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*Numbers Can Be Just What They Have To**

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Benacerraf (1965) tells of two children taught arithmetic on Zermelo-Frankel set theoretic foundations (ZF). They learn that numbers are sets but they differ as to *which* set each number is. Ernie learns each natural number is the set of its predecessors. So 0 is the empty set and 4 is {0,1,2,3}. Johnny learns each is the singleton of its immediate predecessor. For him 0 is also empty but 4 is {3}. This makes no difference to the arithmetic they learn so Benacerraf says the boys have equal claims to know what numbers are. But when contrary claims are equally true, both are false. By an obvious generalization any identification of numbers with sets is wrong. Numbers can not be sets.

More rigorously, Benacerraf calls any set with the structure of the natural numbers (in effect, any set modelling the 2nd order Peano axioms) a "progression". He says arithmetic is "the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions" (1965 p.70). Any progression of ZF sets has uniquely individuating properties, and thus properties irrelevant to arithmetic, and so it can not *be* the numbers of arithmetic.

Benacerraf calls for a theory of "abstract structures" within ZF (1965 p.70). Abstract structures will not be sets. Rather, speaking of the natural numbers as an abstract structure will be a *façon de parler* for the properties common to all progressions. There can be similar treatment of other structures, say the real numbers or whatever. This idea is widely influential under the name of "structuralism" but remains problematic in its particulars. (Compare Hellman 1989, Parsons 1990, Resnik 1981 and 1982 and Shapiro 1983.)

Another point of view, though, says abstract structure is subtle but not so complex, and the irrelevant features of ZF sets are just technicalities. In fact, the structuralist program is already fulfilled or obviated, depending on how you look at it, by categorical set theory (first described in Lawvere 1964). Sets and functions in this theory have only structural properties. There is no need and no place for a further theory of abstract structures.

For example categorical set theory includes progressions, models of the 2nd order Peano axioms, usually called "natural number objects". The plural is used advisedly as there are provably infinitely many of them, all isomorphic to one another. The striking contrast with ZF is that all these natural number objects have *exactly the same properties*. They do not just have the same provable properties but rather, for any property in the theory, it is provable that if one of them has it they all do. Each of these progressions has only the structure that all have in common. So categorical set theory meets the demands Benacerraf puts on a structuralist account of mathematics.

To demonstrate this claim I give axioms for a fragment of categorical set theory, i.e. for a fragment of the theory of the topos of sets. These axioms suffice for elementary arithmetic and for the results proved here. Anyone who knows the topos of sets will see all our remarks also apply to it. For more detail and broader context see McLarty (1992). I also pursue the mock pedagogical form in some detail, to tell against the implication in Feferman (1977) and Mayberry (1977) that the categorical approach is pure formalism, unintelligible except by interpreting it in some 'real' set theory. I hope to show that finitary sets and arithmetic relate as easily to categorical set theory as to ZF.

One last preliminary. We define an "isomorphism" of sets to be a function $f : A \rightarrow B$ with an inverse $f^{-1} : B \rightarrow A$. That is, the composite $f^{-1} \circ f$ is the identity function on A, while $f \circ f^{-1}$ is the identity on B. By an easy proof these are the one-to-one onto functions. The definition is standard in mathematics. A group isomorphism is a group homomorphism with an inverse homomorphism. An isomorphism of topological spaces is a continuous function with a continuous inverse (which requires more than being one-to-one and onto). Classical set theorists rarely speak of "isomorphism" of sets in any sense, so they pose no obstacle to our usage.[1]

Finitist Beginnings

Imagine sisters, Tasha and Brittany, whose parents taught them arithmetic in full logical rigor. Tasha was a few years older and went first, beginning with extreme finitism. She learned to count to ten. She learned to recite the numbers from one to ten and also to count up to ten cows, boxes, or fingers. Then she learned to count to one hundred, and the system for counting to a thousand. She was not so demanding as to ask how much further this went.

This beginning included the idea of a set. Her parents pointed out that if there are four children in the yard and then they separate to go home, they are still four children. There is a *set* containing just those children, and it has four *elements* even when they are not all in one place. In other words the elements of a set need no unifying property beyond being the elements of the set.

