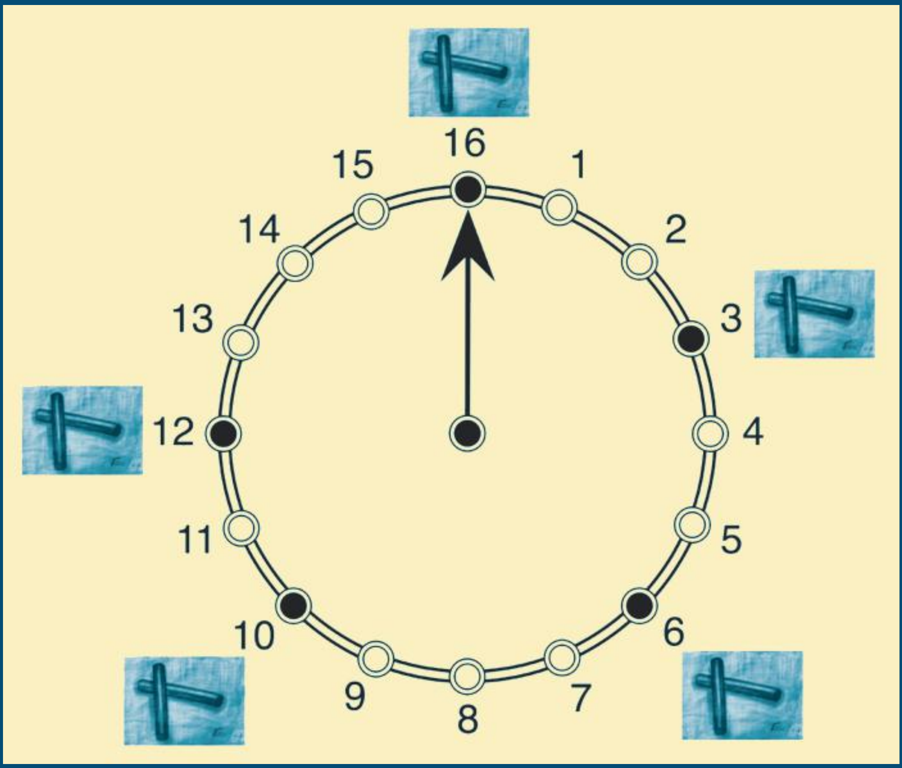
The GEOMETRY of MUSICAL RHYTHM

What Makes a "Good" Rhythm Good?



Godfried T. Toussaint

CRC Press
Taylor & Francis Group
A CHAPMAN & HALL BOOK

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Illustrative drawings by Yang Liu

CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Version Date: 20121203

International Standard Book Number-13: 978-1-4665-1203-0 (eBook - PDF)

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For the love of my life—
Eva Rosalie Toussaint

Contents

Prolegomenon, xi

Author, xvii

Chapter 1 ■ What Is Rhythm? 1

Chapter 2 ■ A Steady Beat 9

Chapter 3 ■ Timelines, Ostinatos, and Meter 13

Chapter 4 ■ The Wooden Claves 17

Chapter 5 ■ The Iron Bells 19

Chapter 6 ■ The Clave Son 23

Chapter 7 ■ Six Distinguished Rhythm Timelines 27

Chapter 8 ■ The Distance Geometry of Rhythm 33

Chapter 9 ■ Classification of Rhythms 39

Chapter 10 ■ Binary and Ternary Rhythms 45

Chapter 11 ■ The Isomorphism of Rhythm and Scale 51

Chapter 12 ■ Binarization, Ternarization, and Quantization of Rhythms 57

Chapter 13 ■ Syncopated Rhythms 67

KEITH'S MEASURE OF SYNCOPATION 70

Chapter 14 ■ Necklaces and Bracelets 73

Chapter 15 ■ Rhythmic Oddity 85

Chapter 16 ■ Off-Beat Rhythms 99

Chapter 17 ■ Rhythm Complexity 107

OBJECTIVE, COGNITIVE, AND PERFORMANCE COMPLEXITIES 109
LEMPEL–ZIV COMPLEXITY 117
COGNITIVE COMPLEXITY OF RHYTHMS 119
IRREGULARITY AND THE NORMALIZED PAIRWISE VARIABILITY INDEX 120

Chapter 18 ■ Dispersion Problems and Maximally Even Rhythms 121

Chapter 19 ■ Euclidean Rhythms 129

Chapter 20 ■ Leap Years: The Rhythm of the Stars 137

Chapter 21 ■ Approximately Even Rhythms 143

Chapter 22 ■ Rhythms and Crystallography 151

Chapter 23 ■ Complementary Rhythms 157

Chapter 24 ■ Radio Astronomy and Flat Rhythms 165

Chapter 25 ■ Deep Rhythms 175

Chapter 26 ■ Shelling Rhythms 187

Chapter 27 ■ Phantom Rhythms 195

Chapter 28 ■ Reflection Rhythms and Rhythmic Canons 207

Chapter 29 ■ Toggle Rhythms 217

Chapter 30 ■ Symmetric Rhythms 225

HOURGLASS DRUMS AND HOURGLASS RHYTHMS 233

Chapter 31 ■ Odd Rhythms 237

Chapter 32 ■ Other Representations of Rhythm 243

ALTERNATING-HANDS BOX NOTATION 243

SPECTRAL NOTATION 244
TEDAS AND CHRONOTONIC NOTATION 245

CHAPTER 33 ■ Rhythmic Similarity and Dissimilarity 247

CHAPTER 34 ■ Regular and Irregular Rhythms 257

CHAPTER 35 ■ Evolution and Phylogenesis of Musical Rhythm 265
GUAJIRA 271

CHAPTER 36 ■ Rhythmic Combinatorics 275

CHAPTER 37 ■ What Makes the Clave Son Such a Good Rhythm? 279
MAXIMAL EVENNESS 279
RHYTHMIC ODDITY 280
OFF-BEATNESS 282
WEIGHTED OFF-BEATNESS 283
METRICAL COMPLEXITY 283
MAIN-BEAT ONSETS AND CLOSURE 283
DISTINCT DURATIONS 284
DISTINCT ADJACENT DURATIONS 284
ONSET-COMPLEXITY AND DISTINCT DISTANCES 284
DEEP RHYTHMS, DEEPNESS, AND SHALLOWNESS 285
TALLNESS 286
PHYLOGENETIC TREE CENTRALITY 286
MIRROR SYMMETRY 287
SHADOW CONTOUR ISOMORPHISM 288

CHAPTER 38 ■ The Origin, Evolution, and Migration of the Clave Son 293

EPILOGUE, 305

REFERENCES, 311

Prolegomenon

It would not be incorrect to say that this book offers a view of musical rhythm through the eyes of mathematics and computer science. However, mathematics and computer science are extremely broad subjects, and it is easy even within these disciplines to give accounts of musical rhythm that bear little resemblance to each other. The mathematics employed may be continuous or discrete, deterministic or probabilistic, algebraic, combinatorial, or geometric. A computer science approach could focus on software packages, programming principles, electronic production, information retrieval, design and analysis of algorithms, or artificial intelligence. To be more accurate, this book offers a description of my personal mathematical and computational predilections for analyzing musical rhythm. As such, it is in part a record of my own recent investigations into questions about rhythm that have inspired me, in particular, questions such as: How does one measure rhythm similarity? How do rhythms evolve over time? And what is it that makes a "good" rhythm good? However, this book is more than that. Before describing my general approach to the study of musical rhythm, as well as the material that is included and left out, it is appropriate to give a brief outline of where I am coming from, what my background is, what my main goals are, who my target audience is, and how I ended up writing this book.

Three of my passions when I was a teenager attending a boarding school in England were geometry, music, and designing, building, and flying model airplanes. The first two are directly related to this book. At St. Joseph's College in Blackpool, I took 3 years of geometry classes in which we proved many theorems taken from Euclid's book: *The Elements*. The beauty of the problems tackled in this book using the straight edge and compass, and the puzzle-like nature of creating new proofs for the theorems therein made a deep and lasting impression on me. At St. Joseph's, I also sang in the school choir, collected 45-rpm records by Buddy Holly, and longed for classical guitar lessons. Unfortunately, I was not permitted to study classical guitar because, as the headmaster explained to my father, the guitar was considered a vulgar instrument forbidden on the school premises, and therefore I had to wait until I graduated from high school before taking up this instrument.

In university, I focused my studies on information theory, pattern recognition, machine learning, and artificial intelligence, eventually obtaining a PhD in electrical engineering at the University of British Columbia. Then, I switched fields, and joined the School of Computer Science at McGill University, where I rediscovered my passion for Euclidean geometry, this time wearing a computational hat, and where my interest in the guitar was supplanted by a new passion for African drumming and percussion. In the year 2000, I began to create an interdisciplinary academic bridge between my professional interests in discrete mathematics and computational geometry, and my leisure time enthusiasm for rhythm and percussion. I started to read the literature on the mathematics of music, and learned that not much research had been done on rhythm, as compared with other aspects of music such as pitch, chords, melody, harmony, scales, and tuning. This discovery propelled me to investigate musical rhythm more ardently using the conceptual and computational tools at my disposal. My computer science mindset spurred me to view rhythm, in its simplest reductionist terms, as a binary sequence of ones and zeros denoting sounds and silences, respectively. My colleagues Michael Hallett and David Bryant at McGill University graciously introduced me to phylogenetic analysis and molecular biology. Where I heard rhythms, I saw DNA sequences, and I was immediately inspired by the possibilities of applying bioinformatics tools to the analysis of musical rhythm. I promptly initiated a research project on the phylogenetic analysis of the musical rhythms of the world. Such a project offered a variety of challenging problems, such as the development of measures of rhythm similarity, and rhythm complexity, to which I applied my knowledge of pattern recognition and information theory.

While at McGill University, I was invited by Stephen McAdams, the director of the Centre for Interdisciplinary Research in Music Media and Technology (CIRMMT), to become one of its members in the group on Music Information Retrieval headed by Ichiro Fujinaga. At that time, I had already been organizing workshops on computational geometry at McGill University's Bellairs Research Institute in Barbados since 1986, and I invited Dmitri Tymoczko, a music theorist and composer at Princeton University, to join us there to explore connections between computational geometry and music. Since then, Dmitri and I have jointly organized annual workshops on mathematics and music at Bellairs.

In 2009, I moved from McGill to Harvard University and I was awarded a Radcliffe Fellowship to extend my project on the phylogenetics of rhythm, at the Radcliffe Institute for Advanced Study. It was there, after giving a public lecture, that I met Luke Mathews of the Department of Human Evolutionary Biology. Our mutual interest in cultural evolution spawned a fruitful ongoing collaboration on the comparison and application of phylogenetics methods. Luke introduced me to Charlie Nunn, the director of the Comparative Primatology Research Group (CPRG) at Harvard, who invited me to join their weekly seminar meetings. At these meetings, I learned about Bayesian phylogenetics techniques, and obtained useful feedback on my ideas for their application to musical rhythm. It was also during my residence at the Radcliffe Institute, in the fall of 2009, that my wife Eva encouraged me to write a book on the geometry of musical rhythm, a project on which I immediately embarked with zest. The following year, I joined the Music Department at Harvard University as a visiting scholar. There, I benefited from a great music library and stimulating colleagues. Christopher Hasty invited me to attend his graduate seminar course on rhythm, and made time available from his busy schedule to discuss my research ideas and offer his opinions and suggestions. I also began an ongoing collaboration with Olaf Post to model the cognitive aspects of rhythm similarity.

Then, in August of 2011, I was offered a position as research professor of computer science at New York University Abu Dhabi, Abu Dhabi, United Arab Emirates, where a teaching-free fall semester gave me time to finish writing the book, and a spring course on computers and music allowed me to test much of the material on a group of undergraduate students with diverse backgrounds.

My goals in writing this book were multidimensional. First of all, I wanted to illustrate how the study of the mathematical properties of musical rhythm generates common mathematical problems that arise in a variety of seemingly disparate fields other than music theory, such as number theory in mathematics, combinatorics of words and automatic sequence generation in theoretical computer science, molecule reconstruction in crystallography, the restriction scaffold assignment problem in computational biology, timing in spallation neutron source accelerators in nuclear engineering, spatial arrangement of telescopes in radio astronomy, auditory illusions in the psychology of perception, facility location problems in operations research, leap-year calculations in astronomical calendar design, drawing straight lines in computer graphics, and the Euclidean algorithm for computing the greatest common divisor of two integers in the design and analysis of algorithms.

Second, I wanted the book to be accessible to a wide audience of academicians and musicians, classical violinists or drummers, with diverse backgrounds and musical activities, with a minimal set of prerequisites. I hope that mathematicians and computer scientists will find interest in the connections I make with other fields, and will obtain motivation for working on open questions inspired by the problems considered here. I also hope the book will be useful to composers, producers of electronic dance music, music technologists, and teachers. Although the book does not include a set of exercises and problems, it should be suitable as a text for an undergraduate interdisciplinary course on music technology, music and computers, or music and mathematics.

Third, I wanted to introduce to the music community the distance approach to *phylogenetic analysis*, and illustrate its application to the study of musical rhythm. In spite of the fact that phylogenetic analysis has been applied to cultural anthropology for some 40 years, most notably to linguistics, but also to other cultural objects such as stone tools, carpet designs, and musical instruments, its application to musical rhythm is just beginning. While I have applied phylogenetic analysis to several different corpora of rhythms from different parts of the world, here I give just one example of my approach using the flamenco meters of southern Spain.

Having described the essential features of what I tried to do with this book, I should add a few words about what I did not cover, and the reasons for leaving such material out. First of all, I am not concerned with the vast domain of music theory that has traditionally used discrete mathematics (especially combinatorics) to analyze chords and scales. This topic, which may be described as "musical set theory," generally involves a number of issues such as various symmetries, relations between complementary sets, and measuring chord similarity. There exist already many books dealing with this musical set theory, and therefore no effort is made to review this area here. It is well known that there exists an isomorphic relation between pitch and rhythm, which several authors have pointed out from time to time. This book is also not a general introduction to what might be called "rhythmic set theory," even though connections (and pointers to the literature) between the rhythmic concepts covered here and the corresponding pitch set theory ideas are made at points scattered throughout the book. Instead, this book offers a more geometric and computational approach to the study of musical rhythm. However, I do not go as far as to propose a "theory of rhythm." Nevertheless, I hope that the tools provided here will help in the eventual development of such a theory.

The study of musical rhythm may also be divided into two general strategies according to how the input rhythms are represented: acoustic and symbolic. In the first approach, rhythms are given as acoustic waveforms recorded or produced with electronic equipment. In this method, rhythm analysis belongs to the domains of acoustics and signal processing, involves problems that include beat induction and automatic transcription, and uses tools including trigonometry and more advanced mathematics such as Fourier analysis. Many books have already been published that deal with these topics, and I have therefore also left all that material out. In the second approach, rhythms are represented as symbolic sequences, much like the text written on this page. In this approach, rhythm analysis falls in the domain of discrete mathematics. My emphasis in writing this book has been on new symbolic geometric approaches, while making connections to existing methods where appropriate.

By means of copious use of illustrative figures, I have tried to emphasize a visual geometric treatment of musical rhythm and its underlying structures to make it more accessible to a wide audience of musicians, computer scientists, mathematicians, cultural anthropologists, composers, ethnomusicologists, psychologists, researchers, academics, trades people, professionals, and teachers. I have emphasized a new methodology, namely, distance geometry and phylogenetic analysis, for research in comparative musicology, ethnomusicology, and the new field of evolutionary musicology, and I have tried to strengthen the bridge between these disciplines and mathematical music theory.

There are many concepts in the book that, mainly for pedagogical reasons, I have illustrated with examples using a select group of six distinguished rhythms that feature prominently in world music, and one in particular, which is known around the world mainly by its Cuban name: the clave son. Thus, this book also includes an ethnological investigation into the prominence of this rhythm. Part of my reasoning for this approach is my belief that if we can understand what makes the clave son such a good rhythm, we will gain insight into what makes rhythms good in general.

I should also mention that this is not a cookbook of only the best methods available for measuring rhythmic similarity, rhythm complexity, syncopation, irregularity, goodness, or what have you. First of all, the field is too new, and the problems too difficult to afford such categorical descriptions. Second, I have written this book wearing a scientific hat, and thus am interested in considering methods that are bad as well as good, with the hope that comparison of these methods will yield further insight into the nature of the problems considered.

One of the main themes of the book is the exploration of mathematical properties of good rhythms. Therefore, the reader will notice upon reading this book that I often introduce a mathematical property of a sequence, then point out one or more musical rhythms used in practice that have this property, and subsequently speculate on how the mathematical property may encapsulate psychological characteristics that contribute to their attractiveness or salience as rhythm timelines. This form of argumentation is not meant to imply that the rhythm exists or is successful merely because of its mathematical properties. For a mathematician, mathematical properties of rhythms are easy to find. What is more rare and interesting from both the musicological and mathematical points of view is to obtain a *characterization* of a given rhythm in the mathematical sense, or in other words, to discover a collection of properties possessed *only* by the rhythm in question. In the book, I illustrate this characterization approach with two examples: the clave son timeline, and the rhythmic pattern used by Steve Reich in his piece *Clapping Music*. While characterizing a rhythm in this way helps to understand the structure of the rhythm, it does not imply that the structure so obtained is the whole story. Indeed, other mathematical characterizations may exist that are even more useful to the musicologist. However, such characterizations suggest avenues for musicological discourse and the design of psychological experiments to determine the perceptual validity of the properties in question.

The book has a relatively large number of short chapters, rather than a few long ones. This feature serves to highlight the importance and variety of the individual topics covered. The organization of the chapters is partly determined by the logical progression of the topics. The first eight chapters should be read in the order given. However, after Chapter 8, the chapters are fairly self-contained, and could be read in almost any order with some judicious flashbacks to fill in a few gaps. The reader will not be handfed the larger narrative of the book at every chapter, and I hope she or he will make a creative effort to connect the dots between the narrative and the individual chapter topics. Furthermore, a long list of books and articles that I found useful and relevant to the topics covered is provided at the end, where the reader may find much additional material at a variety of different levels of exposition.

I would like to thank several ethnomusicologists, music theorists, and drummers who in the past decade have generously lent me their ears, and were open to the rather strange mathematical meanderings of an outsider wandering into their backyards, learning the ropes along the way. Kofi Agawu invited me to participate in his 2012 Workshop on African Music at Princeton University, and made me feel at home among a group of scholars of African Music who offered candid and useful suggestions for my future research. In his book *Representing African Music*, Kofi Agawu asks the question "How not to analyze African Music?" On page 196, he provides the answer: "Any and all ways are acceptable." These words have always been an inspiration to me. Through email correspondence, Jeff Pressing shared his insightful views on the cognitive complexity of rhythm. I met Willie Anku during one of my visits to Simon Fraser University in Burnaby (Vancouver, Canada), where we presented a joint performance and geometric analysis of African timelines, and brainstormed on the benefits of a mathematical analysis of African rhythm. Enrique Pla, the drummer for the group *IRAKERE*, was a model host and drum set teacher in Havana, Cuba, in 2004 before, during, and after our joint presentation of a geometric analysis of Afro-Cuban rhythms at *PERCUBA: The 15th International Percussion Festival*. Simha

Arom patiently taught me about African rhythm and ethnomusicology by sharing his wonderful stories of the listening experiments he performed with the Aka Pygmies of Central Africa. With an open mind, David Locke invited me to lecture in his African Music class at Tufts University. Jay Rahn at York University sent me all his wonderful papers where I found detailed documentation of many of the rhythms that appear in this book. Richard Cohn welcomed me to Yale University at the 2009 conference that he organized there on Mathematics and Computation in Music/The John Clough Memorial Conference. Rolando Pérez-Fernandez explained his theory of the binarization of ternary rhythms and shared the latest data he had collected. At several music–math conferences, I had illuminating discussions with Jack Douthett and Richard Krantz.

I would like to thank Dmitri Tymoczko for reading and evaluating a preliminary version of the manuscript, and through back-and-forth email discussions providing me with a great deal of feedback and numerous germane constructive suggestions for improving the book. I incorporated almost all his suggestions. However, any errors, omissions, and shortcomings present are entirely my responsibility.

I would also like to thank the team at CRC Press of Taylor & Francis for their help with the logistics of the preparation of the book, especially Sunil Nair, Rachel Holt, Jill Jurgensen, Kyle S. Meyer, Jr., and Amber Donley.

My deepest gratitude goes to my wife Eva Rosalie Toussaint, for suggesting that I write this book in the first place, and for her continuous support and encouragement, in spite of the fact that the 3 years of writing stole too many hours of my time that we could otherwise have spent together.

