David Lewis. New work for a theory of universals. *Australasian Journal of Philosophy*, 1983. D 83; 61: 343 377 0004 8402 English.

Daniel Munoz. Defeaters and disqualifiers. (to appear), 2017.

Laurie Ann Paul and Edward Jonathan Hall. *Causation: A user's guide*. Oxford University Press, 2013.

Willard V Quine. A way to simplify truth functions. *The American Mathematical Monthly*, 62(9):627–631, 1955.

Carolina Sartorio. Causes as difference-makers. *Philosophical Studies*, 123(1-2):71–96, 2005.

Carolina Sartorio. On causing something to happen in a certain way without causing it to happen. *Philosophical studies*, 129(1):119–136, 2006.

Carolina Sartorio. Moral inertia. *Philosophical Studies*, 140(1):117–133, 2008.

Philippe Schlenker. Be articulate: A pragmatic theory of presupposition projection. *Theoretical Linguistics*, 34(3):157–212, 2008.

Philippe Schlenker. Local contexts. *Semantics & Pragmatics*, 2(3):1–78, 2009.

Wilfrid Sellars. Philosophy and the scientific image of man. In *Science, Perception and Reality*. Routledge & Kegan Paul, 1963.

Alexander Skiles. Against grounding necessitarianism. *Erkenntnis*, 80(4):717–751, 2015.

Bradford Skow. *Reasons why*. Oxford University Press, 2016.

Brian Skyrms. Zeno's paradox of measure. *Physics, Philosophy, and Psychoanalysis: Essays in Honor of Adolf Grunbaum, ed. R. Cohen and L. Laudan (Dordrecht: D. Reidl, 1983)*, pages 223–254, 1983.

Michael Strevens. Mackie remixed. *Causation and explanation*, 4:93–118, 2007.

Stephen Yablo. Cause and essence. *Synthese*, 93(3):403–449, D 92 1992a.

Stephen Yablo. Mental causation. *The Philosophical Review*, 101(2):245–280, April 1992b.

Stephen Yablo. De facto dependence. *Journal of Philosophy*, 99(3):130–148, 2002.

Stephen Yablo. *Aboutness*. Princeton University Press, 2014.

Stephen Yablo. Parts and differences. *Philosophical Studies*, 173(1):141–157, 2016.

Stephen Yablo. Kment on counterfactuals, 2017a.

Stephen Yablo. Precis of Aboutness. *Philosophical Studies*, 174(3):771–777, 2017b.