Tasha learned to add sets of different things. If she has a set of nine books and Brittany has a set of seven then there is the set of sixteen books that either Tasha has or Brittany does. The girls tolerate no joint or ambiguous ownership of books. She also learned to multiply sets, so that if T is the set of Tasha's books, and B the set of Brittany's, then $T \times B$ is the set of ordered pairs $<t,b>$ with t one of Tasha's books and b one of Brittany's. Counting up showed that $T \times B$ has sixty three elements.

Tasha's workbook gave pictorial examples of disjoint unions up to ten plus ten, showing how many elements each had. She checked these by counting and memorized them as addition facts. She did the same with products up to ten times ten. The examples were pictures of ducks or beach balls or whatever but Tasha saw that they did not depend on the particular elements. She could squint until she could not tell what the elements were and yet still count them.

Then Tasha learned about functions. She was taught, in all due finitist rigor, that a function f from a set A to a set B is any rule assigning a value f(x) in B to each element x of A. If A is the set of children on her block, and B the set of houses on it, there is a function f from A to B where each f(x) is the house x lives in. You can imagine other examples.

She learned every set A has an identity function called 1_A with $1_A(x) = x$ for all x in A. And if C is a singleton set then every set has exactly one function to C. She was shown how any product $T \times B$ has a function p_1 to T, with $p_1(<x,y>) = x$ for every pair $<x,y>$ in $T \times B$, and a similar p_2 to B. And given any function f from A to B, and any g from B to C, there is a composite function $g \circ f$ whose value for any x in A is g(f(x)) in C.

Abstract Sets

Tasha soon knew more arithmetic than she had memorized. She could in fact add five to thirty four, or eight to forty six with a moment's thought, though neither was in her table of addition facts. She could multiply a hundred by two. But to learn general arithmetic she had to learn of the set of all natural numbers. Her parents did not think there were that many cows, ducks or books around nor did they think it mattered. They thought it beneath the dignity of mathematics to worry about such things. Truth to tell, they felt the same about actual or possible inscriptions, space-time points and regions, and anything else of the sort. However nice such things may be, arithmetic, the parents felt, did not depend on them.

Nor did they choose to start with transfinitely iterated set collection (as in any axiom of infinity for ZF). That idea hardly seemed to lie at the beginning of arithmetic. It seemed to lie nowhere outside of some axiomatic set theories. They preferred to stick with set forming operators that Tasha would use in the rest of her mathematics.

They told Tasha that what she had seen so far were *concrete* sets in that they collect concrete individuals. But she already knew that counting and arithmetic did not depend on what the elements were. Now she was to learn about *abstract* sets and functions. These are idealizations of the concrete ones, and can represent them, but they are described by their functions to each other rather than by concrete elements. From here on the words "set" and "function" refer to abstract sets and functions unless we specify concrete ones.

Her parents wrote $f : A \to B$ to say f is a function from the set A to the set B. And they said any $g : B \to C$ has a composite $g \circ f : A \to C$ with f. They did not define $g \circ f$ by an action on elements of A. Rather, they would soon define elements in terms of functions. For now they gave axioms, saying composition is associative and there are identity functions: For any further $h : C \to D$ we have $(h \circ g) \circ f = h(g \circ f)$. Every set A has a function $1_A : A \to A$ such that $f \circ 1_A = f$ and $1_A \circ h = h$ for any functions $f : A \to B$ and $h : C \to A$. In other words, sets and functions form a category.

A further axiom posited a set 1 such that every set has exactly one function to 1. So 1 is an abstract singleton. They defined an *element* of a set A to be any function from 1 to A, that is any $x : 1 \to A$. It follows immediately that 1 has exactly one element. Given an element x of A and a function $f : A \to B$, they could write f(x) for the composite $f \circ x$. In this notation the rule $(g \circ f)(x) = g(f(x))$ is a case of the associativity of composition, while $1_A(x) = x$ is a case of the identity axiom.