Author

Godfried T. Toussaint is a Canadian computer scientist born in Belgium. Presently, he is a research professor of computer science at the New York University Abu Dhabi, Abu Dhabi, United Arab Emirates. After receiving a PhD in electrical engineering from the University of British Columbia in Vancouver, Canada, he taught and did research at the School of Computer Science at McGill University in Montreal, in the areas of information theory, pattern recognition, textile-pattern analysis and design, computational geometry, instance-based learning, music information retrieval, and computational music theory. In 2005, he became a researcher at the Centre for Interdisciplinary Research in Music Media and Technology, in the Schulich School of Music at McGill University. In 1978, he received the Pattern Recognition Society's *Best Paper of the Year Award;* and in 1985, he was awarded a *Senior Killam Research Fellowship* by the Canada Council. In May 2001, he was awarded the *David Thomson Award* for excellence in graduate supervision and teaching at McGill University. He is a founder and cofounder of several international conferences and workshops on computational geometry. He is an editor of several journals, has appeared on television programs to explain his research on the mathematical analysis of flamenco rhythms, and has published more than 360 papers. In 2009, he was awarded a *Radcliffe Fellowship* by the Radcliffe Institute for Advanced Study at Harvard University, for the 2009–2010 academic year, to carry out a research project on the phylogenetic analysis of the musical rhythms of the world. After spending an additional year at Harvard University, in the music department, he moved in August 2011 to New York University Abu Dhabi.

CHAPTER 1

What Is Rhythm?

CREATING MUSIC, LISTENING TO MUSIC, and dancing to the rhythms of music are practices cherished in cultures all over the world. Although the function of music as a survival strategy in the evolution of the human species is a hotly debated topic, there is little doubt that music satisfies a deep human need.[*] To the ancient philosopher Confucius, good music symbolized the harmony between heaven and earth.[†] The nineteenth-century philosopher Friedrich Nietzsche put it this way: "without music life would be a mistake."[‡] And the Blackfoot people roaming the North American prairies "traditionally believed that they could not live without their songs."[§]

Of the many components that make up music, two stand tall above all others: rhythm and melody. Rhythm is associated with time and the horizontal direction in a typical Western music score. Melody, on the other hand, is associated with pitch and the vertical direction. Although rhythm and melody may be studied independently, in music, they generally interact together, and influence each other in complex ways.[¶] Of these two properties, rhythm is considered by many scholars to be the most fundamental of the two, and it has been argued that the development of rhythm predates that of melody in evolutionary terms.[**] The ancient Greeks maintained that without rhythm melody lacked strength and form. Martin L. West writes: "rhythm is the vital soul of music,"[††] the philosopher Andy Hamilton notes that "rhythm is the one indispensable element of all music,"[‡‡] and Ton de Leeuw considers that "rhythm is the highest and most autonomous expression of time-consciousness."[§§]

[*] Huron, D. (2009), Bispham, J. (2006), Cross, I. (1999, 2001).
[†] Lau, F. (2008), p. 119.
[‡] Nietzsche, F. (1889). This quotation comes from the 33rd Maxims and Arrows in the book *Die Götzen-Dämmerung* (*Twilight of the Gods*).
[§] Nettl, B. (2005), p. 23.
[¶] Monahan, C. B. and Carterette, E. C. (1985). However, in many parts of the world, such as India, Iran, and the Arab world, "musical rhythm is a highly artistic element, self-contained in its rich and most intricate composition, and conceived quite independently of the melodic line." See Gerson-Kiwi, E. (1952), p. 18.
[**] Benzon, W. L. (1993).
[††] West, M. L. (1992), p. 129.
[‡‡] Hamilton, A. (2007), p. 122.
[§§] De Leeuw, T. (2005), p. 38.

Joseph Schillinger writes: "The temporal flow of music is primarily a matter of rhythm."[*] Christopher Hasty offers a concise universal definition of music as the "rhythmization of sound."[†] From the scientific perspective, psychological experiments designed to assess the dimensional features of the music space, based on similarity judgments of pairs of melodic fragments, suggest that the major dimensions are rhythmic rather than melodic.[‡] The American composer George Gershwin believed that the public loved his music because of its rhythm, and in analyzing his rhythms, Isabel Morse Jones writes: "Gershwin has found definite laws of rhythm as mathematical and precise as any science."[§]

What *is* rhythm? There is no simple answer to this question. Hasty cautions that "rhythm is often regarded as one of the most problematic and least understood aspects of music."[¶] James Beament echoes this sentiment when he writes: "Rhythm is often considered the most difficult feature of music to understand."[**] For Robert Kauffman, "The difficulties of dealing with rhythm are immense."[††] Wallace Berry writes: "The awesome complexity of problems of rhythmic structure and analysis can be seen when one appreciates that rhythm is a generic factor."[‡‡] Berry goes on to note that another consideration that makes studying rhythm difficult is the fact that meanings ascribed to terms such as "rhythm," "meter," "accent," "duration," and "syncopation" are vague and used inconsistently. Elsewhere, he writes more concisely: "Rhythm is: everything."[§§] In spite of some of these difficulties, or perhaps because of them, many definitions of rhythm have been offered throughout the centuries. Already in 1973, Kolinski wrote that more than 50 definitions of rhythm could be found in the music literature.[¶¶] Before diving into the geometric intricacies of rhythm that are explored in this book, it is instructive to review a few examples of definitions and characterizations of rhythm, both ancient and modern.

Plato: "An order of movement."[***]
Baccheios the Elder: "A measuring of time by means of some kind of movement."[†††]
Phaedrus: "Some measured thesis of syllables, placed together in certain ways."[‡‡‡]
Aristoxenus: "Time, divided by any of those things that are capable of being rhythmed."[§§§]

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[*] Schillinger, J. (2004), p. vi.
[†] Hasty, C. F. (1997), p. 3.
[‡] Monahan and Carterette, *op. cit.*, p. 1. It has also been shown experimentally that rhythmic structures serve a principal function in the perception of melodic similarity (Casey, M., Veltkamp, R., Goto, M., Leman, M., Rhodes, C., and Slaney, M. (2008), p. 687). This view is apparently not held in some legal circles in the context of music copyright infringement resolution. According to Cronin, C. (1997–98), p. 188, "For federal courts at least, originality—the *sine qua non* of copyright—in music lies in melody." He quotes from the case of *Northern Music Corporation v. King Record Distribution Co.* that "Rhythm is simply the tempo in which a composition is written."
[§] Jones, I. M. (1937), p. 245.
[¶] Hasty, *op. cit.*, p. 3.
[**] Beament, J. (2005), p. 139.
[††] Kauffman, R. (1980), p. 393.
[‡‡] Berry, W. (1987), p. 303.
[§§] Berry, W. (1985), p. 33.
[¶¶] Kolinski, M. (1973), p. 494.
[***] *Ibid.*
[†††] Abdy Williams, C. F. (2009), p. 24.
[‡‡‡] *Ibid.*
[§§§] *Ibid.* It is clear that Aristoxenus considers rhythm as a general phenomenon that is not restricted to music, but also includes speech and dance, among other "things."

Nichomacus: "Well marked movement of 'times'."[*]

Leophantus: "Putting together of 'times' in due proportion, considered with regard to symmetry amongst them."[†]

Didymus: "A schematic arrangement of sounds."[‡]

D. Wright: "*Rhythm* is the way in which time is organized within measures."[§]

A. C. Lewis: "Rhythm is the language of time."[¶]

J. Martineau: "Rhythm is the component of music that punctuates time, carrying us from one beat to the next, and it subdivides into simple ratios."[**]

A. C. Hall: "Rhythm is made by durations of sound and silence and by accent."[††]

T. H. Garland and C. V. Kahn: "Rhythm is created whenever the time continuum is split up into pieces by some sound or movement."[‡‡]

J. Bamberger: "The many different ways in which *time* is organized in music."[§§]

J. Clough, J. Conley, and C. Boge: "Patterns of duration and accent of musical sounds moving through time."[¶¶]

G. Cooper and L. B. Meyer: "Rhythm may be defined as the way in which one or more unaccented beats are grouped in relation to an accented one."[***]

D. J. Levitin: "Rhythm refers to the durations of a series of notes, and to the way that they group together into units."[†††]

A. D. Patel: "The systematic patterning of sound in terms of timing, accent, and grouping."[‡‡‡]

R. Parncutt: "A musical rhythm is an acoustic sequence evoking a sensation of pulse."[§§§]

C. B. Monahan and E. C. Carterette: "Rhythm is the perception of both regular and irregular accent patterns and their interaction."[¶¶¶]

M. Clayton: "Rhythm, then, may be interpreted either as an alternation of stresses or as a succession of durations."[****]

B. C. Wade: "A rhythm is a specific succession of durations."[††††]

S. Arom: "For there to be rhythm, sequences of audible events must be characterized by contrasting features."[‡‡‡‡] Arom goes on to specify that there are three types of contrasting features that may operate in combination: *duration*, *accent*, and *tone*

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[*] *Ibid*.

[†] *Ibid*.

[‡] *Ibid*.

[§] Wright, D. (2009), p. 23. See also Hughes, J. R. (2011), for a review of this book in the context of the interdisciplinarity of mathematics and music.

[¶] Lewis, A. C. (2005), p. 1.2.

[**] Martineau, J. (2008), p. 12.

[††] Hall, A. C. (1998), p. 6.

[‡‡] Garland, T. H. and Kahn, C. V. (1995), p. 6.

[§§] Bamberger, J. (2000), p. 59.

[¶¶] Clough, J., Conley, J., and Boge, C. (1999), p. 470.

[***] Cooper, G. and Meyer, L. B. (1960), p. 6.

[†††] Levitin, D. J. (2006), p. 15.

[‡‡‡] Patel, A. D. (2008), p. 96.

[§§§] Parncutt, R. (1994), p. 453.

[¶¶¶] Monahan and Carterette, *op. cit.*, p. 4.

[****] Clayton, M. (2000), p. 38.

[††††] Wade, B. C. (2004), p. 57.

[‡‡‡‡] Arom, S. (1991), p. 202.

color (*timbre*). Contrast in each of these may be present or absent, and when accentuation or tone contrasts are present they may be regular or irregular. With these marking parameters, Arom generates a combinatorial classification of rhythms.*

C. Egerton Lowe writes: "There is, I think, no other term used in music over which more ambiguity is shown." Then he provides a discussion of a dozen definitions found in the literature.†

The reader must have surely noticed that the various definitions enumerated above emphasize different properties of the term "rhythm." Some definitions imply that a rhythm must be "good" to qualify as being a rhythm. Leophantus, for example, insists that the durations that make up a rhythm must exhibit due *proportions* and *symmetry*. While the question of what makes a "good" rhythm good is a central concern in this book, the definition adopted here is neutral on this issue. A more relevant property of the definitions listed above discriminates between rhythms that are either general or specific. Indeed, the *Harvard Dictionary of Music* makes such a distinction explicit. Its general definition of rhythm is: "The pattern of movement in time." Its specific definition of "a rhythm" is: "A patterned configuration of attacks."‡ When we listen to a piece of music such as "Hey, Bo Diddley," we hear many instruments, each playing a different rhythm. Some instruments are playing solos with rhythmic patterns that vary, while other instruments repeat rhythms throughout. The singer adds yet another layer of rhythm. What is the rhythm of such a piece? The answer to this question corresponds to the *general* definition of rhythm given by the *Harvard Dictionary of Music*, and is exceedingly difficult to ascertain. This book is not concerned with either the objective rhythmic signal, or its subjective perception, that results when a *group* of rhythms is played and heard simultaneously, but rather with the *specific* definition of a rhythm given by the *Harvard Dictionary of Music*: a "patterned configuration of attacks." Furthermore, the focus is on a particular class of distinguished rhythms: those that are repeated throughout most or all of a piece of music.

Apart from the many conceptual definitions of rhythm listed above, we all know experientially what rhythm is, because it is a natural phenomenon, an inherent aspect of nature. Even before we come into this world, while we are still in the womb, we are already bathed in the steady comforting rhythm of our mother's thumping heartbeat and her smooth breathing.§ Figure 1.1 (top) shows a greatly simplified schematic diagram of the waveform that shows up in an electrocardiogram (EKG) of a beating heart. The horizontal axis measures time, and the evenly spaced spikes indicate instants of time at which a healthy heartbeat is heard. Since we are only interested in the points in time at which

* Rivière, H. (1993). Riviere proposes an alternative classification of rhythms in terms of the parameters: *intensity, timbre*, and *duration*. See also the commentary by Arom (1994).

† Lowe, C. E. (1942), p. 202.

‡ Randel, D. M., Ed. (2003), p. 723.

§ Ayres, B. (1972), describes research that uncovers a significant correlation between preferences for regular rhythms and infant carrying practices that involve bodily contact with the mother. Wang, H-M., Lin, S.-H., Huang, Y.-C., Chen, I.-C., Chou, L.-C., Lai, Y.-L., Chen, Y.-F., Huang, S.-C., and Jan, M.-Y. (2009), showed that listening to certain rhythms can also change the interbeat time intervals of the heart of the listener.

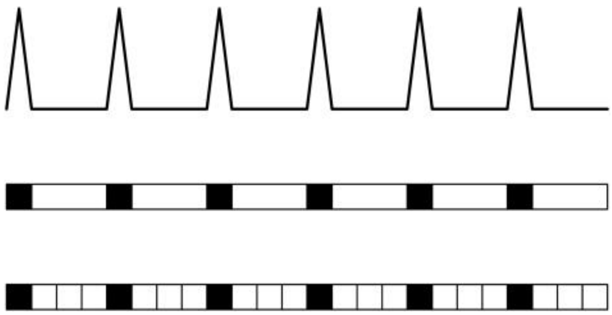

FIGURE 1.1 The heartbeat represented as a binary sequence.

the spikes occur (and not their height), the waveform may be represented as a string of elements (Figure 1.1 (middle)) in which a mark is made wherever a spike occurs. However, the white rectangles representing the spaces between the spikes are now longer than the black squares representing the spikes. It is most convenient for the analysis of rhythmic patterns to divide these long interspike intervals (silences) into smaller silent units that have the same duration as the sounded units (Figure 1.1 (bottom)). In this way, the heartbeat has been reduced to a *pulsation*, a binary string (sequence) of evenly spaced pulses, some of which are sounded (attacks) while others remain silent (rests). The term "pulse" is used in this book to denote the location at which a sound or attack may be realized. This representation of rhythm is also called box notation.* A convenient way to write box notation in running text is to use the symbol [x] for a black square (attack) and the period symbol [**.**] for a white square (rest). Thus, the rhythm at the bottom of Figure 1.1 may be written in box notation as [x . . . x . . . x . . . x . . . x . . . x . . .].

It is often said that rhythm is in the mind and not in the acoustic signal. Grosvenor Cooper and Leonard B. Meyer write: "Rhythmic grouping is a mental fact, not a physical one."† This statement should of course not be taken literally. Besides the fact that at present, we still do not know that the mind is not physical, and that there exist machines such as functional magnetic resonance imagery (fMRI) and magnetoencephalography (MEG) that record physical manifestations of rhythm in the brain, there is little doubt that whether or not grouping is physical, what we perceive as rhythm emanates from an acoustic physical signal. Although William Sethares is right to point out that "many of the most important rhythmic structures are present only in the mind's ear,"‡ the converse is also true: many of the most important rhythmic structures are present only in the physical signal or the symbolic score. Indeed, music psychologists have found compelling evidence that everyone,

* Kaufman Shelemay, K. (2000), p. 35. Koetting, J. (1970), p. 117, is greatly responsible for popularizing box notation among ethnomusicologists, which he called Time Unit Box System (TUBS). Although, Koetting credits Philip Harland, the assistant head of the UCLA drum ensemble at the time, as the originator of TUBS, this notation has been in use in Korea for hundreds of years; see the paper by Lee, H.-K. (1981). The TUBS system notates only the time or duration information of rhythms. This is not a problem for the timelines considered here, where the attacks are almost always isotonic. However, in African drumming, the timbre of the drums is also important. Therefore, the TUBS system has been extended by Serwadda, M. and Pantaleoni, H. (1968) and Ngumu, P.-C. and A. T. (1980) to take into account information other than interattack durations, that may be contained in the drum attacks.

† Cooper and Meyer, *op. cit.*, p. 9.

‡ Sethares, W. A. (2007), p. 75.

whether trained musicians or not, can discriminate among the styles of Western classical music based solely on the variability of the durations of the notes, as measured by their standard deviation and pairwise variability index (nPVI).[*] Therefore, it is more accurate to characterize rhythm as a manifestation of a *process* that emerges from the amalgamation of a physical signal with perceptual and cognitive structures of the mind. Such a broad definition naturally leaves open the door to consideration for analysis any of the multitude of complex features that make up rhythmic patterns. As such, rhythm may be studied at any level in between these two extremes, ranging from the purely mathematical and scientific[†] approaches to the experiential "mythopoetic explanations,"[‡] as well as its spiritual roots.[§] Knowledge gained from such studies helps us to understand the totality of rhythm.

In this book, musical rhythm is studied predominantly at one extreme of the above panorama: rhythm is considered purely in durational terms as a symbolic binary sequence of isochronous elements representing sounds and silences. Simha Arom writes: "In the absence of accentuation or differences in tone color, contrasting durations are the only criterion for the determination of rhythm."[¶] This is the simplest definition of rhythm possible. Since rhythm is considered to be such a difficult topic, it behooves us to understand it at this level well, especially when exploring and evaluating new tools, before moving on to higher ground. We should first understand precisely what we lose by confining ourselves to such a skeletal definition of rhythm, as well as how much we can gain from it. In this book, I attempt to demonstrate what we can gain by combining geometric methods with such a simple and unambiguous objective definition of rhythm, "rhythm as technical concept, rhythm as a precise, quantifiable process."[**] Some researchers have argued that rhythm must be studied in a cultural and social context.[††] This is a perfectly valid endeavor, particularly so if the main interest is sociology or anthropology, just as it is of interest to study Einstein's theory of general relativity or his equation $E = mc^2$, in a sociocultural context. However, the physical laws of the universe are independent of culture, and arguably so are the physical laws of rhythmic patterns. John McLaughlin put it this way: "The mathematics of rhythm are universal. They don't belong to any particular culture."[‡‡] Here, we take the position of Kofi Agawu with respect to the application of analytic methods to ethnomusicology, which may be extended to musicology in general: "Given the relative paucity of analyses, erecting barriers against one or another approach seems premature."[§§] Furthermore, although thinking about musical rhythm in a mathematical way, using

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[*] Dalla Bella, S. and Peretz, I. (2005), p. B66.
[†] Cross, I. (1998).
[‡] Cook, N. (1990).
[§] Redmond, L. (1997).
[¶] Arom, *op. cit.*
[**] Agawu, K. (1995), p. 3.
[††] Avorgbedor, D. (1987), p. 4.
[‡‡] Prasad, A. (1999). This quotation is part of the answer of John McLaughlin to Anil Prasad's interview question: "How did you go about balancing the mathematic equations of Indian rhythmic development with the less-studied, more chaos-laden leanings of jazz?"
[§§] Agawu, *op. cit.*, p. 196.

mathematical terminology,[*] may be quite inharmonious with certain cultural traditions,[†] it facilitates another goal of this book, the exploration of geometric rhythmic universals.[‡]

It is possible that in the past, rhythm has been difficult to dissect, precisely because its myriad definitions have been too vague, or too general, or because it has not received enough attention from a purely objective mathematical point of view. To quote Curt Sach's, "Rhythm weakens the more we widen its concept and scope."[§] It is hoped that the analysis presented in this book, of "rhythms" as purely mathematical, culturally independent, binary symbolic sequences, will stimulate future progress in the systematic and comparative study of rhythm as a whole in the context of a *world music theory*.[¶]

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[*] Rahn, J. (1983), p. 33, discusses the problems inherent in three approaches that deal with terminology when analyzing world music: the use of Western terms, the use of non-Western terms, and the avoidance of both, necessitating the introduction of new terminology. Needless to say, all three approaches have their drawbacks. Nevertheless, one may argue that the third approach makes more sense, and that the mathematical language is the most objective. Once such a terminology is agreed upon, both Western and non-Western terms may be translated to the mathematical terms on equal footing.

[†] Agawu, K. (1987), p. 403.

[‡] Honingh, A.K. and Bod, R. (2011, 2005).

[§] Sachs, C. (1953), p. 17.

[¶] Tenzer, M. (2006), p. 33, Hijleh, M. (2008), p. 88.

CHAPTER 2

A Steady Beat

IMAGINE A STEADY HEARTBEAT going boom, boom, boom, boom, … or a grandfather clock ticking *tick*, *tick*, *tick*, *tick*, … without end. It is natural for most people to consider these sequences to be examples of the simplest kinds of rhythms. They certainly satisfy several of the definitions of rhythm given in Chapter 1, such as those of Baccheios the Elder, Nichomacus, Didymus, Parncutt, Wade, and Wright. However, some music theorists and musicologists[*] would say that this isochronous sequence is not a rhythm at all because it contains no discernible audible *pattern*, by which they mean that in such a sequence there are no contrasting features. As H. Riviere asserts, "A succession of sounds of equal duration, with invariable intensity and identical timbre, do not constitute a rhythmic event."[†] A more appropriate term for such a sequence is arguably a *pulsation*. Nevertheless, it has been suggested that in the context of human behavior, musical pulsation (periodic production) is a uniquely human trait that appears to have evolved specifically for music.[‡] Furthermore, other musicologists express a contrary view; the folklorist Alan Lomax refers to pulsations as "one beat" rhythms.[§] In this book, pulsations are considered to be *bona fide* members of the family of rhythms.

The human brain would quickly tire of a monotonous sequence of isochronous sounds. Indeed, music psychologists discovered more than a hundred years ago that if a human subject in a laboratory setting is presented with a sequence of identically sounding evenly spaced ticks such as *tick, tick, tick, tick, tick, tick,* …, the mind, being so thirsty for patterns, perceives the sequence as *tick,* ***tock****, tick,* ***tock****, tick,* ***tock***, … instead.[¶] In other words, the mind converts the repetition of single-sound *ticks* into the repetition of two-tone *tick-****tock*** patterns.[**] The same phenomenon, called the *perceptual center*, has been observed with

[*] I use the term *musicologist* not in the narrow sense, restricted to Westerm music history, but rather in the broadest sense possible to denote a scholar of music in any musically relevant discipline in either the sciences or humanities.
[†] Rivière, H. (1993), p. 243. See also Sachs, C. (1953), p. 16, for further references on this view as well as the opposition.
[‡] Bispham, J. (2006), p. 131. See Swindle, P. F. (1913), for early arguments about whether the human skill of producing pulsations is inherited or acquired.
[§] Lomax, A. and Grauer, V. (1964).
[¶] Bolton, T. (1894).
[**] Parncutt, R. (1994), p. 418, refers to this perceptual grouping of isochronous sound events as *subjective rhythmization*.

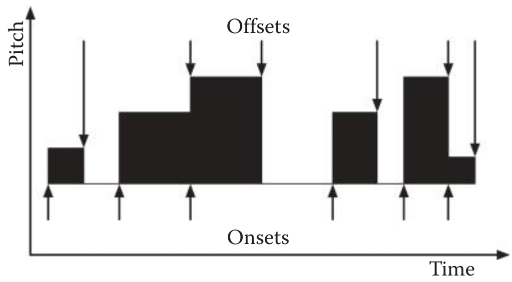

FIGURE 2.1 *Idealized onsets* and *offsets* of notes.

speech rhythm, and is hypothesized to be a *rhythm universal*: "a sequence of spoken digits with evenly spaced acoustic onsets was judged to be uneven by listeners."* These psychological phenomena underscore, in the simplest possible manner, the fact that the rhythms perceived by the human mind are not veridical representations of the written score or its realization by the human voice or a musical instrument. Rhythm perception emerges from the interplay between the bottom-up, data-driven, outer stimuli emanating from the world, and the top-down, conceptually driven, inner response mechanisms of the mind's ear. In spite of this, it is useful to focus only on the written score, which is more objective than human perception.† In the geometric analysis of musical rhythm developed in this book, the psychological aspects of music perception take second place, as do the acoustic aspects of music production,‡ focusing instead on purely durational symbolically notated rhythm. Therefore, in this setting, to create a more interesting rhythm out of the pulsation *tick, tick, tick, tick, tick, tick, …*, we should use two different tones (timbres) to create *tick, **tock**, tick, **tock**, tick, **tock**, …*. This could also be accomplished by accenting every other tick in some other way, such as making it louder. However, the sequence would be an even more interesting rhythm if the durations between two adjacent beats were not all the same. Such durations are called *inter-onset intervals*. This terminology comes from a more general setting in which melodies are represented by sustained notes that start and end at fixed positions in time, as pictured in Figure 2.1. The starting and ending times of the notes are the *onsets* and *offsets*, respectively. In the case of pure rhythms consisting of beats (attacks), it is assumed that there are no sustained notes, and thus we dispense with the offsets altogether. In this setting, the inter-onset-duration intervals are simply the durations between two consecutive attacks. In the physical world where the notes are acoustic signals, the notes would of course not look like the isothetic§ rectangles depicted in Figure 2.1, with vertical

* Hoequist, C. J. (1983), p. 368.

† In spite of the fact that certain aspects of rhythm are psychological in nature, Dahlig-Turek, E. (2009), obtained very useful results on the evolution of the morphology of Polish rhythms using mathematical features of the pure durational patterns, thereby ignoring all other information of the rhythms. On p. 131, he writes: "Thanks to the applied method, it was possible to back up the discussion on 'Polish rhythms' using solid ('objective') arguments rather than emotional ('subjective') statements, typical of many previous studies."

‡ Dannenberg, R. B. and Hu, N. (2002). Recognition of musical structure from audio recordings is a central problem in music technology that uses tools that are often quite different from the tools used here. Indeed, one of the main goals in analyzing audio signals is the transcription of music to the types of symbolic representations covered here.

§ A geometric object such as a rectangle or polygonal chain is isothetic, provided all its sides are either vertical or horizontal.

lines denoting the onsets. Instead, the acoustic signal would be a much more complicated waveform.[*] Furthermore, with acoustic input in the real world, the exact placement of the attacks exhibits deviations caused by factors such as interpretation,[†] time-warping due to expressive timing on the part of the performer,[‡] the physical distance between the drummer and some of the drums,[§] or by purposeful design, in order to test theories of perception.[¶] Here, however, we are dealing with idealized symbolic notated rhythms, and hence the model adopted is justified.

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[*] For a tutorial on techniques for detecting onsets in acoustic signals, see Bello, J. P., Daudet, L., Abdallah, S., Duxbury, C., Davies, M., and Sandler, M. B. (2005), and the references therein. Acoustic onset detection differs from perceptual onset detection. Indeed, Wright, M. J. (2008), argues that musicians do not learn to make their physical attacks have a certain rhythm, but rather to make their perceptual attacks exhibit that rhythm. He proposes maximum likelihood estimation methods to estimate perceptual attack times.

[†] During, J. (1997), p. 26. See also Repp, B. H. and Marcus, R. J. (2010), for a discussion of perceptual illusions concerning onsets and offsets, such as the *sustained sound illusion* (SSI): a continuous sound that seems longer than the silence of equal duration.

[‡] Benadon, F. (2009b), p. 2.

[§] Alén, O. (1995), p. 69.

[¶] Gjerdingen, R. O. (1993), p. 503, explores “smooth” rhythms as probes of rhythmic entrainment. “Smooth” rhythms have no well-defined time points that determine the onsets and offsets.

Timelines, Ostinatos, and Meter

In much traditional, classical, and contemporary music around the world, one hears a distinctive and characteristic rhythm that appears to be an essential feature of the music, that stands out above the other rhythms, and that repeats throughout most if not the entire piece. Sometimes this essential feature will be merely an isochronous pulsation without any recognizable periodicity. At other times, the music will be characterized by unique periodic patterns. These special rhythms are generally called *timelines*.[*] Timelines should be distinguished from the more general term *rhythmic ostinatos.* A rhythmic ostinato (from the word *obstinate*) refers to a rhythm or phrase that is continually repeated during a musical piece. Timelines, on the other hand, are more particular ostinatos that are easily recognized and remembered, play a distinguished role in the music, and also serve the functions of conductor and regulator, by signaling to other musicians the fundamental cyclic structure of the piece. Thus, timelines act as an orienting device that facilitates musicians to stay together and helps soloists navigate the rhythmic landscape offered by the other instruments. Indeed, Royal Hartigan considers timelines akin to the "heartbeat" of the music.[†] Wendell Logan refers to the timeline as a *life-line*.[‡] Since timelines repeat over and over in a cyclic manner, they are periodic, and thus it is natural to represent them on a circle.[§] Figure 3.1 illustrates two examples of well-known timelines: one is very short consisting of three onsets in a cycle of eight pulses, and the other is relatively long, made

[*] Agawu, K. (2006), p. 1, also refers to a *timeline* as a "bell pattern, bell rhythm, guideline, time keeper, *topos*, and phrasing referent," and characterizes it as a "rhythmic figure of modest duration that is played as an ostinato throughout a given dance composition." According to Kofi Agawu, the term *timeline* was introduced in 1963 by Kwabena Nketia. *ibid*, p. 3. Nketia (1962), p. 78, characterizes a Time Line as a "short but persistent" rhythm that acts as "constant point of reference." He considers the simplest types of timelines to be isochronous and isotonic sequences performed with a gong or handclapping. Flatischler, G. (1992), p. 119, uses the term "guideline" for the same concept. Agawu's 2006 paper discusses timelines at length and provides many useful references on the topic.

[†] Hartigan et al. (1995), p. 63.

[‡] Logan, W. (1984), p. 193.

[§] Anku, W. (2000), Benadon, F. (2007). London, J. (2000) also finds it useful to represent meters on a circle.

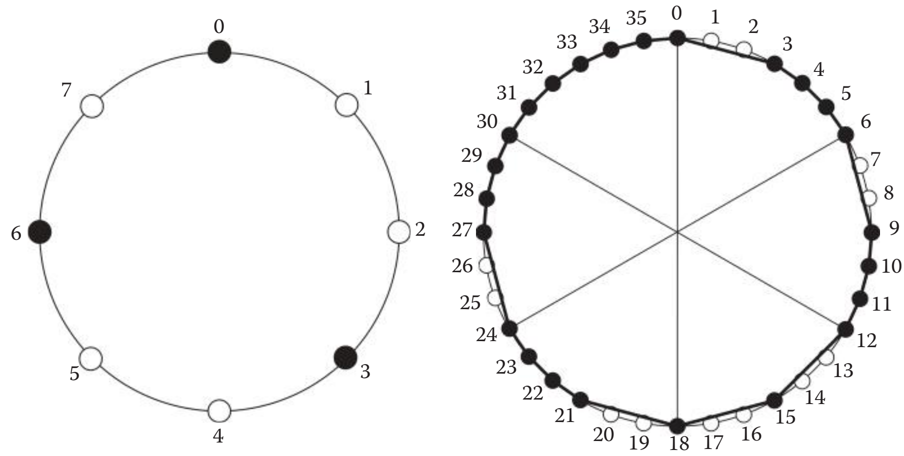

FIGURE 3.1 The Cuban *tresillo* timeline (left) and the ostinato in Ravel's *Bolero* (right).

up of 24 onsets in a cycle of 36 pulses. The onsets (or sounded pulses) are shown as black-filled circles and the silent pulses as white-filled circles. It is assumed throughout the text that a rhythm starts at the pulse labeled zero, and that time flows in a clockwise direction.

The timeline on the left in Figure 3.1 is made up of three adjacent inter-onset intervals (or durations). The first duration occurs from the first onset at pulse zero to the second onset at pulse three, and thus has duration equal to three units. Similarly, the second and third inter-onset intervals have durations of three units (from pulse three to six) and two units (from pulse six to zero), respectively. Another useful notation for rhythms, which complements the circle notation just described, consists of expressing the sequence of adjacent inter-onset intervals present in the rhythm as a list. This rhythm may thus be notated as [3-3-2]. Breslauer refers to this representation of rhythms as "durational patterns."* In this book, the terms "durational pattern" and "inter-onset interval structure" will be used interchangeably.

The durational pattern [3-3-2], popular in Central Africa, is most famously known as the Cuban *tresillo* (*tres* in Spanish means *three*), and is widely used in the circum-Caribbean.† However, it forms part of almost every music tradition throughout the world,‡ and dates back historically to at least thirteenth-century Baghdad, where it was called *al-thakil*

* Breslauer, P. (1988), p. 2. In Breslauer's notation, this rhythm would actually be written as [3 3 2] rather than [3-3-2]. The latter modification is used here to avoid confusion because some timelines may have inter-onset intervals of duration greater than 10 units, and the *dash* symbol provides greater iconic value than the *space*. Hook, J. L. (1998), develops an algebra of durational patterns, and applies it to the analysis of the music of Messiaen.

† Johnson, H. S. F. and Chernoff, J. M. (1991), p. 67, Gerard, G. (1998), p. 69, Floyd, Jr., S. A. (1999), p. 9, Uribe, E. (1993), p. 126. It is the foundation pattern played on the bass drum for the Baiaó music of Brazil. It is also used in the Jamaican *mento* song "Sly Mongoose." See Logan, W. (1984), p. 194. Manuel, P. (1985), p. 250, analyzes its influence in salsa music as the typical Afro-Cuban bass line. Sandroni, C. (2000), p. 61, analyzes the influence that the tresillo rhythm had in Latin America.

‡ Apel, W. (1960), Toussaint, G. T. (2005c), Leake, J. (2009). Acquista, A. (2009) documents the influence that the tresillo has had on a variety of different rhythmic styles.

al-thani.[*] It is a traditional rhythm played on the banjo in bluegrass music.[†] It was used extensively in the American rockabilly music of the 1950s for bass, saxophone, or piano. It is sometimes referred to as the *habanera* rhythm or *tumbao* rhythm, although the term habanera usually refers to the four-onset rhythm [3-1-2-2], which is less syncopated than the tresillo since it inserts an additional attack in the middle of the cycle.[‡] The habanera rhythm is also known as the *tumba francesa*.[§] If to the habanera a fifth onset is inserted at pulse five, then the resultant rhythm [3-1-1-1-2] is the *bomba* from Puerto Rico.[¶] On the other hand, if the third onset of the tresillo pattern is deleted, one obtains the prototypical Charleston rhythm: [x . . x . . . .].[**] A couple of typical examples of rockabilly songs that use the tresillo pattern are *Shake, Rattle, and Roll* by Bill Haley, and *Hound Dog* by Elvis Presley. More recently in 2005, the Greek singer Helena Paparizou won the *Eurovision* contest with a song titled *My Number One* written by Christos Dantis and Natalia Germanou,[††] which used the tresillo timeline on the snare drum. We will return to this important rhythm frequently with further details.

The timeline on the right in Figure 3.1, with inter-onset interval structure [3-1-1-1-3-1-1-1-3-3-3-1-1-1-3-1-1-1-1-1-1-1-1-1], is the ostinato percussion pattern from Ravel's *Bolero*, usually played on one or more snare drums.[‡‡] Also shown in Figure 3.1 are the three lines connecting the six main beats in the cycle at pulses 0, 6, 12, 18, 24, and 30.

In the circular representation of rhythms illustrated in Figure 3.1, the length of the arc between two consecutive pulses represents the duration of one unit of time. Thus, the duration between the first and second onsets is three units. The sum of the three durations, 3 + 3 + 2, equals eight. For our purposes, it does not matter much how long in absolute terms the unit actually is in real time, although the entire circle of most timelines usually lasts between 1 and 3 s. The circle could have been divided into 16 pulses instead of eight, yielding the inter-onset interval structure [6-6-4]. However, since we will be interested mainly in the *relative* durations of inter-onset intervals, and less in the *tempo* (speed) at which a rhythm is played, we will in general use the smallest number of pulses that will

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[*] Wright, O. (1978). p. 216. The cycle of *al-thakil al-thani* actually has sixteen pulses and thus the complete rhythm is [3-3-2-3-3-2].

[†] Keith, M. (1991), p. 126.

[‡] Brewer, R. (1999), p. 303, refers to the tresillo [3-3-2] as the habanera rhythm. However, according to Orovio, H. (1992), p. 237, and Rey, M. (2006), p. 192, [3-1-2-2] is the habanera, and the tresillo is a habanera-derived rhythm. Note that the habanera rhythm is the same as the drumstick time-keeping rhythm played with a stick on the side of a drum in the *sabar* drumming of Senegal. See Tang, P. (2007), p. 98, Sandroni, C. (2000), p. 61, hypothesizes that the *habanera* [x . . x x . x .] was born independently in different parts of Latin America from a marriage of the African *tresillo* [x . . x . . x .] with the Spanish-Portuguese pattern [x . . . x . x .], since it is the *resultant* (union) of both rhythms. On the other hand, Manuel, P. (1985), p. 250, writes: "Its European and predominantly bourgeois origin is obvious." It is interesting to note that if the *habanera* is rotated by a half-cycle to obtain [x . x . x . . x], it becomes the *whaî* rhythm of the *kanak* people of New Caledonia in the South Pacific. See Ammann, R. (1997), p. 242.

[§] Fernandez, R. A. (2006), p. 9.

[¶] *Ibid*., p. 7.

[**] Kleppinger, S. V. (2003), p. 82.

[††] Released in Greece by Sony BMG Music Entertainment on March 24, 2005.

[‡‡] Tanguiane, A. S. (1994), p. 478, Tanguiane, A. S. (1993), p. 149, Blades, J. (1992), p. 374. Tanguiane tested a machine perception model with a set of experiments aimed at recognizing the rhythm timeline of Ravel's *Bolero*. Asada, M. and Ohgushi, K. (1991), on the other hand, tested and analyzed the human perceptual impressions of the eighteen pieces in Ravel's *Bolero*.

accommodate the required integer inter-onset intervals. Such pulses are also called *elementary pulses*. The number of elementary pulses used in the cycle is sometimes called the *form number* or *cycle number* of the rhythm.[*] It should be noted that some books define the elementary pulses as the *smallest* time units (inter-onset interval) present in a rhythm. For example, the smallest time unit in the tresillo timeline [3-3-2] is two. However, here the elementary pulse for this rhythm is one, and is defined as the *largest* time unit that evenly divides into all the inter-onset intervals. Thus, the largest number that evenly divides both two and three is one.

In ethnomusicology, the use of the word *timeline* is generally limited to asymmetric durational patterns of sub-Saharan origin such as the tresillo in Figure 3.1 (left). In this book, however, the term is expanded to cover similar notions used in other cultures such as *compás* in the flamenco music of Southern Spain,[†] *tala* in India,[‡] *loop* in electronic dance music (EDM),[§] and just plain rhythmic ostinatos in any type of music. A word is in order concerning the ubiquitous related concept referred to as *meter* in Western music. There is slightly less vagueness present in the many definitions of meter published, as there is of the definitions of rhythm listed above.[¶] There is also much discussion about the differences between meter and rhythm.[**] Meter is usually defined in terms of a hierarchy of accent patterns, and considered to be more regular than rhythm.[††] Some music, such as sub-Saharan African music is claimed to have only pulsation as a temporal reference, and no meter in the strict sense of the word.[‡‡] Christopher Hasty's book titled *Meter as Rhythm*[§§] considers meter to be a special case of rhythm. In this book, the word "timeline" is expanded to include all those meters used in music around the world, that function as time-keepers, or ostinatos, and determine the predominant underlying rhythmic structure of a piece. Here, meter is viewed as just another rhythm that may be sounded or merely felt by the performer or listener, and it is also represented as a binary sequence. While consideration of a metric context is indispensable for a complete understanding of rhythm, the underlying assumption in this book is that it can also be profitable to focus on purely inter-onset durational issues.

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[*] Kubik, G. (2010a), p. 42.
[†] Parra, J. M. (1999).
[‡] Morris, R. (1998).
[§] Butler, M. J. (2006), p. 90.
[¶] See for example Arom, S. (1991), p. 184, and Temperley, D. (2002), p. 77, for contrasting views.
[**] London, J. (2000, 2004), Arom, S. (1989), Kenyon, M. (1947), Ku, L. H. (1981), Palmer, C. and Krumhansl, C. L. (1990), Lehmann, B. (2002).
[††] Chatman, S. (1965).
[‡‡] Arom, *op. cit.*, p. 206.
[§§] Hasty, C. F. (1997).

CHAPTER 4

The Wooden Claves

IMAGINE A SCORE OF DRUMMERS playing loud dance music at a festival in a village somewhere in sub-Saharan Africa. If the musician playing the timeline is to fulfill the role of a conductor and time-keeper, then all the drummers, including those playing far from each other and separated by other drummers, and especially the soloists who will improvise during their flights of fancy, must be able to hear the timeline, so that when they depart on their rhythmic improvisational adventures, they can find their way back to home base. For this reason, the instruments that play the timeline are designed so as to produce a sound that cuts through the intense booming of all those drums. Traditionally, these instruments consist of either two sticks, 20–30 cm in length, made of a very hard wood such as ebony, that are clicked together, or a metallic object usually made of iron, such as a bell that is struck with another piece of metal or stick of wood. Sometimes, a pair of axe-blades are chinked together.[*] In Afro-Cuban music, the wooden sticks are called *claves*. Clave is the Spanish word for *key* or *code*. The charcoal drawing in Figure 4.1 illustrates a typical pair of wooden claves.

In Cuba, the quintessential timeline rhythm played with the claves is also called *clave*, suggesting that the *key* to the piece of music lies in the timeline rhythm itself. This term has been extended to refer to other similar rhythms from Brazil and elsewhere.[†] The name *claves* was also accorded to groups of singers who in the nineteenth-century Havana would play in the streets during carnival festivities. Eventually, the songs the musicians played during these occasions were also called *claves*.[‡] Today, the word *clave* has taken even broader significance. Chris Washburn, for example, considers the term to refer to the rules that govern the rhythms played with the claves.[§] Bertram Lehman regards the clave as a concept with wide-ranging theoretical syntactic implications for African music in general,[¶] and for David Peñalosa, the *clave matrix* is a comprehensive system for organizing music.[**]

[*] Jones, A. M. (1954b), p. 58.