They added the following axioms:

Products) For any sets A and B there is a set $A \times B$ and a pair of functions $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ such that: Given any set S and pair of functions $f : S \to A$ and $g : S \to B$ there is a unique function $<f,g> : S \to A \times B$ with $p_1<f,g> = f$ and $p_2 \circ <f,g> = g$.

Equalizers) For any sets A and B and functions f and g both from A to B, there is a function $e : E \to A$ with $f \circ e = g \circ e$ and such that: For any function $h : T \to A$ with $f \circ h = g \circ h$ there exists a unique $u : T \to E$ with $e \circ u = h$.

Coproducts) For any sets A and B there is a set $A + B$ and a pair of functions $i_1 : A \to A + B$ and $i_2 : B \to A + B$ such that: Given any set S and pair of functions $f : A \to S$ and $g : B \to S$ there is a unique function $\binom{f}{g} : A + B \to S$ with $\binom{f}{g} \circ i_1 = f$ and $\binom{f}{g} \circ i_2 = g$.

Nontriviality) The functions i_1 and i_2 from 1 to $1 + 1$ are not the same.

The axioms so far were simple enough abstractions of ideas Tasha already applied to finite concrete sets. The product $A \times B$ corresponds to her familiar products. By the definition of element, an element of $A \times B$ is any $<x,y>$ where x is an element of A and y an element of B. When subsets are defined below, the equalizer $e : E \to A$ of f and g will turn out to be the subset of A containing those elements x with $f(x) = g(x)$. The coproduct $A + B$ is a disjoint union. Nontriviality guarantees that $1 + 1$ has two different elements. It will be used later to show the number 0 is not a successor.[2]

The next axiom requires the idea of a subset or, more precisely, of an abstract subset inclusion. A subset inclusion into a set A is defined to be *any* one-to-one function $i : S \to A$. That is, any function i such that, for any elements x and y of S, if $i(x) = i(y)$ then $x = y$. We often write $i : S \rightarrowtail A$ to show that i is a subset inclusion. As a standard abuse of language from category theory we refer to either i or S as a "subset" of A.[3]

Given any subset $i : S \rightarrowtail A$ and any element $x : 1 \to A$ we say x is in the subset i if there is some $h : 1 \to S$ with $x = i \circ h$. In other words, the element x factors through the subset inclusion. For every set A the identity $1_A : A \rightarrowtail A$ is a subset and contains every element of A. It is easy to see that any equalizer $e : E \to A$ for functions f and g on A is a subset, by the uniqueness of u in the definition of an equalizer, and it contains just those elements x of A with $f(x) = g(x)$.

Given two subsets $i : S \rightarrowtail A$ and $j : T \rightarrowtail A$ we say i is *included* in j, and write $i \subseteq j$, if there is some function $h : S \to T$ such that $i = j \circ h$. That is, if i factors through j. So for example every subset i of A is included in 1_A since $i = 1_A \circ i$. We say i is *equivalent* to j as a subset of A, and write $i \equiv j$, if both $i \subseteq j$ and $j \subseteq i$. Clearly, if $i \subseteq j$ then every element x of A in i is also in j. The converse fails in most categories, and so characterizes sets. So Tasha got another axiom:

1 Generates) For any subsets i and j of a set A, if every element x of A in i is also in j, then $i \subseteq j$.

It follows immediately that $i \equiv j$ if and only if i and j have exactly the same elements of A in them. An easy theorem shows that a function is fully determined by its effects on elements:

THEOREM 1: For any sets A and B and functions f and g both from A to B, if $f(x) = g(x)$ for every element x of A then $f = g$.
PROOF: If $f(x) = g(x)$ for every element x of A then every element of A is in the equalizer $e : E \rightarrowtail A$ of f and g. Thus $1_A \subseteq e$. But then from $f \circ e = g \circ e$ we conclude $f \circ 1_A = g \circ 1_A$ and so $f = g$.■

To represent a concrete set, say a set D of ducks, by an abstract set A is to associate each element of D with an element of A. That is, each duck is associated to a function from 1 to A. We might as well identify each duck with such a function (nothing in the axioms says functions are not ducks). Then for any subset $S \rightarrowtail A$ each duck in A is either in S or is not. Subsets of A are determined up to equivalence by the ducks in them. If the elements of B are (or are associated to) certain beachballs, then the elements of $A \times B$ are (or are associated to) pairs $<d,b>$ with d a duck and b a beachball, and so on. Of course stipulating that the elements of A are certain ducks is stipulating that A has a non-structural property. Only the purely abstract part of set theory is purely structural.