[†] Toussaint, G. T. (2002).

[‡] Ortiz, F. (1995), p. 5. Mauleón, R. (1997) traces the origins and development of the clave in world music.

[§] Washburne, C. (1995, 1998).

[¶] Lehmann, B. (2002).

[**] Peñalosa, D. (2009).

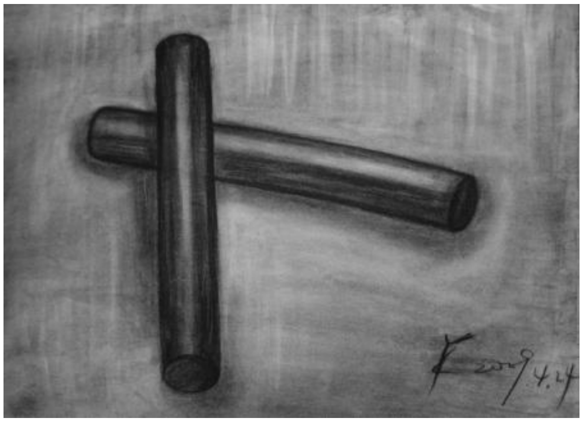

FIGURE 4.1 A pair of wooden *claves*. (Courtesy of Yang Liu.)

As simple as this instrument looks, there is a certain amount of technique required to bring out its magical sound, that will frustrate the novice. It will not do to just hold a stick in each hand and strike the ends together. Such an approach will produce a short dull dry sound. First, one clave must be laid on one tightly cupped hand balanced between the wrist and the tips of the fingers. In this configuration, this clave, called the "female" in Cuba, is not only free to resonate, but the cupped hand provides a chamber much similar to a miniature kettle drum that acts as an amplifier. This resting clave is struck near its center by the first clave, called the "male."[*] If the result is a sound that appears to be produced by a material that resonates somewhat like crystal and a little like metal, then success has been achieved.

There exist similar instruments in other cultures around the world. For example, the Australian aboriginal people strike together a similar pair of wooden sticks called *clap sticks*. Clap sticks are larger and heavier than claves, tapered to a rounded point at their ends, and often decorated with aboriginal dot-art. Instruments such as the claves and clap sticks belong to the family of instruments called *clappers* that have a long history dating back more than 5500 years ago to China, where they were originally made of bone.[†]

[*] Orovio, H. (1992), p. 110.
[†] Logan, O. (1879), p. 690.

CHAPTER 5

The Iron Bells

In addition to hard wood, a variety of bells and gongs made of metal are used for playing rhythm timelines in traditional African and Afro-Cuban music.[*] Perhaps, the most noteworthy metal bell is the *gankogui*, a hand-held iron instrument composed of two bells of different pitches, attached together as shown in Figure 5.1. The instrument is held with one hand without muting the sound, and the bells are struck with a wooden stick. The first onset of the timeline is often played on the low-pitched bell and the remaining onsets on the high-pitched bell. These bells come in a variety of sizes that produce a wide range of tones and textures.

In the drum ensemble music of the Ewe people of Ghana, there is a popular dance music called *gahu*, which makes use of the following 16-pulse timeline played with the first onset on the large bell and the other four on the small bell [x . . x . . x . . . x . . . x .].[†]

The *dawuro* (also called *atoke*,[‡] banana, or boat bell) is shaped somewhat like a canoe or taco shell, as pictured in Figure 5.2. To play it, the bell is balanced delicately in the palm of one hand, and the edges of the bell are struck with a metal rod. The sound is a piercing reverberation that resembles a whistle, and cuts through a score of drums. Furthermore, by muting the sides of the bell with the thumb after striking it, a variety of interesting sustained sound effects may be produced. Two traditional timelines played with this bell, and used in the *adowa* drum music of the Ashanti people of Ghana are [x . x . x . . x . x x .] and [x . . . x . . x . x . .].[§]

The *frikyiwa* illustrated in Figure 5.3 is a type of metal castanet used to play the timeline in the *sikyi* rhythms of the Ashanti people of Ghana in West Africa.[¶] The walnut-shaped object is held with the middle finger inserted under the bridge that

[*] Bells are sometimes also made of nonmetallic materials. Söderberg, B. (1953), p. 49, documents bells made from fruit shells in Central Africa, and Fagg, B. (1956), p. 6, describes rock gongs discovered in the rocky hills of Nigeria that were used as percussion instruments.

[†] Locke, D. (1998), p. 17.

[‡] *Ibid.*, p. 13.

[§] Hartigan, R., Adzenyah, A., and F. Donkor, F. (1995), p. 35.

[¶] *Ibid.*, p. 16.

FIGURE 5.1 The iron *gankogui* double-bell. (Courtesy of Yang Liu.)

connects the two halves of the object. The ring-shaped part is worn on the thumb, which is used to strike the object. Although made of metal, the sound produced by the frikyiwa is similar to that made by the hard wooden claves. A typical timeline played on the frikyiwa in the *sikyi* rhythms of the Ashanti people of Ghana is given by [. . . x . x . x].

FIGURE 5.2 The iron *dawuro* bell. (Courtesy of Yang Liu.)

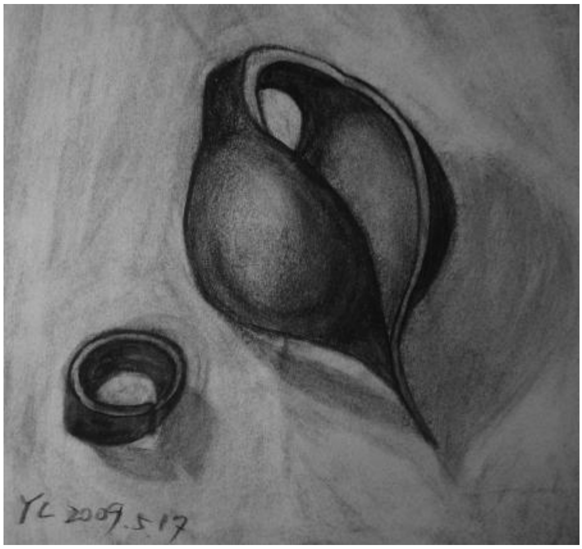

FIGURE 5.3 The *frikyiwa*, a metallic castanet-like bell. (Courtesy of Yang Liu.)

Although the metal bell is used in African music predominantly to play timelines to mark the time for drumming ensembles, it is sometimes used as a "master instrument"* in its own right, without any other musical instruments. An example in Ghana is the *gamamla*, played with five gankogui bells of varying sizes. A different rhythmic ostinato is played on each bell, which uses both the high and low tones, resulting in a unique and rich overall composition.†

* Nzewi, O. (2000), p. 25.
† Klőwer, T. (1997), p. 175.

CHAPTER 6

The Clave Son

As we have seen earlier, the Cuban tresillo has duration pattern [3-3-2] consisting of three attacks in a cycle of eight pulses. At an abstract level, all rhythms can be classified into families described by these two numbers: the *onset-number* and the *pulse-number*.* Let the integers k and n, denote, respectively, the onset-number and pulse-number of any rhythm. Among the timelines used in traditional and contemporary music all over the world, the values of k and n vary greatly across cultures. In Western music, n is usually less than or equal to 24. The fourth century BC Greek statesman Aristides wrote that it is not possible to perceive rhythm when n is greater than 18. However, the Aka Pygmies of central Africa use timelines with $n = 24$. Furthermore, some of the largest values of n are found in Indian classical music, where the timelines are called *talas* and the value of n can be as high as 128.† In Western and sub-Saharan African music, the value of n is usually an even number. In Eastern Europe, North Africa, and the Middle East, n is often an odd number, and sometimes a *prime* number such as 5, 7, or 11 (a number that can be divided without remainder only by *itself* and 1). In the Black Atlantic region, on the other hand, the pulse number is almost never a prime number.‡

For reasons that will be explored in this book, the value of k is usually slightly higher or slightly lower than *half* the value of n. In sub-Saharan Africa, a value of $n = 12$ is preferred. The most popular value of n in the world appears to be 16. Furthermore, for rhythms with 16 pulses, a value of k equal to five seems to be preferred. Specifying the values of k and n allows the calculation of the number of theoretically possible rhythms in the family. For example, the number of rhythms with five onsets and 16 pulses corresponds to the number of ways of selecting five items from among 16. This is equal to the number of different two-symbol sequences that contain k ones and $(n - k)$ zeros. There is a well-known combinatorial formula for this type of calculation that yields the solution.§ The formula is given by (16!)/((5!)(11!)), where the symbol "!" is pronounced *factorial*, and $k!$ denotes the product of

* Cohn, R. (1992a), p. 195, refers to the pulse number of a rhythm as the span.
† Šimundža, M. (1987, 1988) and Clayton, M. (2000).
‡ Pressing, J. (2002), p. 289.
§ Keith, M. (1991), p. 17.

the k terms $(k)(k-1)(k-2)(k-3)\ldots(1)$. Evaluating this formula yields the number 4368. Note that the number 11 in this formula is the difference between the number of onsets and the number of pulses. Figure 6.1 shows, in box notation, a dozen arbitrary members of this large family of rhythms.

Of course, not every rhythm in this family is considered to be a "good" rhythm, in the sense that it has been adopted as a timeline pattern in traditional music somewhere on the planet. Indeed, the ancient Greek music scholar Aristoxenus of Tarentum wrote in his *Elements of Rhythm* in the sixth century BC that not every division of a time span yields a rhythm that is rhythmical, by which he meant "good." Aristoxenus reserved the term *eurhythmical* for those rhythms that were *beautifully* rhythmical. However, he did not provide any algorithms for generating eurhythmic rhythms, and was content to emphasize that for examples of eurhythmic rhythms one should turn to the compositions of the great masters.* In contrast, one of the main goals of this book is precisely the design of algorithms for generating good rhythms using simple mathematical principles.

Figure 6.2 displays an illustrious member of the family of rhythms with five onsets and 16 pulses.† In Cuba, it goes by the name *clave son*.‡ It has inter-onset-interval sequence [3-3-4-2-4]. In addition to marking the onsets at positions 0, 3, 6, 10, and 12 with black-filled

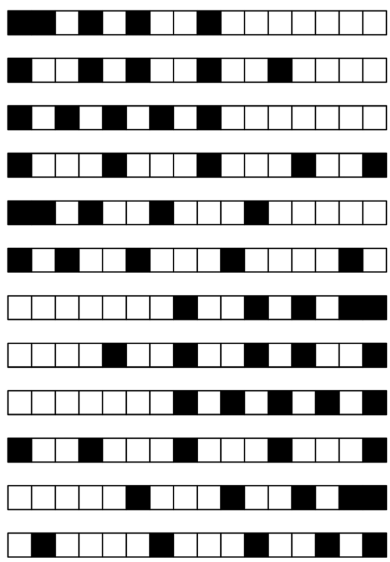

FIGURE 6.1 A dozen examples of the 4368 rhythms with five onsets and 16 pulses.

* Abdy Williams, C. F. (2009), pp. 34–35.

† Vurkaç, M. (2011), p. 30, Toussaint, G. T. (2011), Waterman, R. A. (1948), p. 36, Kauffman, R. (1980), Table 1, p. 409, Gerard, G. (1998), p. 33.

‡ Chernoff, J. M. (1979), p. 145, Johnson, H. S. F. and Chernoff, J. M. (1991), p. 67, Fernandez, R. A. (2006), p. 15. Robbins, J. (1990), p. 189, explores the social and historical contexts in which the son evolved, and offers some possible reasons for its great success. In the United States, the clave son is known as the Bo Diddley beat, in Ghana, it is called the *kpanlogó*, and in thirteenth-century Baghdad, it was called *al-thaqūl al-awwal*, but my daughter Stephanie calls it the *knock-of-death*, because I use it on her bedroom door every morning to wake her up at 6:00 a.m.

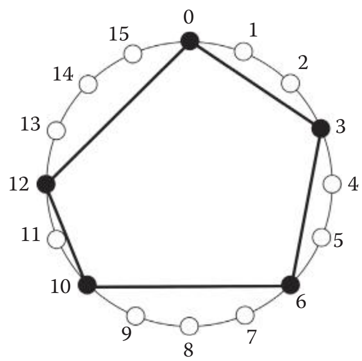

FIGURE 6.2 The *clave son* in polygon notation. (From Toussaint, G. T. 2011. *Percussive Notes*, November Issue, pp. 52–59. With permission.)

circles, Figure 6.2 shows how a rhythm may be represented as a convex polygon by connecting each onset to its adjacent onset in time with a straight line segment, in this case yielding a pentagon (five-sided polygon). Such a representation is useful for a multitude of purposes, as will become clear in subsequent chapters.

A natural question that comes to mind, besides the reason for the choices of $n = 16$ and $k = 5$ to begin with, is how, out of the 4368 myriad possible rhythms with five onsets from among 16 pulses, this particular configuration of inter-onset intervals [3-3-4-2-4] managed to become such a catchy and widely used rhythm.[*] What is so seductive about this rhythm that has been described as "the elegantly insinuating syncopated rhythm that defines Cuban *son* and *salsa*"[†] that it should win the hearts and minds of people all over the world? Attempting to answer this question using a wide variety of tools ranging from geometry to musicology and psychology is one of the main themes of this book.

Studying the evolution of rhythms is not an easy task. Until very recently, many musical traditions in the world were oral, and so the rhythms used by musicians were handed down from teacher to apprentice without leaving written records. This is particularly so in the sub-Saharan African tradition, which did not give importance to who was responsible for creating rhythms, or when they were created.[‡] In addition, sailors, soldiers, and tourists throughout history traveled constantly back and forth from one place to another, carrying with them their songs, musical instruments, and rhythms that were either borrowed intact or perhaps transformed by intercultural exchanges.[§] A few historical markers about the clave son are known. According to Peter Manuel, the rhythm was common in Afro-Cuban

[*] Vurkaç, M. (2012), Toussaint, G. T. (2011).

[†] Zigel, L. J. (1994), p. 131.

[‡] Arom, S. (1988), p. 1.

[§] For example, traditional Turkish rhythms with five and seven pulses have been imported into popular Western music: Dave Brubeck's *Take Five* has a five-pulse meter, and Pink Floyd's *Money* has a seven-pulse meter. See Keith, M. (1991), p. 126.

music since the 1850s.[*] In the early twentieth century, it was played in eastern Cuba in the area of Santiago in a style of music called *son*. This music made its way to Havana in the early part of the twentieth century. Between 1930 and 1960, many Latin musicians such as Tito Puente incorporated the clave son and other Afro-Cuban rhythms into popular music based on European harmonies.[†] During the 1950s, the son traveled northward from Cuba to the ports of New Orleans and New York. In New Orleans, it influenced rockabilly musicians such as Bo Diddley, and in New York, with the Puerto Rican influence, it was transformed to what is called *salsa*. Finally, from New York, the rhythm infiltrated virtually all parts of the world.[‡]

In Ghana, the clave son rhythm is known as the *kpanlogo* timeline, where it is played on an iron bell.[§] There is evidence that the rhythm traveled with *gome* music from Central Africa to Ghana in the early part of the twentieth century.[¶] The earliest historical documentation of this rhythm appears in a book about rhythm written by the Persian scholar Safi al Din in the middle of the thirteenth century.[**] It is also noteworthy that in his writings about rhythm, Safi al Din used a circular notation similar to that used here. Apart from these snapshots, it is difficult to determine where the rhythm originated from, and how it traveled between Persia, Central Africa, and Cuba. We will return to this topic later in the book after exploring its structure, its relationship to other rhythms, and the phylogenetics tools needed to explore the evolution and migration of musical rhythms.

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[*] Manuel, P., with Bilby, K. and Largey, M. (2006), p. 50. Mauleón, R. (1997), p. 9, whose thesis examines the evolution of the clave son rhythm, and expounds its worldwide influence on a variety of musical genres, also traces the emergence of this pattern in Cuba to the nineteenth century.

[†] Goines, L. and Ameen, R. (1990), p. 6.

[‡] Washburne, C. (1997), p. 66, documents the influence of the clave son timeline on jazz music. Rey, M. (2006) illustrates with many examples the incorporation of the clave son rhythm on the art music of Cuba. See also Quintero-Rivera, A. G. and Márquez, R. (2003).

[§] Rentink, S. (2003), p. 45.

[¶] *Ibid*, p. 35.

[**] Safi al-Din al-Urmawî (1252).

Six Distinguished Rhythm Timelines

Given that the number of possible rhythms with five onsets and 16 pulses is 4368, a natural question that arises is: how many of these rhythms are used as timelines in traditional music practice around the world? In other words, what is the competition? Naturally, many of the rhythms in this family, such as [x x x x x. . . . .] and [. x x x x x. . . . .], for example, do not appear to be very interesting as timelines. Therefore, one might venture a guess of perhaps a number between 50 and 100. However, even this number is too large. In the ethnomusicology literature, it is difficult to find more than a dozen traditional 5-onset, 16-pulse timelines. Of these, six have made a significant mark as timelines in the music of the world. These distinguished rhythms are shown in box notation in Figure 7.1, and for pedagogical reasons I have chosen them to illustrate many of the concepts explored in this book. This list is in no way suggestive that the other 4362 rhythms are not good for some function in music. For example, by permuting the durations in these six rhythms, one may obtain other rhythms that could serve quite well as timelines in some contexts. Many more could be used as rhythmic solos and variations in a large variety of music, and would sound attractive to the modern ear. However, these six are the rhythms that have been adopted over vast expanses

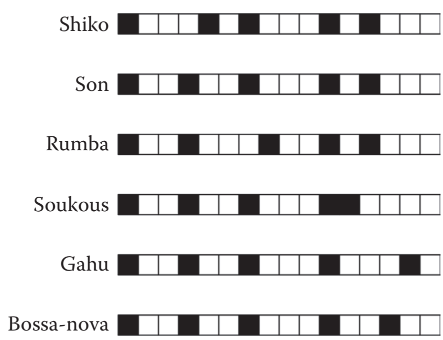

FIGURE 7.1 The six distinguished timelines with five onsets and 16 pulses.

of historical time to serve as *timelines* in traditional music. Because they are distinguished in this sense, they are worthy to be studied in depth. When a rhythm is described as "good" in this book, the word is intended to denote that it is effective as a timeline, as judged by cultural traditions and the test of time.

A word is in order concerning the names attached to these rhythms. As noted earlier, what is called the *clave son* in Cuba is called the *kpanlogo bell pattern* in Ghana.[*] All these rhythms have different names in different parts of the world where they are used. The names adopted here reflect the terminology perhaps most well known in the Western popular media, and I use them here purely for convenience rather than the establishment of any historical priority or cultural entitlement.

The timeline at the top in Figure 7.1, shown in polygon notation in Figure 7.2, is a common rhythm in West Africa and the Caribbean. In Nigeria, it goes by the name shiko. It has a durational pattern [4-2-4-2-4]. Note that these intervals are divisible by two, and thus the rhythm could be notated as [2-1-2-1-2], or in box notation as [x . x x . x x .].[†] In this form, it is commonly called by its Cuban name, the *cinquillo*.[‡] This timeline is played in the *moribayasa* rhythm among the *Malinke* people of Guinea and in the *Banda* rhythm used in *voudoo* ceremonies in Haiti. In Cuba, it is played on a wooden block in the *makuta* rhythm. This rhythm necklace is also started on the second and fourth onsets. For example, the *timini* rhythm in Senegal is [x x . x x . x .], which is equivalent to starting the shiko on the second onset. This pattern is played on a bell for the *adzogbo* dance of the *Fon* people of Benin, and is also frequently encountered in the Persian Gulf region.[§] On the other hand, the *kromanti* rhythm of Surinam is [x x . x . x x .], which is equivalent to starting the shiko on the fourth onset.

The shiko timeline is also found as the first part of longer rhythms. A well-known Arabic rhythm, the *wahda kebira* given by [X . x x . x x . X . X . x . x x] contains the shiko

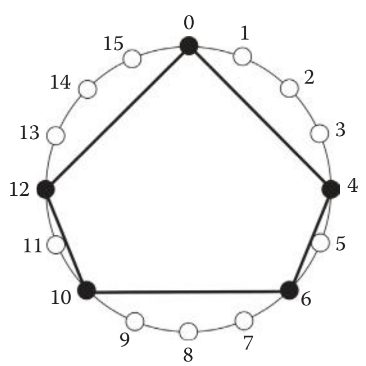

FIGURE 7.2 The *shiko* timeline in polygon notation.

[*] S. Rentink, S. (2003) and Unruh, A. J. (2000).

[†] As mentioned earlier, the emphasis in this book is in the *proportions* of the inter-onset durations of rhythms rather than their absolute values, as is done by Pearsall, E. (1997).

[‡] Fernandez, R. A. (2006), p. 7, Gerard, G. (1998), p. 69. Sandroni, C. (2000) analyzes the influence that the *cinquillo* and *tresillo* had on Latin American music.

[§] Olsen, P. R. (1967), p. 32.

as its first part. Since this is played on a drum, rather than a bell, the uppercase X's denote low-pitched sounds (*dum*) and the small x's high-pitched slaps (*tek*). Cyclic shifts of the shiko timeline also appear in other longer rhythms. For example, the *kassa* from Guinea given by [x x . x . x x . x . x . x . x .] has the kromanti as its first bar. Finally, we remark that starting the shiko on the second onset is also a popular pattern found in Arabic rhythms played on a drum. Here again some notes are low sounds whereas others are high pitched. The *maqsum* is given by [X x . x X . x .], and the *baladi* by [X X . x X . x .]. The *masmudi* is a slow baladi, and the *sáidi* has duration intervals [X X . X X . x .].[*] The durational pattern in these last three Arabic rhythms is the same, and it is only the pitch (or timbre) of the drum notes that varies from one rhythm to another.

The timeline with inter-onset intervals [3-4-3-2-4] shown in Figure 7.3 in polygon notation has also traveled through much of the world and goes mainly by its Cuban name: *clave rumba*.[†] The rumba is one of the most well-known Afro-Cuban folkloric song-and-dance styles popular at large feasts. There are three styles of rumba music: the 12-pulse *fume-fume*[‡] and the two 16-pulse rhythms: the fast *guaguancó* and the slower *yambú*. It is these two last styles that use the clave rumba timeline, which is played on the wooden claves.[§] This pattern is also used in several other Cuban rhythms such as the *conga de comparsa* and the *mozambique*, both employed mainly for carnivals. The same timeline pattern is played on a bell in a processional music of the *Ibo* and *Yoruba* peoples of Nigeria.

Rhythm did not travel with the slaves along a one-way street from Africa to America. After the Second World War, Cuban music became popular in Central Africa. Rumba in particular hit a sympathetic chord with dancers, and was speeded up to create what was first called *congolese* and later became *soukous*. The word "soukous" comes from the French word *secouer* meaning to "shake," and what may account for the strong heartbeat pulse,

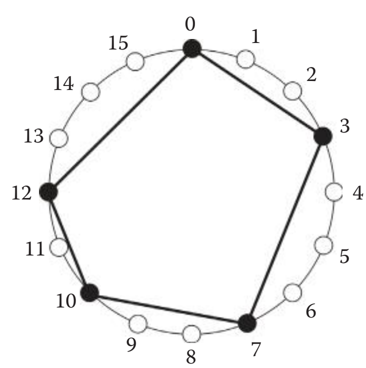

FIGURE 7.3 The *rumba* clave in polygon notation.

[*] Wade, B. C. (2004), p. 70.

[†] Johnson, H. S. F. and Chernoff, J. M. (1991), p. 68, Crook, L. (1982). This timeline is sometimes called the clave *guaguancó*. See Gerard, G. (1998), p. 83. It is also played in the Afro-Cuban religious *batá* drumming, Moore, R. and Sayre, E. (2006), p. 128.

[‡] Klőwer, T. (1997), p. 176.

[§] In the guaguanco style of rumba, the clave pattern is played starting on the second half of the measure thus [. . x . x . . . x . . x . . x .], Manuel, P. with Bilby, K. and Largey, M. (2006), p. 29.

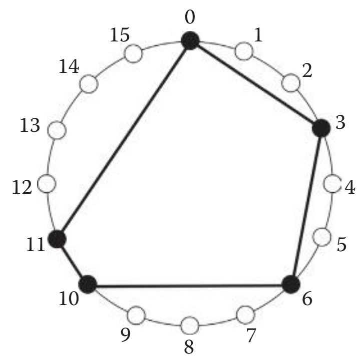

FIGURE 7.4 The *soukous* timeline in polygon notation.

is the timeline pattern shown in Figure 7.4, with its last two onsets close together. Gary Stewart writes about soukous that "Its intricate, invigorating rhythms set feet to tapping."[*] This pattern is usually played on either a wood block or a snare drum.

It is not surprising that Africans would resonate well with a new music that had African roots. Neither is it surprising that they would speed it up to suit their more energetic dances. What may seem surprising at first glance is that they would change the rhythm itself to such an extent that the resulting timeline ended up closer to the son than the rumba. To transform the son to soukous, one merely plays the last onset of the son one pulse earlier. On the other hand, to transform the rumba to soukous, one must play both the third and fifth onsets of the rumba one pulse earlier. Even when the tempo of the rumba is increased, the relatively late third onset has the subjective effect of slowing it down because the second inter-onset interval is longer than the first. Advancing this third onset by one pulse as in the son and soukous gives the resulting rhythms a more rolling drive.

Gahu is a polyrhythmic drumming music of the *Ewe* people of Ghana.[†] The word "gahu" means either "money dance" or "airplane." It appears to have been created by *Yoruba* speakers of Benim and Nigeria as a form of satirical commentary on the modernization in Africa, and was first taken to Ghana in the early 1950s. The Gahu timeline is played on a *gankogui* double bell. It has inter-onset intervals [3-3-4-4-2], and is shown in polygon notation in Figure 7.5.

The timeline with inter-onset intervals [3-3-4-3-3] shown in Figure 7.6 in polygon notation is sometimes called the *bossa-nova* rhythm[‡] and is played in the slower bossa-nova music, the faster *samba* music, and in Afro-Brazilian folk music from Bahia.[§] The bossa-nova, an offspring of samba, is a style of music that was developed in the late 1950s in Rio de Janeiro by musicians such as Joao Gilberto and Stan Getz. The bossa-nova clave was originally played either with the claves or wood block. However, in more contemporary

[*] Stewart, G. (1989), p. 19.
[†] Locke, D. (1998), Agawu, K. (2003), p. 81, Reich, S. (2002), p. 60.
[‡] Kernfeld, B. (1995), p. 23.
[§] Van der Lee, P. (1998) and Morales, E. (2003).

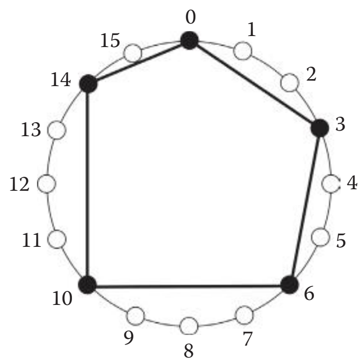

FIGURE 7.5 The *gahu* timeline in polygon notation.

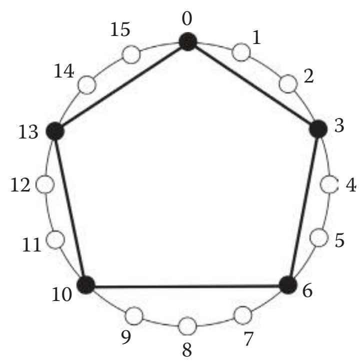

FIGURE 7.6 The *bossa-nova* timeline in polygon notation.

music, it is played on cymbals or a snare drum, such as in Dave Brubeck's *Bossa Nova U.S.A.* (Columbia—CS 8798, Vinyl, LP, U.S. 1963).

Several rotations of this pattern are sometimes used in traditional African and Brazilian music, including the rotations [3-3-3-4-3] and [3-3-3-3-4], as well as the 48 pulse [3-3-3-3-3-3-3-3-3-3-3-3-3-3-3-3].* The [3-3-3-3-4] version is used in the form of a synthesized rhythmic ostinato sound by the Manchester-based British electronic dance music group *808 State*, in their piece titled *Cubik*. Their rather strange name was inspired by one of the earliest commercially available programmable drum machines, the Roland TR-808 *Rhythm Composer*.†

One property that all six distinguished rhythms have in common is that they all use adjacent inter-onset durations with values equal to one, two, three, or four. Interestingly, ancient Chinese philosophers at the time of Confucius regarded these four numbers as "the source of all perfection."‡ Of course, there are many other rhythms with five onsets among 16 pulses that use only these four durations. In this book, we will expound on what additional features contribute to making these six rhythms so distinguished.

* Béhague, G. (1973), p. 221.
† Butler, M. J. (2001).
‡ Jeans, J. (1968), p. 155.

CHAPTER 8

The Distance Geometry of Rhythm

In *The Birth of Tragedy*, Friedrich Nietzsche wrote: "Music is like geometric figures and numbers, which are the universal forms of all possible objects of experience."[*] Traditionally, musicologists have analyzed music from the numerical, historical, sociological, psychological, anthropological, neurological, as well as music theory and praxis points of view. The geometric approach used here permits a new kind of analysis of rhythms that yields novel insights, and thus augments the traditional tools employed by musicologists.[†] This is not to imply that geometry has not been utilized in the past as a music-theoretic tool. Indeed, geometric images have served multiple purposes for illustrating a variety of musical concepts since antiquity.[‡] The circular notation for cyclic rhythms goes back at least to the thirteenth century Baghdad.[§] In modern times, geometric structures in two and higher dimensions are applied to a variety of different aspects of music analysis with increasing frequency.[¶] Furthermore, the visualization of rhythms as cyclic polygons allows instant recognition of many structural features of the rhythms that are more difficult to perceive with standard Western music notation or even box notation. For example, suppose we want to know whether the clave son has the *palindrome* property: that it contains an onset from which one can start playing the rhythm either forward or backward so that it sounds the same. With Western music notation, the novice requires some reflection to come up with the answer. On the other hand, with polygon notation, the answer

[*] Johnson, I. (2009), p. 16.

[†] See Toussaint, G. T. (2003, 2004a, 2005a,b) for a more detailed and deeper discussion of computational geometric tools that may be exploited by musicologists.

[‡] Christensen, T. (2002), p. 280.

[§] Liu, Y. and Toussaint, G. T. (2010a).

[¶] Tymoczko, D. (2011), Bhattacharya, C. and Hall, R. W. (2010), Hall, R. W. (2008), Hook, J. (2006), Rappaport, D. (2005), and Yust, J. (2009) explore the geometry of harmony. McCartin, B. J. (1998), Don, G. W., Muir, K. K., Volk, G. B., and Walker, J. S. (2010), Hodges, W. (2006), Cohn, R. (2000, 2001, 2003), and Andreatta, M., Noll, T., Agon, C., and Assayag, G. (2001) explore rhythmic canons. See also Mazzola, G. (2002, 2003), Honingh, A. K. and Bod, R. (2005), and Wild, J. (2009).

is instantly revealed. Consider the six distinguished timelines described previously, and pictured in polygon notation in Figure 8.1.

The son polygon has a solid line connecting the pulse at position three with the pulse at position 11. This is an axis of mirror symmetry for the polygon. The polygon looks the same on both sides of this line. Therefore, the son rhythm has the palindrome property when started on the second onset (at pulse position three). Note that among the six timelines, two other rhythms have the palindrome property, the shiko and the bossa-nova, both of which have mirror symmetry about the vertical line connecting pulses zero and eight. This example highlights the ease with which humans perceive spatial symmetry, especially with the polygon notation, compared to Western music notation. On the other hand, without these visual aids, "it is extremely difficult to perceive temporal symmetry."*

The rhythm polygons in Figure 8.1 contain two other markers of noteworthy geometric and musical properties. All but the rumba have a dashed line connecting some pairs of onsets: the shiko, son, and soukous each has one such line, the gahu has two, and the bossa-nova has three. Geometrically, these lines determine isosceles triangles, together with the two adjacent edges on the polygons. Musically, such a triangle indicates that there are two adjacent inter-onset intervals of the same duration. Three of the rhythm polygons contain

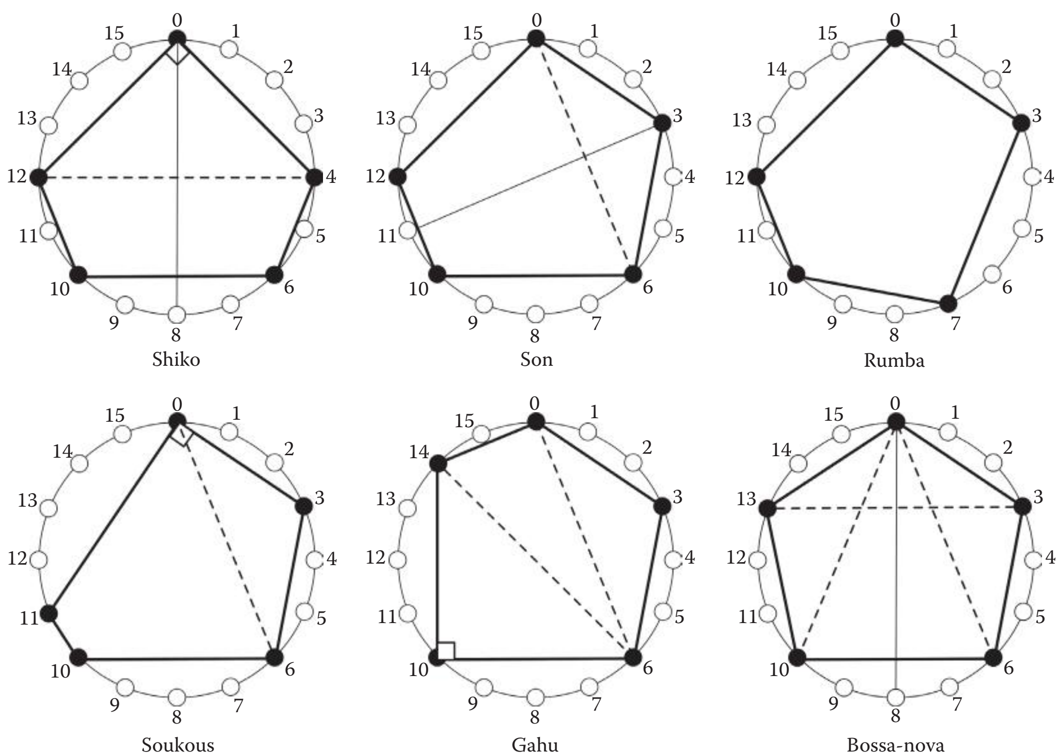

FIGURE 8.1 The geometry of the six distinguished timelines.

* Handel, S. (2006), p. 188.

vertices that make an interior right angle (90°), indicated by the small squares. The shiko and soukous have a right angle at their first onset at pulse zero. The gahu has a right angle at the fourth onset at pulse 10. Geometrically, a right angle indicates that there exists a pair of onsets diametrically opposite to each other.* Musically, this means that there are two onsets that break the cycle into two equal half cycles, introducing a certain degree of *regularity*, an important musical property that we shall return to in more detail later in the book.

These geometric properties can provide new explanations that complement certain conclusions that musicologists have made about rhythms, on the basis of musicological properties alone. For instance, consider the clave rumba and the clave son, which differ only in the position of the third onset. Musicologists agree that the clave rumba rhythm is more complex than the clave son rhythm because the rumba has its third onset on a weak beat (pulse seven) and a silence on a strong beat (pulse six), whereas the opposite is true for the son.† These two properties of rhythms (the presence of onsets on weak beats, and the absence of onsets on strong beats) are features of the musicological notion called *syncopation*.‡ Comparing the two rhythm polygons in Figure 8.1, we observe that, unlike the son, the rumba has no isosceles triangles and no axes of mirror symmetry. Therefore, from a purely geometric point of view, the rumba is less structured and thus more complex than the son.

When the mind is presented with a rhythm such as the clave son that is repeated continuously throughout a piece of music, and that has a cycle that lasts only a few seconds, it is natural to ask whether it perceives durations other than those that occur between adjacent onsets. There exists plenty of evidence, and consensus, that the "conscious present" (also called "specious present"§) lasts for about 3 s. This phenomenon is known as the "three-second window of temporal integration."¶ Therefore, it is most likely that the mind also perceives (perhaps unconsciously) the durations between *all* the other pairs of onsets, in rhythms that last less than 3 s.**

A list of all the inter-onset intervals is called the full *interval content* of the rhythm. Figure 8.2 shows each of the five onsets of the clave son connected to all the others with straight lines labeled with numbers. The line connecting the first onset at pulse zero with the third onset at pulse six has the label "6" attached to it, indicating that the time duration between these two onsets is six units. This number is the shortest distance along the circle that connects pulse zero to pulse six. Note that the *clockwise* distance along the circle

* Patsopoulos, D. and Patronis T. (2006), p. 59. The theorem asserting that if in a triangle inscribed in a circle one of its sides determines the diameter of the circle, then the angle opposite that side is a right angle, is attributed to Thales, the pre-Socratic Greek philosopher from of Miletus, who is considered by many to be the "Father of the Scientific Method" for introducing the notion of a mathematical proof by means of deductive reasoning.

† Velasco, M. J. and Large, E. W. (2011), p. 185.

‡ Fitch, W. T. and Rosenfeld, A. J. (2007), Longuet-Higgins, H. C. and Lee, C. S. (1984).

§ Phillips, I. (2008), p. 182.

¶ Pöppel, E. (1989), p. 86.

** There is as yet no experimental evidence that durations between nonadjacent attacks play a role in the perception of rhythm similarity. The analog question in the pitch domain has been investigated experimentally using chords, throwing doubt on the perceptual validity of some music theoretical assumptions such as octave equivalence: see Gibson, D. (1993). Nevertheless, experiments performed by Quinn, I. (1999) show that the relations between nonadjacent pitch tones (the *combinatorial* model) do affect the judgments of perceptual similarity, but to a lesser degree than the relations between adjacent tones (the *note-to-note* model).

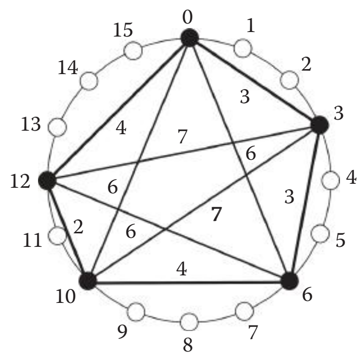

FIGURE 8.2 The full *interval content* of the clave son rhythm.

starting from pulse six and ending at pulse zero is 10. However, we use the shorter of the two distances (be it clockwise or counterclockwise) as the distance between the pair of onsets. In geometry, such a distance is called the *geodesic* distance.*

Therefore, the full interval content of the clave son contains 10 distances in total, some of which occur more than once. One numerical way to represent the interval content of a rhythm is by listing how many times each possible distance occurs. In the case of 16-pulse rhythms, the possible distances range from one to eight, and therefore the interval content may be written as (0, 1, 2, 2, 0, 3, 2, 0). This is sometimes called the *interval vector* of the rhythm.† A more visually compelling representation of the interval vector is as a histogram. Figure 8.3 shows the histograms of the six distinguished timelines. Useful information about the rhythms may be gleaned from the properties of their interval content histograms. For example, shiko uses only four different distances: two, four, six, and eight. On the other hand, gahu uses seven different distances ranging from two to eight. In a later chapter, we shall explore the concept of rhythm complexity, and its application to the creation of music, and compare a variety of measures of mathematical, perceptual, and performance complexities. In the present context, one measure of the complexity of a rhythm is the total number of different distances that it generates. Therefore, one would expect the gahu to be more complex than the shiko, and perhaps more challenging to learn as well. The difference between son and rumba is not as pronounced; son contains five different intervals and rumba six. Nevertheless, it is observed again that this property of histograms (their density of occupied cells) suggests, like the previous geometric features, as well as syncopation, that the rumba is more complex than the son. Furthermore, the higher the number of different distances that a rhythm contains, the flatter the histogram will tend

* Points that lie on a circle are called *cyclotomic* sets in crystallography, where they serve as models of one-dimensional periodic molecules of crystals. The actual models are straight line segments of one period, but the ends are tied together into a circle to facilitate the visualization of all the geodesic distances between the pairs of points (atoms), Buerger, M. J. (1978). See also Senechal, M. (2008) for related material. Tymoczko, D. (2009) analyses the relationship between three different musical distances and the musico-geometrical spaces they inhabit.

† Lewin, D. (2007), p. 98. The term *interval vector* is normally used to describe the pitch intervals in chords and scales. The terms *interval-class content, interval function*, and *pitch class content* are also used: see Isaacson, E. J. (1990), Lewin, D. (1959, 1977), Rogers, D. W. (1999), and Block, S. and Douthett, J. (1994). Much additional work has been done exploring interval vector relations in the pitch domain for chords and scales.