What Numbers Have To Be

What the natural numbers have to do is support recursive definitions. That is, a set N of natural numbers must have an element 0 and a successor function $s : N \to N$ with this property: Given any set A with a selected element x and a function $f : A \to A$ there is a unique function $u : N \to A$ such that $u(0) = x$ and $u(s(n)) = f(u(n))$ for every n in N. We say u is defined by the recursion data x,f. Any triad N,0,s with this property is called a *natural number object*. The idea is due to Lawvere.

Actually, arithmetic requires slightly more than this. The numbers must support recursive definitions with parameters. For any parameter set P and parametrized initial condition $x : P \to A$ and function $f : A \times P \to A$ there must be a unique function $u : N \times P \to A$ such that $u(0,p) = x(p)$ and $u(s(n),p) = f(u(n),p)$ for every element p of P and n of N. Such a natural number object is called *stable*. If any natural number object is stable so are they all (within a given category). This follows from Theorem 3 below, whose proof does not use stability.

The final axiom Tasha got at this stage was:

Infinity) There is a stable natural number object N,0,s.

Tasha learned that numbers are elements of N. Thus $1 = s(0)$ and $2 = s(s(0))$ and so on. Do not confuse the number 1 with the singleton set 1. Tasha could count elements of sets by pairing them with these numbers. And for every number n there is a subset $[n] \rightarrowtail N$ containing those numbers less than n.[4] So she could also say a set A has n elements if it is isomorphic to [n].

She learned to define addition by parametrized recursion data $0 + m = m$ and $s(n) + m = s(n + m)$. She saw that this agreed with the addition facts she had learned before. She defined multiplication by $0 \cdot m = 0$ and $s(n) \cdot m = (n \cdot m) + m$. This agreed with her old multiplication facts, as far as they went. And addition and multiplication still corresponded to disjoint unions and products of sets, only now for all finite sets.

She noticed there is a function $f : N \to 1 + 1$ defined by $f(0) = i_1$ and $f(s(n)) = i_2$. Since $i_1 \neq i_2$ she concluded that 0 is not s(n) for any n. A little manipulation showed there is an immediate predecessor function $h : N \to N$, defined by $h(0) = 0$ and $h(s(n)) = n$, and so she concluded that s was one-to-one. She even showed that any subset of N including 0 and closed under successor is all of N. More precisely:

THEOREM 2: Let $i : I \rightarrowtail N$ be any subset of N such that 0 is in I and there is some $p : I \to I$ with $i \circ p = s \circ i$. In other words the restriction p of s to I takes all its values in I. Then $i \equiv 1_N$.

PROOF: Certainly $i \subseteq 1_N$. Define $h : N \to I$ by $h(0) = 0$ and $h(s(n)) = p(h(n))$. Then $i \circ h(0) = 0$ and $i \circ h(s(n)) = s((i \circ h(n)))$, so $i \circ h$ has recursion data 0,s. Thus $i \circ h = 1_N$, and so $1_N \subseteq i$.■

Thus she verified the Peano axioms for N,0,s, and she was on her way to recursive function theory, though her parents did not push her too far along just yet. When she turned 16 they taught her about power sets, completing the axioms for the topos of sets (and rendering the coproduct axiom redundant). That changed nothing from the point of view of 'indiscernibility' as discussed below.