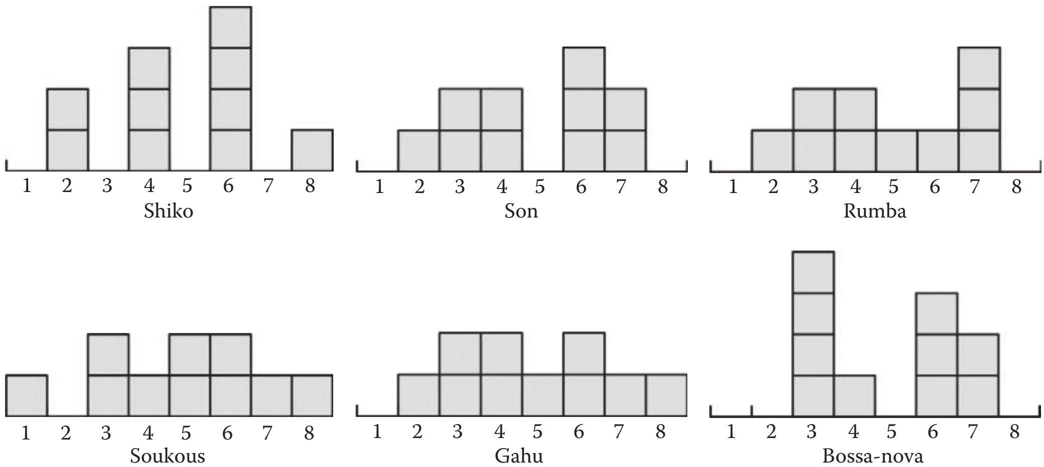

FIGURE 8.3 The *full* interval content histograms of the six distinguished timelines.

to be, since the histogram bins have to spread themselves out. Therefore, the shape of the histogram is relevant to a variety of musical properties of rhythms. We will return to consider shape properties of interval content histograms other than flatness later in the book.

It is instructive to compare the histograms that contain *all* the inter-onset intervals shown in Figure 8.3, with the histograms that contain only the *adjacent* intervals, shown in Figure 8.4. The latter histograms are equivalent to Pearsall's duration sets.* Although these histograms have their strengths, here, we see one of their weaknesses. The son, rumba, and gahu all have identical histograms, and thus cannot be distinguished from each other based only on this information. Indeed, these three rhythms are permutations of their adjacent inter-onset intervals. It will be seen, however, that even the full interval histograms have some serious drawbacks for characterizing rhythms, and thus, for one of the most important problems in musicology, the measurement of rhythm similarity.

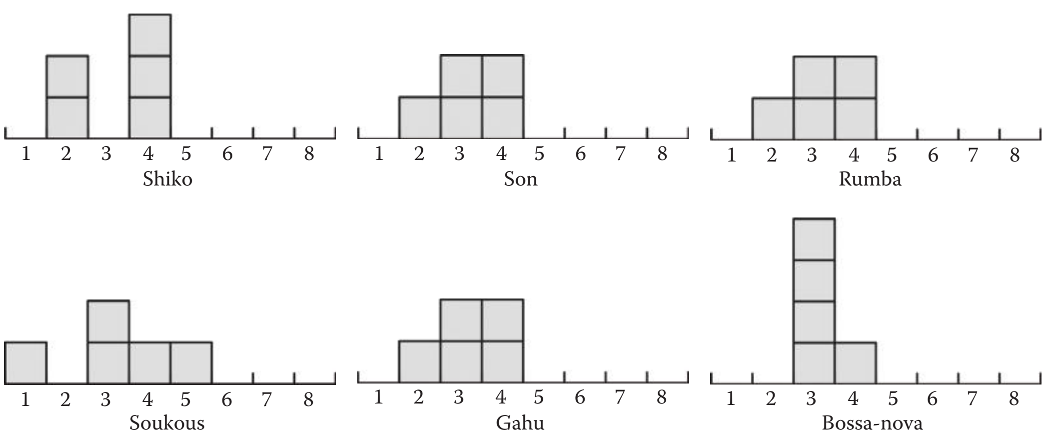

FIGURE 8.4 The *adjacent* interval content histograms of the six distinguished timelines.

* Pearsall, E. (1997).

CHAPTER 9

Classification of Rhythms

THE CLASSIFICATION OF OBJECTS into categories appears to be a universal preoccupation of human beings all over the world. Besides seeming to provide untold pleasures in creating order around us, classification assists us in an uncountable number of more specific ways. For one, it improves our ability to remember large amounts of information. It aids librarians to catalogue musical material for archiving and efficiently retrieving information.[*] It helps doctors prescribe the right medication if they classify a patient's disease correctly. In the domain of music, musicologists classify almost everything they can: musical instruments,[†] drum sounds,[‡] music notes on score sheets,[§] music patterns,[¶] folk tunes,[**] scales,[††] chords,[‡‡] keys,[§§] meters,[¶¶] spans,[***] complexity classes of meters,[†††] rhythms,[‡‡‡] Indian *talas*,[§§§] melodies,[¶¶¶] contours,[****]

[*] Casey, M., Veltkamp, R., Goto, M., Leman, M., Rhodes, C., and Slaney, M. (2008).

[†] Kartomi, M. J. (1990) tackles the classification methods used to classify instruments all over the world in the past and present, whereas Lo-Bamijoko, J. N. (1987) focuses on the classification of Igbo musical instruments in Nigeria, based on two considerations: how the instruments are played, and the function that the instruments serve in a cultural context. See also Lawergren, B. (1988) and Hornbostel, E. M. von and Sachs, C. (1992).

[‡] Herrera, P., Yeterian, A., and Gouyon, F. (2002).

[§] Bainbridge, D. (2001).

[¶] Coyle, E. J. and Shmulevich, I. (1998).

[**] Elschekova, A. (1966), Kolata, G. B. (1978), Lomax, A. (1959), Lomax, A. and Grauer, V. (1964), Lomax, A. (1972).

[††] Clough, J., Engebretsen, N., and Kochavi, J. (1999).

[‡‡] Callender, C., Quinn, I., and Tymoczko, D. (2008), Rowe, R. (2001), p. 19.

[§§] Temperley, D. (2002).

[¶¶] Christensen, T. (2002), p. 660.

[***] Cohn, R. (2002b), p. 194, classifies time spans as pure if the pulse numbers are a power of a single prime number. Thus, the number 16 is pure because it may be written as 2^4, where 2 is a prime number. Similarly, the number 9 is pure because it may be written only as 3^2, where the number 3 is a prime number. The former is a "pure duple" span whereas the latter is a "pure triple" span. On the other hand, the number 12 is not pure because it does not afford such an expression.

[†††] London, J. (1995), p. 70.

[‡‡‡] Arom, S. (1989), p. 92, Toussaint, G. T. (2003), Arom, S. (2004), p. 41, classifies 44 *aksak* rhythms. Butler, M. J. (2006), p. 81, classifies the rhythms of electronic dance music into three categories: even, diatonic, and syncopated.

[§§§] Morris, R. (1998), Akhtaruzzaman, Md. (2008), Akhtaruzzaman, Md., Rashid, M. M., and Ashrafuzzaman, Md. (2009).

[¶¶¶] Vetterl, K. (1965). See Bhattacharya, C. and Hall, R. W. (2010) for a classification of North Indian *thaats* and *raags*.

[****] Morris, R. D. (1993), p. 220.

genres,[*] styles,[†] dance music,[‡] and other types[§] of music. Clearly, classification is a primary concern in almost all aspects of music. For each of these applications, there exist suitable features, and a variety of tools available for classification. Simha Arom classifies the family of *aksak* rhythms from the Balkan region into three classes depending on the properties of the number of pulses contained in the rhythm's cycle.[¶] All aksak rhythms are composed of a string of inter-onset durations of lengths two and three, which Arom calls *binary* and *ternary* cells. He calls an aksak rhythm *authentic* provided that its pulse number is a *prime* number. Some authentic aksak rhythms include [2-3], [2-2-3], and [2-2-2-3-2-2], with pulse numbers 5, 7, and 13, respectively. A rhythm is *quasi-aksak*, provided that its pulse number is *odd*, *not prime*, and *divisible* by *three*. Some instances of quasi-aksak rhythms are [2-2-2-3] and [2-2-2-2-3-2-2], with pulse numbers 9 and 15, respectively. Finally, a rhythm is called *pseudo-aksak* if its pulse number is even. Thus, the rhythms [2-3-3], [2-2-3-3], and [2-2-2-3-3], with pulse numbers 8, 10, and 12, respectively, are pseudo-aksak. Arom lists 33 confirmed and documented aksak rhythms that are played in practice, that have pulse numbers ranging from 5 to 44, excluding the numbers 6, 20, 31, 36, 38, 40, and 43. In this chapter, I illustrate several general approaches to solving the problem of classification, by using the six distinguished timelines as a pedagogical toy exemplar.

Geometric properties such as those described in Chapter 8 can be used to design methods for the categorization and automatic classification of rhythms. Starting from the acoustic signal produced by an instrument, there are several stages in any musical rhythm recognition system. A fundamental and difficult first step is the analysis of the acoustic waveform, to detect and estimate the locations of the onsets. Once these onsets are established, a matching is sought between the query rhythm to be classified, and the stored templates. This matching problem is made easier if the underlying *fundamental beat* is also known. By fundamental beat is meant a series of perceived salient pulses marking equal durations of time. Intuitively, the fundamental beat is what most people do unconsciously when they tap their feet to music that is playing in the background. However, this problem of automatically determining by computer, at what points in time people tap their feet, also known as *beat induction*, is not an easy problem.[**] One approach is to look for a match over all cyclic shifts between the unknown pattern and the stored templates, by means of a *decision tree*.[††] This idea is illustrated in Figure 9.1 for the six distinguished timelines, without knowledge of where the "start" of the rhythm is. However, here the input is not acoustic,

[*] McKay, C. and Fujinaga, I. (2006), Correa, D. C., Saito, J. H., and Costa, L. F. (2010).

[†] Blum, S. (1992) and Backer, E. (2005). See Lomax, A. (1959), Lomax, A. and Grauer, V. (1964), Grauer, V. A. (1965), Lomax, A. (1968), and Lomax, A. (1972) for an ambitious program to classify all the folk song styles of the world. Cilibrasi, R., Vitányi, P., and de Wolf, R. (2004) are able, surprizingly, to classify types of music disregarding all music information, by merely using data compression algorithms from information theory.

[‡] Chew, E., Volk, A., and Lee, C.-Y. (2005).

[§] Levitin, D. J. (2008). Dan Levitin classifies the music of the world into six types (friendship, joy, comfort, knowledge, religion, and love) and explains how music contributed to the evolution of society, science, and art.

[¶] Arom, S. (2004), p. 45. See also Brăiloiu, C. (1951).

[**] Desain, P. and Honing, H. (1999), Dixon, S. (1997).

[††] Breiman, L., Friedman, J. H., Olshen, R. A., and Stone, C. J. (1984).

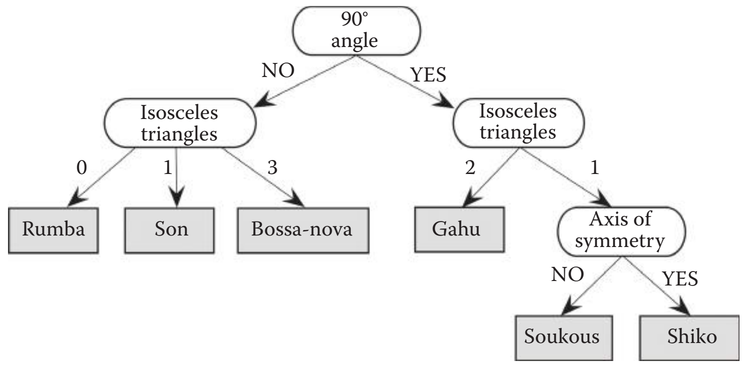

FIGURE 9.1 A decision tree classification using geometric features of the rhythm polygons.

but assumed to be known and symbolically notated.* From this input, it is straightforward to compute the polygonal representations of the rhythms.

A decision tree classifier may be designed in a variety of ways using different features. The first tree described in the following uses geometric features of the rhythm polygons (refer to Figure 9.1). First determine if the polygon has a 90° angle. If the answer is NO, then we know that the rhythm must be the rumba, son, or bossa-nova. These three cases can be differentiated by computing the number of isosceles triangles contained at vertices of the polygon: zero for rumba, one for son, and three for bossa-nova. If the polygon does contain a 90° angle, then we also compute the number of isosceles triangles. If there are two isosceles triangles, we have found the gahu rhythm. Otherwise, if there is only one isosceles triangle, then determine if there exists an axis of symmetry. If the answer is YES, we have identified the shiko; otherwise we have found the soukous. Note that no measurement depends on knowing which is the starting note of the rhythm because these properties are invariant to rotations of the polygons.

Another approach measures *global* shape features of the full inter-onset interval histograms of Figure 8.3 to obtain a decision tree such as the one pictured in Figure 9.2. Here, two measurements are made: the height of the histogram and the number of connected components that make up the histogram. A component is considered connected if it consists of a set of cells of height at least one, not separated by an empty cell. The clave son has a histogram of height three (see Figure 8.3), and is made up of two connected components, one consisting of cells with distances two, three, and four, and another of cells with distances six and seven.

A third approach uses the presence and absence of certain distances in the interval histogram, as illustrated in Figure 9.3. In this case, the clave son is identified by the fact that it contains a distance of two, but not distances equal to five or eight.

* Wright, M., Schloss, W. A., and Tzanetakis, G. (2008) present a variety of tools for analyzing rhythm in audio recordings. In particular, they propose an original method for beat tracking in Afro-Cuban music by using knowledge specific to this type of music, namely, the clave pattern itself. Their technique highlights the benefits that may be acrued by incorporating domain-specific knowledge about the musical style and culture under study.

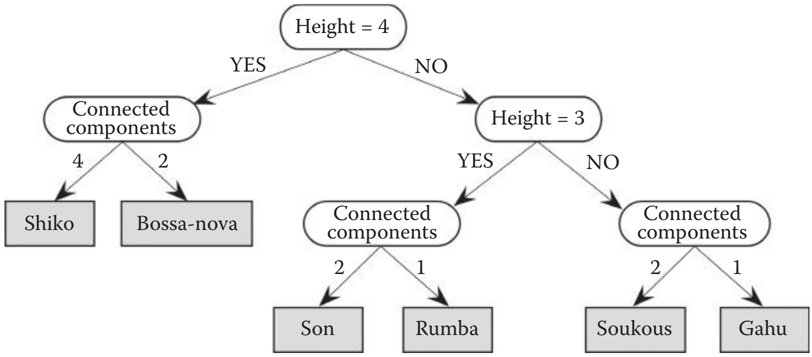

FIGURE 9.2 A decision tree classification using global shape features of the interval histogram.

As a final illustration of decision trees, we may use the *number* of distinct inter-onset distances that occur in the histogram, as well as the *ranges* that these values take. One possible decision tree that uses these features is shown in Figure 9.4. Here, the range is calculated as the difference between the highest and lowest values of distances present in the histogram.

Other types of geometric features may be computed from the polygons that represent the rhythms, including features based on symmetry properties as well as statistical moments,* and moments of inertia.† Furthermore, once a set of features has been chosen, there exist a plethora of metric equations that can be used to obtain classifications.‡ Such features may also be used to generate rhythms for use in performances.§

An alternate approach to classification constructs proximity trees using a suitable measure of distance or dissimilarity.¶ This method is illustrated in Figure 9.5 for the *aksak* rhythms listed above, and the six Afro-Cuban timelines, using a proximity tree called *BioNJ*, frequently employed in bioinformatics.** In this approach, the distance between

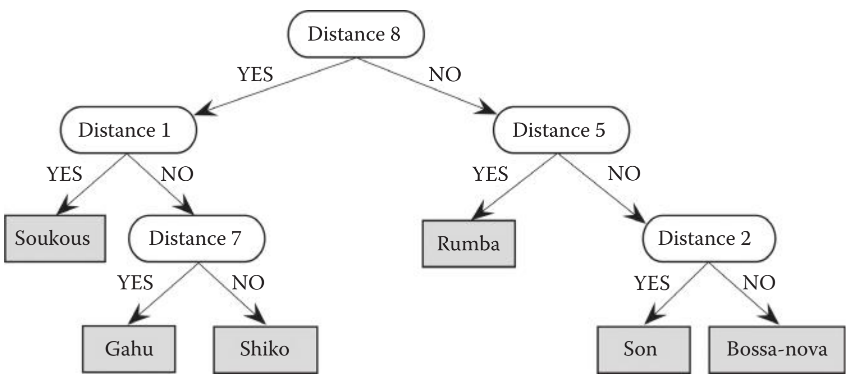

FIGURE 9.3 A decision tree classification using inter-onset interval distances.

* Boveiri, H. R. (2010), p. 17.
† Toussaint, G. T. (1974).
‡ Polansky, L. (1996).
§ Sampaio, P. A., Ramalho, G., and Tedesco, P. (2008).
¶ Jaromczyk, J. W. and Toussaint, G. T. (1992), Toussaint, G. T. (2005d), Toussaint, G. T. (1980, 1988).
** Huson, D. H. and Bryant, D. (2006), Dress, A., Huson, D., and Moulton, V. (1996), Gascuel, O. (1997), Huson, D. H. (1998).

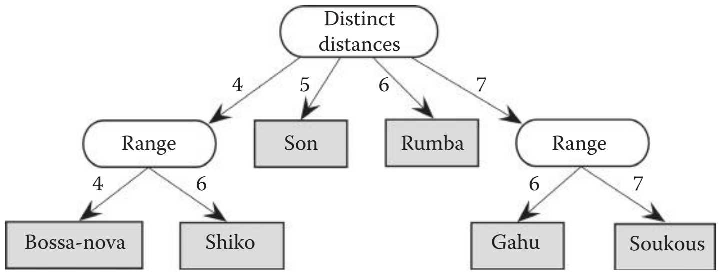

FIGURE 9.4 A decision tree classification using histogram-distinct distances and their range.

every pair of rhythms is first calculated (here with a distance measure called the *edit distance* that we shall revisit later in more detail). The tree is then computed using the resulting distance matrix. In Figure 9.5, the rhythms are designated by the labeled nodes in the tree, and the edit distance between each pair of rhythms is portrayed by the shortest path in the tree, between the corresponding nodes. From this tree, it can be clearly observed that the aksak and Afro-Cuban rhythms form two separate groups. This is no surprise since the Afro-Cuban rhythms have 16 pulses, the aksak rhythms have fewer than 16 pulses, and the edit distance is sensitive to this parameter. Within the Afro-Cuban rhythms, the bossa-nova, gahu, and soukous form their own separate subgroup. The "Q" and "P" prefixes in the labels of the aksak rhythms denote quasi-aksak and pseudo-aksak, respectively, whereas the postfixes denote the numbers of pulses of the rhythms. Note that the edit distance does not group the three different types of aksak rhythms into separate branches of the tree.

A third general approach to classification uses probability, statistics, and Bayesian decision theory. This topic, however, is beyond the scope of this book, and the reader is referred to the references listed.*

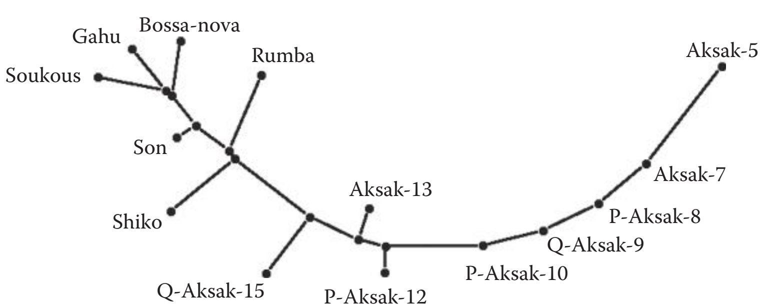

FIGURE 9.5 The *BioNJ* proximity tree of the Afro-Cuban and aksak rhythms.

* Lin, X., Li, C., Wang, H., and Zhang, Q. (2009), Temperley, D. (2007), Gómez, E. and Herrera, P. (2008) apply probabilistic methods to compare music audio recordings from Western and non-Western traditions by automatically extracting tonal features. Toussaint, G. T. (2005d) provides a survey of nonparametric methods for machine learning and data mining, including nearest-neighbor methods that approximate Bayesian decisions by incorporating proximity graphs. For the application of proximity graph methods used in discriminating Western from non-Western music, see Toussaint, G. T. and Berzan, C. (2012d). The textbook by Duda, R. O. and Hart, P. E. (1973) provide an excellent introduction to a variety of statistical methods for classification.

CHAPTER 10

Binary and Ternary Rhythms

THE RHYTHMS CONSIDERED IN Chapters 5 through 9 were determined by cycles that had either 8 or 16 pulses. Such rhythms are here called *binary* rhythms, as are those with cycles of 2, 4, or 32 pulses. Note that all these numbers can be evenly divided by two, but not by three. Rhythms with cycles of 16 pulses are popular all over the world. In addition to 16, there is another number of pulses that also figures prominently in music of many parts of the world, most notably in sub-Saharan Africa and southern Spain, and this is the number 12. Such rhythms are here called *ternary* rhythms. Rhythms with 3, 6, and 24 pulses also belong to the family of ternary rhythms. The smallest binary and ternary rhythms with two and three pulses (also called *duple* and *triple* rhythms), and their combinations, form the building blocks of most rhythms of the world, leading some scholars to label them as music universals. The reverend A. M. Jones writes: "When Europeans sing or play, their music will consist of rhythms which are essentially duple or triple or a combination of both. So it is with Africans. It is a fundamental natural law of rhythm and is therefore universal."[*]

The most illustrious ternary rhythm timelines in sub-Saharan Africa and the Caribbean have either five onsets or seven onsets, with durational patterns [2-2-3-2-3] and [2-2-1-2-2-2-1], respectively, in a cycle of 12 pulses. They are shown as polygons in Figure 10.1. Both rhythms have been called the *standard* pattern in the literature.[†] E. D. Novotney reviews the evolution of the terminology used for these rhythms and proposes his own: the *five-stroke key pattern* and the *seven-stroke-key pattern*, respectively, on the basis that they play in sub-Saharan African music, the same function that the "clave" rhythms play in Cuban music, and the word "clave" means "key."[‡] The rhythm on the left is sometimes called the *fume-fume* timeline[§] and

[*] Jones, A. M. (1949), p. 293.

[†] Agawu, K. (2006), King, A. (1960), p. 51, Kubik, G. (1999), p. 53.

[‡] Novotney, E. D. (1998), p. 165.

[§] Klőwer, T. (1997), p. 176. This timeline pattern is also a clapping rhythm used by Ewe children in Ghana, who sometimes dance four evenly spaced steps marking out the rhythm [x . . x . . x . . x . .]. See also Kubik, G. (2010b), p. 55, Kubik, G. (2010a), p. 45, Jones, A. M. (1959), p. 3, Collins, J. (2004), p. 29, Akpabot, S. (1972), p. 62, and Logan, W. (1984), p. 194. Bettermann, H., Amponsah, D., Cysarz, D., and Van Leeuwen, P. (1999), p. 1736, call this rhythm by the name of *inyimbo*.

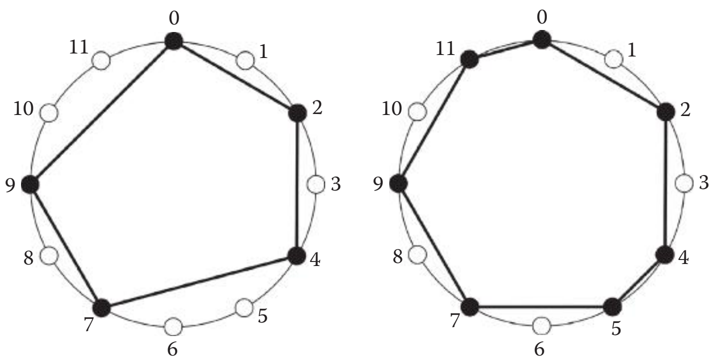

FIGURE 10.1 The *fume-fume* and *bembé* ternary timelines.

the one on the right the *bembé* pattern.* The fume-fume is often used as a clap pattern in West African music, and A. M. Jones writes that: "This little clap-pattern is quite charming."†

The reader has surely noticed that the ternary fume-fume clave has a geometric structure very similar to that of the binary clave son. The two rhythms are shown together in Figure 10.2 for easy comparison.‡ In a subsequent chapter, both rhythms will be mapped to a 48-pulse clock to quantify the absolute values of the differences between their corresponding attack points. For now, it suffices to remark that visually the two pentagons are almost identical in shape and orientation. To be more precise, both rhythms contain an isosceles triangle rooted at the second onset, and both exhibit mirror symmetry about an axis that bisects the isosceles triangle at its apex. Furthermore, both contain *obtuse* isosceles triangles, which means that the triplets (onsets at pulses zero, two, and four in the

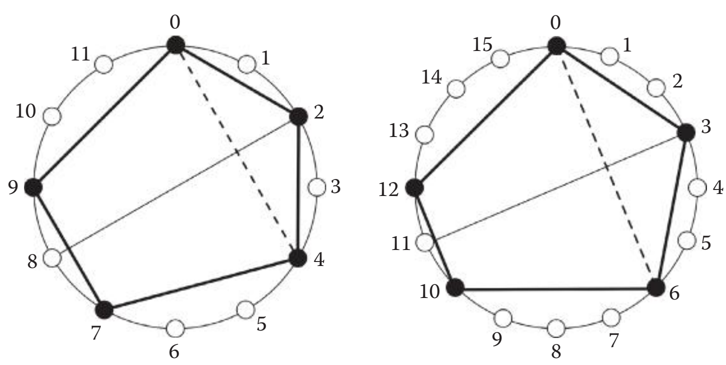

FIGURE 10.2 The similar geometry of the *fume-fume* and *son* clave rhythms.

* King, A. (1960), p. 51, Malabe, F. and Weiner, B. (1990), p. 8, Logan, W. (1984), p. 194, Waterman, R. A. (1948), p. 28, Temperley, D. (2000, 2004), p. 269, Kauffman, R. (1980), p. 397, Stone, R. M. (2005), p. 82, Kubik, G. (2010a), p. 44, Johnson, H. S. F. and Chernoff, J. M. (1991), p. 67, Chernoff, J. M. (1979), p. 145.

† Jones, A. M. (1954a), p. 33.

‡ Both of these rhythms are played in the Highlife popular dance rhythm of West Africa. See Chernoff, J. M. (1979), p. 145.

fume-fume and at pulses zero, three, and six in the son) are well separated from the twins (7 and 8 in the fume-fume, and 10 and 12 in the son) by the diagonals (11, 5) and (15, 7), respectively.

There are other musicological structural similarities between the two rhythms. For example, both numbers 12 and 16 may be evenly divided into four equal durations (quarter measures) without requiring additional pulses, by selecting the "north," "south," "east," and "west" pulses numbered 0, 3, 6, and 9 in the ternary case and 0, 4, 8, and 12 in the binary case. These are the four most salient locations for regular metric beats in families of rhythms with 12 and 16 pulses. The fume-fume and son rhythms both have their first and last onsets on their "north" and "west" metric pulses, respectively. Since both regular meters, [3-3-3-3] and [4-4-4-4], can be easily aligned with each other, and the two rhythms are so similar, they can easily be interchanged during the performance of a piece, as is done in the *Highlife* music of West Africa. Jeff Pressing calls such timelines with unequal values of pulses in their cycles, but with similar inter-onset interval structures, *transformational analogues*,[*] and Fernando Benadon explores their use as compositional and analytical expressive transformations of each other.[†]

In addition to the fact that the two rhythms are quite similar to each other with respect to the exact locations of their attacks, they are in fact identical to each other if they are represented by their *rhythmic contours*. The rhythmic contour of a rhythm is obtained by coding the change in the durations of two adjacent inter-onset intervals using 0, +1, and −1 to stand for equal, greater, and smaller, respectively. The durational patterns of the fume-fume and son timelines are, respectively, [2-2-3-2-3] and [3-3-4-2-4]. Therefore, both rhythms have the same rhythmic contour: [0, +1, −1, +1, −1]. Rhythmic contours are relevant from the perceptual point of view because humans have an easier time perceiving qualitative relations such as "less than" or "greater than" or "equal to" than quantitative relations such as the second interval is four-thirds the duration of the first interval. It has also been found that often the reduced information contained in the contour is sufficient to effectively describe certain types of music.[‡] On the other hand, two rhythms with the same contour may also sound quite different, as is the case for the 16-pulse and 11-pulse rhythms with inter-onset intervals [4-3-2-3-4] and [3-2-1-2-3], respectively.[§] Therefore, used in isolation or in a context where the intervals can vary widely, the rhythmic contour suffers from

[*] Pressing, J. (1983), p. 43.

[†] Benadon, F. (2010).

[‡] Hutchinson, W. and Knopoff, L. (1987), p. 281, hypothesize that music style may be effectively described and discriminated on the basis of syntactic structures of three symbols used to code rhythmic contours, namely R (repetition), S (shortening), and L (lengthening), in effect, a three-letter alphabet for temporal groupings.

[§] See Marvin, E. W. (1991) for the application of rhythmic contours to composition analysis. Contours have also been explored in the pitch domain, where they are called pitch *contours* or *melodic contours*. See Schultz, R. (2008) for the application of melodic contours to the analysis of the *nonretrogradable* structure in the birdsong music of Olivier Messiaen. A structure is nonretrogradable if it is palindromic, that is, has the same structure when played forward or backward. Freedman, E. G. (1999), p. 365, writes that "musically experienced listeners can recognize both the contour and interval information, whereas musically inexperienced listeners rely predominantly on the contour information." See also Callender, C., Quinn, I., and Tymoczko, D. (2008) for a more recent discussion on contour in the pitch domain.

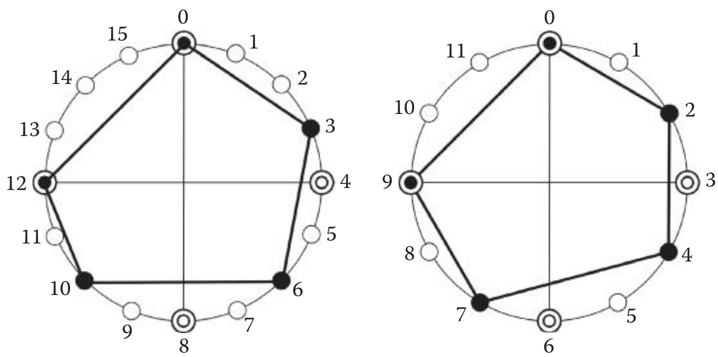

FIGURE 10.3 The clave *son* (left) and *fume-fume* (right) embedded in a duple (regular) meter.

severe drawbacks as a representation from which to extract meaningful rhythmic similarity features.*

John Chernoff has suggested that for all practical purposes there is not much difference between the binary and ternary versions of the "standard" African bell pattern (fume-fume) when perceived relative to their underlying duple metric beats, [4-4-4-4] and [3-3-3-3], and that these regular beats play a perceptually important role. However, while there is little doubt that these duple metric beats influence perception, it is not at all clear that this influence propels the listeners' judgments of the two versions toward greater similarity. It may be argued to the contrary, that the duple underlying beats flesh out rather than camouflage their differences. Figure 10.3 shows the binary and ternary versions of the five-onset standard pattern superimposed on their duple metric structures, to more accurately examine their perceptual role. For either rhythm, imagine playing the metric beats (each highlighted with a ring) with the left hand on a bass drum, and the rhythm with the right hand on a woodblock. Let us denote with the letters **R**, **L**, and **U** the events consisting of striking the instruments with the right hand, left hand, and both hands in unison, respectively. While it is true that the sequence of onsets that describes the union of the metric beats and rhythm onsets yields the same alternating pattern for both the clave son and the fume-fume, namely [**U**-**R**-**L**-**R**-**L**-**R**-**U**], and although both rhythms start and end on the first and last beats of the cycle, these properties by themselves are not sufficient to engender greater perceived similarity. On the contrary, feeling the duple meter makes the listener more keenly aware of the differences in the placements of the third and fourth onsets of the rhythms, which in the clave son falls squarely in the middle of the interbeat intervals, whereas in the standard pattern fall closer to the beats, creating greater syncopation. In the fume-fume pattern, the third onset is twice as close to the second beat than to the third beat, and the fourth onset is twice as close to the third beat than to the fourth beat.

Furthermore, examples may be constructed of quite dissimilar rhythms that satisfy both these properties. To this end consider the two rhythms in Figure 10.4. The left rhythm

* Whether or not contours will play a significant role in music theory, they have already spawned interesting problems in computer science. Demaine, E. D., Erickson, J., Krizanc, D., Meijer, H., Morin, P., Overmars, M., and Whitesides, S. (2008) consider the problem of reconstructing rhythms from the full and partial contour information.

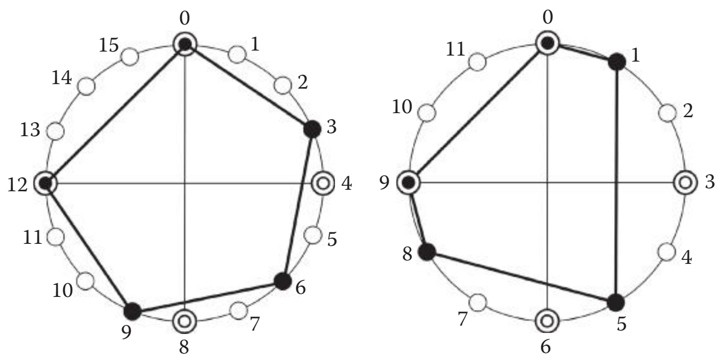

FIGURE 10.4 A rotation of the bossa-nova rhythm (left) and a mutation of the fume-fume (right) embedded in a duple meter.

is a rotation of the bossa-nova clave, and the right is a mutation of the standard pattern, in which the second onset is moved from pulse two to pulse one, and the fourth onset is moved from pulse seven to pulse eight. Both rhythms have the same onset-to-meter placement pattern [**U**-**R**-**L**-**R**-**L**-**R**-**U**], and both have their first and last onsets on the first and last beats of the cycle, but they sound quite different from each other.

In closing this chapter, it should be noted that the terms binary and ternary are also used in music to describe the form of the compositions as a whole. In this context, form refers to the manner in which sections of the piece are structured. In the binary form, two main sections of the work are repeated to create patterns such as AABB, whereas in the ternary form, the sections are organized in patterns such as ABA.*

* Wright, D. (2000), p. 27.

The Isomorphism of Rhythm and Scale

Twelve is a very special number. There are 12 months in a year. The Chinese use a 12-year cycle in their calendar. Near the equator, there are 12 h of daylight, and 12 h of darkness. There are 12 days of Christmas, and we buy a dozen eggs. To a mathematician, 12 is the sum of the three smallest integers $a = 3$, $b = 4$, and $c = 5$, which satisfy the famous Theorem of Pythagoras ($a^2 + b^2 = c^2$). Here, $3 + 4 + 5 = 12$, and $3^2 + 4^2 = 5^2$ or $9 + 16 = 25$. From the musical point of view, 12 has the important property that it is a small number that contains many divisors other than one and 12, in particular two, three, four, and six. For comparison, the larger number 16 only has two, four, and eight as divisors other than 1 and 16. Twelve is also the number of different pitches in the chromatic scale or octave of the modern piano keyboard that consists of 12 pitch intervals called semitones (see Figures 11.1 and 11.2). Since a note that is transposed by an octave may be considered to be the same note, we may wrap the piano octave onto a circle consisting of 12 intervals, as in Figure 11.3 using the labels C, C#, D, E*b*, E, F, F#, G, G#, A, B*b*, B.* The white keys on the modern piano keyboard correspond to the seven pitches without the sharps and flats, that is, C, D, E, F, G, A, B, as shown in polygon notation† in Figure 11.3 (left), and have been immortalized in song with the words *Do-Re-Mi-Fa-Sol-La-Ti*, which will bring us back to *Do*. Note that this polygon is identical to the *bembé* ternary rhythm timeline polygon of Figure 10.1 (right). Thus, the bembé rhythm and the diatonic scale are *isomorphic* to each other, they are the same pattern of long and short intervals, one expressed in time intervals and the other in pitch intervals.‡

* Note that C# = D*b*, D# = E*b*, F# = G*b*, G# = A*b*, and A# = B*b*.

† McCartin, B. J. (2007), p. 2420, refers to such polygons as pitch-class polygons.

‡ Pressing, J. (1983). Rahn, J. (1983) devotes Chapter 5 to the issue of isomorphism of pitch and time. See also Rahn, J. (1987) and Carey, N. and Clampitt, D. (1996).

FIGURE 11.1 A modern piano keyboard. (Courtesy of Yang Liu.)

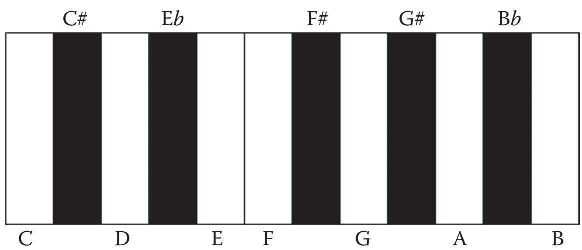

FIGURE 11.2 The diatonic and chromatic scales on the piano keyboard.

The diatonic scale is a *heptatonic* (seven-note) scale.* For this reason, the interval pattern [2-2-1-2-2-2-1] is sometimes called the "diatonic pattern," even when referring to rhythm.† The pentatonic (five-note) scale is found throughout the world, prompting Simha Arom to suggest that it is a musical universal.‡ One of the favorites contains no semitones, and consists of the tones C, D, E, G, and A, as shown in polygon notation§ in Figure 11.3 (right). In ancient classical Chinese music, this was the only scale used until the Chou dynasty

* Loy, G. (2006), p. 17. The diatonic scale is considered to be the prototype of all scale systems in the West. See Kappraff, J. (2010) for a mathematical treatment of ancient scales.

† Temperley, D. (2004), p. 281.

‡ Arom, S. (2001), p. 28. For contrary views concerning the pentatonic scale, see Bradby, B. (1987). There exists a fair amount of variability in the number of tones in scales used around the world. A more constant universal of scales appears to be the preference for unequal step intervals. See Trehub, S. (2001), p. 435. See Meyer, L. B. (1998) for biologically constrained music universals.

§ Representing musical scales with a "clock-face" is quite common in the literature; see Jeans, J. (1968), p. 163. Krenek, E. (1937) was one of the first writers to represent chords and scales as polygons as done here, and thus such polygons are also referred to as Krenek diagrams. See also McCartin, B. J. (1998), Rappaport, D. (2005, 2007), Ashton, A. (2007), p. 43, and Martineau, J. (2008), p. 17.

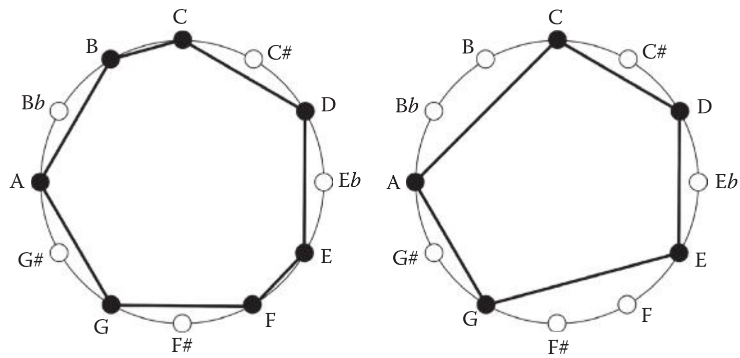

FIGURE 11.3 The *diatonic* (left) and *pentatonic* (right) scales.

more than 3000 years ago.[*] Observe that this scale is isomorphic to the fume-fume timeline of Figure 10.1 (left). Also worthy of note is that if this scale is rotated by a half-circle, one obtains the complement of the diatonic scale, or the black keys of the modern piano keyboard, corresponding to the white circles in Figure 11.3 (left).

Another noteworthy pair of isomorphic rhythm-scale structures consists of the eight onsets in a cycle of 12 pulses shown in Figure 11.4. The rhythm on the left with durational pattern [2-1-2-1-2-1-2-1] is a common rhythmic ostinato used in many parts of the world. It is played on the *kenkeni* drum for the traditional circumcision song *kéné foli* in Guinea.[†] The *kanak* people of New Caledonia call it the *pilou* rhythm.[‡] It is also the "ancestral" pattern obtained from a phylogenetic analysis of all the rotations of the rhythmic ostinato pattern [x x x . x x . x . x x .] used in Steve Reich's *Clapping Music*.[§] Its isomorphic pitch counterpart shown on the right in Figure 11.4 is the *octatonic* scale.[¶] It is one of the four

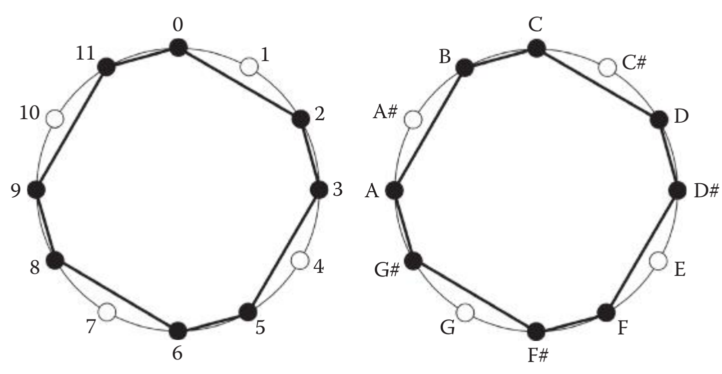

FIGURE 11.4 The *kéné foli* rhythm (left) and the *octatonic* scale (right).

[*] Kennedy, M. (1994), p. 541. Certainly, by the year 433 BCE the 12-note chromatic scale was well established in China, as evidenced by the bronze bells of Hubei discovered in 1977 (see Benson, D. (2007), p. 139).

[†] Konaté, F. and Ott, T. (2000), p. 89.

[‡] Ammann, R. (1997), p. 241.