The Girls Discuss Arithmetic

Brittany got the same education as Tasha, except that she was fond of calligraphy and having seen primes used in her parents' math books she called her natural number object N′,0′,s′. The girls found they agreed entirely on arithmetic, and on which sets had n elements for each number n. But they wondered whether in fact Tasha's N,0,s were the same as Brittany's N′,0′,s′. Britanny quickly found:

THEOREM 3: N and N′ are isomorphic.
PROOF: Define $v : N' \to N$ by recursion data 0,s, and define $w : N \to N'$ by 0′,s′. Then $v \circ w(0) = 0$ and $v \circ w(s(n)) = s(n)$, so $v \circ w$ has data 0,s and the only function from N to N with these data is 1_N. Similarly $w \circ v = 1_{N'}$.■

That seemed like progress, but then Tasha noticed there are provably infinitely many natural number objects. Given N,0,s she could recursively define infinitely many isomorphisms from N to itself and she had this theorem:

THEOREM 4: If $h : N \to M$ is an isomorphism then M, h(0), $h \circ sh^{-1}$ is also a natural number object.
PROOF: Take any $x : 1 \to A$ and $f : A \to A$, and the corresponding $u : N \to A$. Then $u \circ h^{-1}: M \to A$ has $u \circ h^{-1}(h(0)) = x$ and $u \circ h^{-1}(h \circ s \circ h^{-1}(m)) = f(u \circ h^{-1}(m))$ for all m in M. Uniqueness is left to the reader.■

COROLLARY: Different isomorphisms h and k from N to N give different natural number objects N,h(0),$h \circ s \circ h^{-1}$ and N,k(0),$k \circ s \circ k^{-1}$.
PROOF: Trivially, h has recursion data h(0), $h \circ s \circ h^{-1}$, while k has k(0), $k \circ s \circ k^{-1}$. Different functions have different data.■

In other words any isomorphic rearrangment of the succesor function in a natural number object gives a new successor function and so a new natural number object. Tasha concluded that even if she assumed N = N′ she could not prove $0 = 0'$ or $s = s'$. On the other hand the girls found nothing they could prove about N,0,s to distinguish it from N',0',s'. That was hardly surprizing, since they only knew N,0,s and N',0',s' were both natural number objects.

Then they found something much stronger. They found N,0,s and N',0',s' provably indiscernible. Take any formula P(X,f,g) in their set theory with no constants and no free variables except the set variable X and the function variables f and g. The girls could prove the biconditional

$$P(N,0,s) \leftrightarrow P(N',0',s')$$

The two natural number objects provably have all the same properties. Even if P(N,0,s) is undecided, neither provable nor refutable, its equivalence to P(N',0',s') is still a theorem.[5]

In fact there is a predicate NNO(X,f,g) saying X,f,g is a natural number object, and the girls could prove the generalized indiscernibility result

$$[NNO(X,f,g) \ \& \ NNO(Y,h,k)] \rightarrow [P(X,f,g) \leftrightarrow P(Y,h,k)]$$

All natural number objects are indiscernible in this theory. They provably have all the same properties.

The girls used a proof theoretic argument based on switching N,0,s and N',0',s'. They formalized the following definition of an operator F in their set theory.

If A and B are any sets other than N or N', f and g any functions, and $v : N' \rightarrow N$ the isomorphism defined in Theorem 3, then:

$$FA = A \quad \text{and} \quad FN = N' \quad \text{and} \quad FN' = N$$

If	$f: A \rightarrow B$	then	$Ff = f$
If	$f: A \rightarrow N$	then	$Ff = v^{-1} \circ f: A \rightarrow N'$
If	$f: N \rightarrow B$	then	$Ff = f \circ v: N' \rightarrow B$
If	$f: N \rightarrow N$	then	$Ff = v^{-1} \circ f \circ v: N' \rightarrow N'$
If	$f: N \rightarrow N'$	then	$Ff = v \circ f \circ v: N' \rightarrow N$

and similar clauses with N' in place of N and switching the roles of v and v^{-1}.

They easily proved $F0 = 0'$ and $Fs = s'$ and vice versa. Two more facts are crucial. First, F preserves the atomic relations of the set theory. That is, provably in the set theory, if $f : C \rightarrow D$ then $Ff : FC \rightarrow FD$, and $F(g \circ f) = Fg \circ Ff$, and $F(1_C) = 1_{FC}$ for any sets C and D and functions f and g.[6] Second, $C = FFC$ and $f = FFf$ for any set C and function f. So every set and function is provably a value of F as well as having a value under F.

Given any formula Q of the set theory form Q_F by putting F before every constant and free variable in Q. Then $Q \leftrightarrow Q_F$ is provable. For quantifier free formulas Q use the first and second facts on F. To add quantifiers notice that, by the second fact, a formula holds for all (some) sets C iff it holds for all (some) sets FC, and similarly for functions.

The first result, $P(N,0,s) \leftrightarrow P(N',0',s')$, is the case where Q is $P(N,0,s)$. The meta-proof can be adapted to variables X,f,g and Y,h,k with the antecedent assumption that NNO(X,f,g) & NNO(Y,h,k) to yield the generalized indiscernibility result.

Indiscernibility is no special feature of natural number objects. Any two isomorphic objects are indiscernible in this set theory. Let Isom(X,Y) be the formula saying X and Y are isomorphic, that is saying there is a function from X to Y with an inverse. Let P(X) be any formula with no constants and no free variables but the set variable X. Then a similar simpler argument shows this is provable:

$$\mathrm{Isom}(X,Y) \rightarrow (\ P(X) \leftrightarrow P(Y)\ )$$

These results say in unusual technical detail what category theorists often say briefly. Isomorphic objects in any category have all the same properties.

Other Set Theories

There is no such indiscernibility in ZF. A model of the Peano axioms in ZF is, like a natural number object, actually a triple N,0,s of a set and element of it and a successor function. Suppose all we know of N,0,s and N′,0′,s′ in ZF is that both model those axioms. Then, as in categorical set theory we can prove N and N′ isomorphic, we can not prove whether or not $N = N'$, and anything we can prove about N,0,s we can also prove about N′,0′,s′. But we can not prove indiscernibility.

In ZF most properties P(N,0,s) of models of the Peano axioms are undetermined by the Peano axioms. Some models of the Peano axioms have P and not others. Let P(N,0,s) say that every element of N is a singleton, for instance. If P(N,0,s) is undetermined then the case of the generalized indiscernibility statement using P is provably false. No model of the Peano axioms (or of any axioms) in ZF has only the properties that all have in common. That is Benacerraf's point. But the point fails for categorical set theory. Sets there, like Benacerraf's numbers, have only structural relations.

An indiscernibility result analogous to ours does hold in 2nd order logic. See the Elementary Equivalence Theorem in Hellman (1989) p.41. But 2nd order logic and our fragment of set theory are both too weak for substantial mathematics. Hellman (1989 p.44ff) shows how a good deal of mathematics can be jammed into 2nd order logic by codings. At the same time he shows how unpleasant this is for any but formalist purposes, and in his Chapter 3 he argues that it really will not work even for all the mathematics needed in modern physics.

The problem is to get a strong enough set theory, without the anti-structuralism of ZF itself.[7] To do that we can extend the axioms above to the axioms for the topos of sets. The extension requires only a power set axiom, stated in the style of those above but slightly more involved.[8] These axioms have the strength of Zermelo set theory (i.e. ZF without choice or replacement) with only bounded quantifiers allowed in the separation axiom. The axiom of choice is easily added, as are unbounded separation or replacement if you want them. See McLarty (1992) Chapter 22. At that point "all of mathematics" is available in this set theory just as it is in ZF, but with the same structuralist indiscernibility result as we proved for the weaker fragment.

Sorting It Out

Tasha and Brittany agreed with Benacerraf that "Arithmetic is therefore the science that elaborates the abstract structure that all progressions have in common merely in virtue of being progressions" (1965 p.70). So it was the science of natural number objects. Brittany was puzzled by another quote she found, saying a mathematical structure should be seen as "the form of a possible system of related objects, ignoring the features of the objects that are not relevant to the interrelations" (Shapiro 1983 p.535). Her education had not taught her any way for the elements of a natural number object to have any features but their arithmetic interrelations. What was there to ignore?

When Tasha got to college she heard of a boy there, Ernie, who had also learned arithmetic with foundations. She met him one day over lunch. Ernie and his friend Johnny had learned ZF and had concluded that numbers can not be sets so when Tasha spoke of the set of natural numbers he immediately asked what she thought that was. As she began explaining indiscernibility he got confused. "Just tell me," he said, "what set is 0?" She said "0 is not a set, it is an element of N". He answered "But elements of sets are sets!" and she got confused. They found they had very different ideas about sets.

She asked him to describe his approach and he did, pointing out that there was nothing like her indiscernibility in it. He told her every set was uniquely determined by its elements, which are themselves sets.

Aiming to see how this worked she asked "You mean that you define the natural numbers as a certain specific set?"

"Well no," he answered, "The natural numbers aren't a set, they are a structure. You see they aren't uniquely defined".

"So they are like my sets?"

"Yes."

She asked if it was the same for the real numbers, or the Euclidean plane, and he said it was. He said all of those are abstract structures, handy ways of talking about sets but not themselves sets and actually not objects at all.

"So the advantage of your set theory is that mathematicians never work with your sets!" she said amazed.

"You could say that," Ernie agreed, "since mathematicians usually work with structures. But I would say the advantage is that my sets are legitimate objects by Leibniz's law and yours aren't".

Tasha had always concentrated on mathematics and the only Leibniz law she knew was the one in calculus for derivatives of products. Ernie had to explain "It is a metaphysical principle that there can't be distinct objects with no difference between them". She found this unmathematical and unconvincing. She thought her set theory refuted it.

"Why do you say sets *have to* be uniquely defined when you also say the numbers and spaces that we want to talk about *can not* be?" she wanted to know.

Ernie stood by his claim that this was merely responsible metaphysics.

For her part, Tasha went on to study logic and learned a number of set theories including ZF. She got good enough at manipulating them in a formal way. But none ever made as much sense to her as the one she had grown up with.

Notes

*John Mayberry suggested several of the ideas here and provoked others. I thank Helen Lauer, Michael Resnik, and Stewart Shapiro for extensive comments. Tasha Dixon advised me on addition facts, and Brittany Dixon on counting to ten.

[1]Set theorists working in ZF generally reserve the word "isomorphism" for ordered sets and model structures: An order isomorphism is an order preserving function with an order preserving inverse. An isomorphism of models is a structure preserving function with a structure preserving inverse.

[2]It is instructive to see that, except for Nontriviality, all the axioms here including the axiom of infinity are true in a model with 1 the only set and its identity 1_1 the only function.

[3]Given the axiom to follow, called 1 Generates, a function is one-to-one iff it is monic in the usual sense of category theory.

[4]Consider the function $f : N \times N \to N$ taking each pair $<p,q>$ to $p + q + 1$, and the constant function taking each pair to n. Their equalizer $[n] \rightarrowtail N \times N$ contains those $<p,q>$ with $p + q + 1 = n$. The projection $[n] \rightarrowtail N$ onto the first coordinate is also one-to-one, i.e. it is a subset, and it contains p iff $p<n$.

[5]The meta-theorem would fail if $P(X,f,g)$ were $0 = f \,\&\, s = g$, or if it were $f = x \,\&\, g = y$ for free variables x and y. In each of those cases the biconditional would imply $0 = 0' \,\&\, s = s'$, which is not provable. The point is that natural number objects do "differ" in that each is itself and is not the others; but do not differ in terms of any property stateable without specifying particular objects.

[6]We take the product, equalizer, and coproduct axioms as describing relations among sets and functions rather than as introducing operators. We do not allow, for example, an operator $\_\times\_$ taking sets A and B to a selected product $A \times B$ since F might not preserve the selected values.

[7]Hellman does this by what Parsons (1990) calls "eliminative structuralism" working with ZF as a formal theory and quantifying over models of it. Categorical set theory is not eliminative. It works with sets themselves structurally described, not with a formal theory and models of it.

[8]See, for example, Pitts (1991), which also gives an impressively concise survey of topos theory very much from a logician's viewpoint. Compare Exercise 13.16 in McLarty 1992.

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