[§] Colannino, J., Gómez, F., and Toussaint, G. T. (2009).

[¶] McCartin, B. J. (2007), p. 2424.

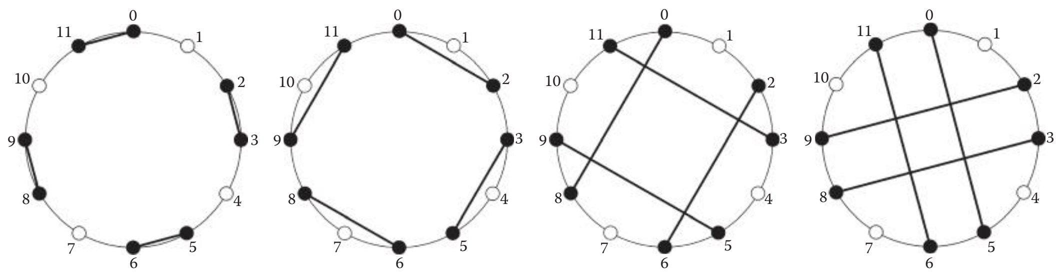

FIGURE 11.5 Dyad partitionings of the octatonic pattern with $d = 1, 2, 4$, and 5.

most important scales used in jazz, where it is called the *diminished* scale.[*] This pattern possesses many rotational and mirror symmetries, indeed, as many as those possessed by a square.[†] It has four axes of mirror symmetry about the lines through pulse pairs (1, 7) and (10, 4), and the orthogonal bisectors of the four short edges of the octagon, and it can be rotated by four different angles to correspond with itself.[‡]

The octatonic durational pattern also has two noteworthy geometric extremal properties. The first is that among all the *maximal-area* eight-note scales, it is the only one whose complementary four-note scale (a diminished seventh chord) is a *maximal-area* four-note scale.[§] The second extremal property concerns the partitioning of the onset points into unions of disjoint pairs of points separated by distance *d*. Such pairs are called *dyads* in the context of scales and chords. In this case, the possible *a priori* values that *d* can take are the values one, two, three, four, five, and six. Figure 11.5 shows the partitionings of the octatonic pattern into unions of *disjoint* dyads of durations one, two, four, and five. The reader is invited to generate the remaining partitionings. The extremal combinatorial-geometric property of the octatonic pattern concerns dyads, and is known as Cohn's theorem in music theory.[¶] It states that the octatonic pattern is the only one among 12-point cycles that can be partitioned into dyads of all six durations. Brian McCartin obtained a simple geometric proof of Cohn's theorem.[**]

Jeff Pressing has expounded on several fascinating parallels between pitch and time. However, Justin London maintains that the two are not isomorphic concepts.[††] Milton Babbitt, while recognizing the limitations of the pitch-time analogy, nevertheless developed methods for transferring pitch-class operations to the rhythmic domain.[‡‡] Indeed, exploring the extent to which the comparative analysis of pitch and rhythm provides insight that can be transferred from one modality to the other is a fruitful endeavor. On the one hand, it is nice to be able to apply the tools developed for one domain to the other,

[*] Levine, M. (1995).
[†] Cohn, R. (1991).
[‡] In the pitch domain, mirror symmetry (or reflection) is called *inversion*, and rotation is termed *transposition*. See Tymoczko, D. (2011), p. 35. These symmetries are very important for music. See also Coxeter, H. S. M. (1968).
[§] Rappaport, D. (2007), p. 327.
[¶] Cohn, *op. cit*.
[**] McCartin, *op. cit*., p. 2431.
[††] London, J. (2002).
[‡‡] Christensen, T. (2002), p. 720.

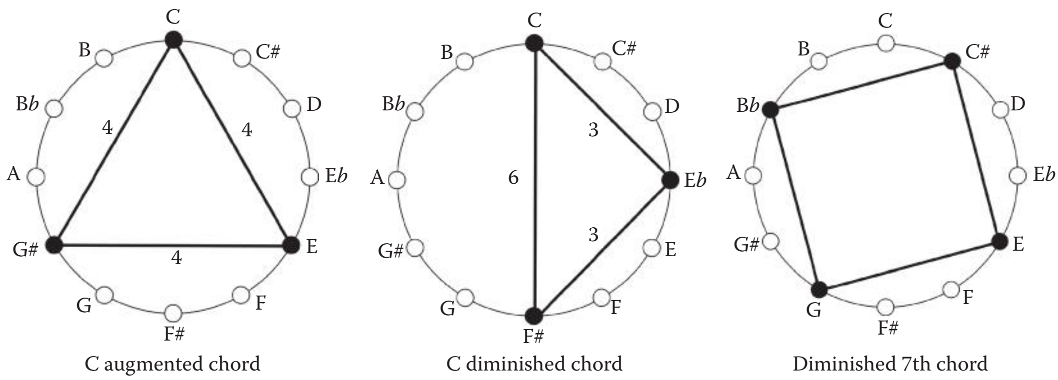

FIGURE 11.6 These three dissonant chords determine stable rhythms.

and if some concepts do not transfer successfully, these provide us with insight about their differences. Furthermore, there exist a variety of tools that are equally applicable to both domains.*

We close this chapter with a simple example in which a concept is not transferable from the pitch to the time domain. Consider the question of consonance and dissonance in pitch and rhythm. The three chords shown in Figure 11.6 sound quite dissonant as chords in the pitch domain. On the other hand, as rhythms in the time domain they are stable.

* Amiot, E. and Sethares, W. A. (2011).

Binarization, Ternarization, and Quantization of Rhythms

Imagine a seventeenth century sailor crossing the Atlantic Ocean on a galleon full of Aztec and Inca gold, on the regular run from the port of Havana in Cuba to the port of Seville in Spain. Let us assume that this sailor was brought up in a village where the popular music always incorporated rhythms that used a 16-pulse cycle. One day, some freed slaves from sub-Saharan West Africa show up on the ship, with drums and an iron bell, and they play the fume-fume rhythm [x . x . x . . x . x . .]. Since this rhythm is so similar to the clave son [x . . x . . x . . . x . x . . .], it is quite possible that our sailor would perceive it as being the clave son. To more accurately compare these two rhythms, it helps to put them together on the same clock diagram so that both complete cycles take the same amount of real time. For this, it is convenient to use a clock with a number of pulses that is divided evenly (without remainder) by both 12 and 16. The smallest such number is 48: it is equal to 4×12 and 3×16. Figure 12.1 (left) shows the son and fume-fume rhythms embedded on such a 48-pulse clock. The son is indicated with the larger white circles on pulses 0, 9, 18, 30, and 36, whereas the fume-fume is made up of the smaller black circles on pulses 0, 8, 16, 28, and 36. As pointed out in Chapter 10, the first and last onsets of both rhythms are in unison. The second onsets differ by 1/48th of a cycle, and the other two onsets differ by 1/24th of a cycle. If we assume that the entire cycle lasts about 2 s, it means the second onsets differ by 1/24th of a second, and the two others by 1/12th of a second. Thus, even in absolute terms, the rhythms may be considered to be quite similar in terms of their corresponding onset alignments.

The ternary bembé timeline and its binarized version* are shown in the right diagram of Figure 12.1. For the ternary rhythm, the onsets consist of small black circles on pulses 0, 8, 16, 20, 28, 36, and 44, which are connected by dotted lines. The binarized rhythm (solid lines) has onsets indicated with larger white circles on pulses 0, 9, 18, 21, 30, 36, and 45. For these two rhythms also, the onsets at the first and last of the four main beats coincide, the

* Johnson, H. S. F. and Chernoff, J. M. (1991), p. 68.

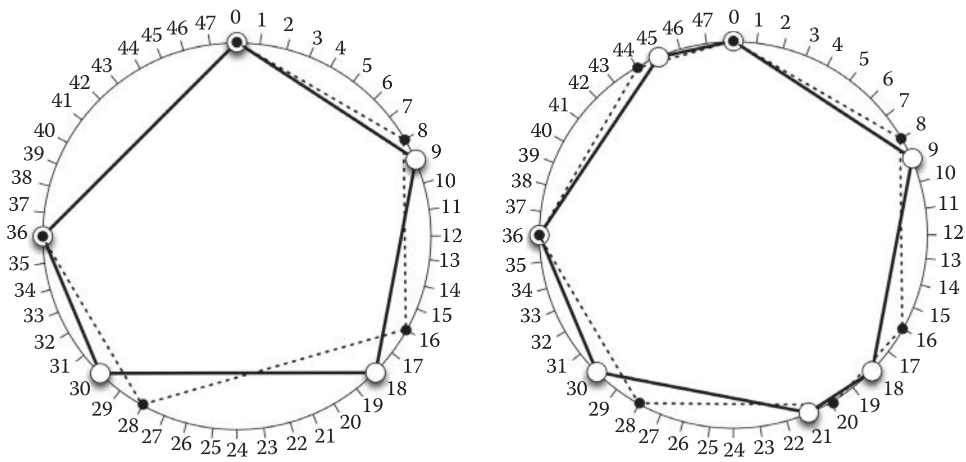

FIGURE 12.1 Left: The binary son (solid lines) and ternary fume-fume (dotted lines) on a 48-pulse clock. Right: The binary bembé (solid lines) and its ternary version (dotted lines).

second, fourth, and seventh ternary onsets precede their binary counterparts by 1/48th of a cycle, and the third and fifth onsets by 1/24th of a cycle.

The Cuban ethnomusicologist Rolando Pérez Fernández hypothesized that the African ternary rhythms were binarized by means of cultural blending caused by human migrations, and that the ternary fume-fume rhythm was, thus, converted to the binary clave son rhythm.[*] One might wonder if such a hypothetical perceptually and culturally based transformation may be translated into distance geometric terms. At first glance, one might even venture an intuitive guess that the onsets in the ternary 12-pulse clock should snap to the nearest pulses of the binary 16-pulse clock.[†] To obtain insight into the nature of such a geometric process, refer to Figure 12.2, where the two clocks are drawn one inside the other. From these diagrams, it may be observed that snapping the ternary onsets to their nearest binary pulses does not convert the fume-fume timeline to the clave son, nor the ternary bembé to its binary counterpart. Instead, one possible snapping rule that yields the desired result is that (1) if the onset of the ternary rhythm coincides with a pulse of the binary rhythm, then the onset stays where it is, and (2) if the onset of the ternary rhythm falls anywhere in between two binary pulses then it snaps to the binary pulse that follows it (rounding up). These operations are indicated in Figure 12.2 with arrows pointing from the onsets of the binary rhythms to the corresponding pulses of the ternary cycles.[‡] This

[*] Pérez-Fernández, R. A. (1986), p. 105. That the fume-fume was binarized to the clave son when it emmigrated to the new world from West Africa is only one possible historical scenario. It is also possible that the clave son, already established in Baghdad in the thirteenth century, was exported to West Africa, where it mutated to the ternary fume-fume. Yet a third possibility is that the binary and ternary forms of this pattern were born independently in different places.

[†] In the context of music transcription systems, Nauert, P. (1994), p. 229, calls this type of snapping *fixed quantization*. For a completely different approach to quantization that uses Bayesian decision theory, see Cemgil, A. T., Desain, P., and Kappen, B. (2000). For a recent original and promising quantization algorithm that uses near division, see Murphy, D. (2011).

[‡] Loy, G. (2007), p. 31. The snapping operation is a special case of the more general concept of rhythm quantization. See Gómez, F., Khoury, I., Kienzle, J., McLeish, E., Melvin, A., Pérez-Fernandez, R., Rappaport, D., and Toussaint, G. T. (2007) for a more detailed analysis of the snapping rules applied to binarization of ternary rhythms, and ternarization of binary rhythms.

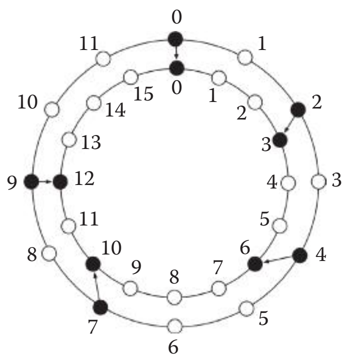

FIGURE 12.2 Binarization of the *fume-fume* timeline to the *clave son* (left) and the ternary *bembé* to its binary counterpart (right), obtained by rounding up.

example appears to challenge the efficacy of the nearest pulse hypothesis as a geometric model of the perceptual process underlying the binarization of the fume-fume, and rhythm quantization in general. But, before speculating further on this issue, it should be noted that there exists an equivalence relation between the nearest integer of a number, and rounding that number down to the next integer, as explained below.

Figure 12.3 shows a circle with 16-unit pulses marked on it. Each unit is also divided into three smaller intervals for convenience. Consider the two points on the circle corresponding to the arrows a and b, occurring at locations 0.67 and 4.33. Rounding these two numbers to their nearest integers takes a to 1.0 and b to 4.0. Now add 0.5 to both numbers to obtain $a + 0.5$ and $b + 0.5$, occurring at locations 1.17 and 4.83, respectively. Rounding down these two numbers to the next lowest integer also yields the numbers 1.0 and 4.0, respectively. This will always happen as long as the notion of nearest is well defined, that is, as long as an attack does not occur exactly halfway between two pulses.*

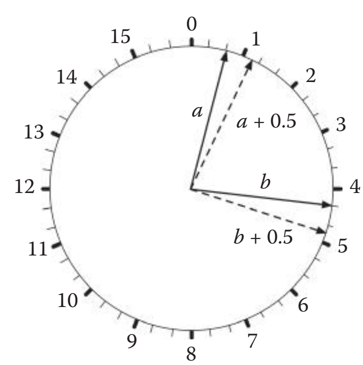

FIGURE 12.3 Rounding a number x to the *nearest* integer is the same as rounding *down* the number $x + 0.5$.

* Cargal, J. M. (1988), Chapter 3, p. 2.

The above observations about the problems incurred when an attack of the ternary rhythm lies either exactly on a pulse or halfway in between two pulses of the 16-point lattice, makes one wonder what happens with all the remaining infinitude of positions at which a ternary rhythm may lie. How many different nearest-pulse binarizations of the fume-fume exist when none of its onsets lies either on a pulse or halfway between two adjacent pulses in the 16-point lattice? Is the clave son one of these binarizations? How good are the others? And how are they related to the clave son? The answers to these questions may be obtained by examining Figure 12.4. For convenience, the circle is partitioned into 48 integers, making up the 16 binary pulses required for binarization. This provides two fiducial points (indicated by thin marks) in between every pair of adjacent pulses in the 16-pulse cycle. The 16 pulses are shown in bold lines, at 0, 3, 6, 9, and so on. In the top left diagram, the fume-fume (in dashed lines) is positioned in its standard mode at pulses 0, 8, 16, 28, and 36, to give the duration pattern [2-2-3-2-3]. The binarization of this mode has attacks at positions 0, 9, 15, 27, and 36, yielding the duration pattern [3-2-4-3-4]. Proceeding from left to right and top to bottom, the remaining four diagrams each shows the fume-fume rotating clockwise by 1/48th of a cycle. In all the diagrams, the rhythms obtained by snapping the fume-fume rhythm using the nearest-pulse rule are indicated with white circles connected by solid lines. When the fume-fume is rotated by 1/48th of a cycle to positions 1, 9, 17, 29, and 37, as in the upper right diagram, the binarization obtained is the clave son at positions 0, 9, 18, 30, and 36 with duration pattern [3-3-4-2-4].* Rotating the fume-fume another 1/48th of a cycle to positions 2, 10, 18, 30, and 38 yields, in the middle left diagram, the binarization at positions 3, 9, 18, 30, and 39, with duration pattern [2-3-4-3-4]. Rotating the fume-fume again another 1/48th of a cycle to positions 3, 11, 19, 31, and 39 yields, in the middle right diagram, the binarization at positions 3, 12, 18, 30, and 39 having duration pattern [3-2-4-3-4]. Finally, rotating the fume-fume 1/48th of a cycle to positions 4, 12, 20, 32, and 40 yields for a second time, at the bottom diagram, the binarization at positions 3, 12, 21, 33, and 39 with duration pattern [3-3-4-2-4]. Note that it is not necessary to rotate the fume-fume one complete revolution around the circle. By the rotational symmetry of the 16-pulse cycle, once a particular binarization is obtained for a second time, the set of binarizations thus far observed will be repeated. Indeed, the fourth pattern in the series is already a rotation of the first, with duration pattern [3-2-4-3-4]. Even so, since each rotation always takes the fume-fume to a position in which at least one of its onsets coincides with a pulse, the reader may wonder if some binarizations could be missed by not considering the rotations for which no onsets lie on a pulse and none lie halfway between two consecutive pulses. A rhythm with these properties is said to be in *general position*. By contrast, a configuration of a rhythm in which at least onset coincides with a lattice pulse will be called *anchored*.

First note that if the fume-fume is in general position, each onset will have its nearest pulse located either ahead of it (clockwise) or behind it (counterclockwise). Now assume that the rhythm is rotated in a clockwise direction by some distance d^*, which is small enough so that no onset crosses over a pulse. Then each onset will be rotated by that same

* Any smaller but positive rotation will also yield the clave son as a binarization.

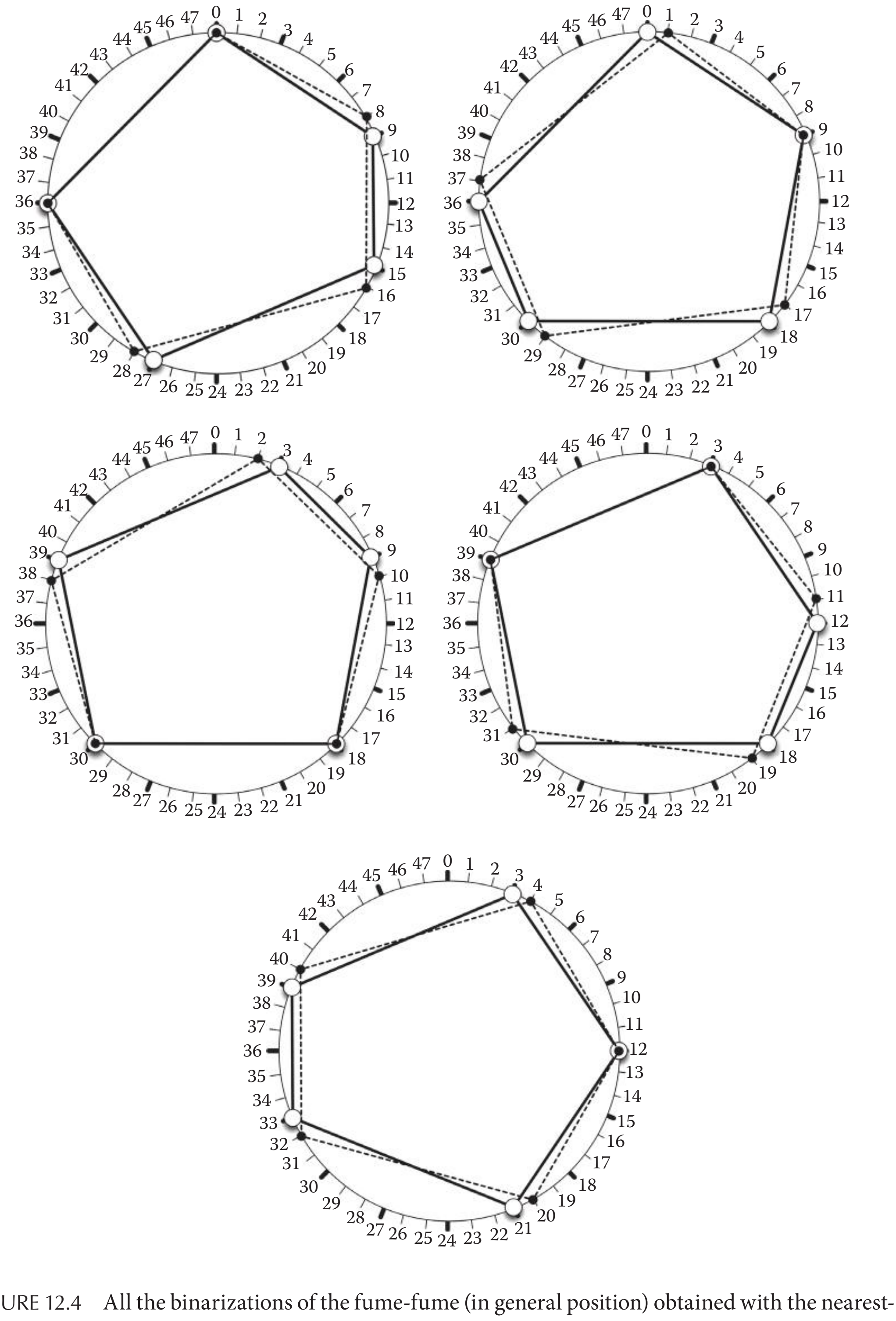

FIGURE 12.4 All the binarizations of the fume-fume (in general position) obtained with the nearest-pulse snapping rule.

distance d^*. Furthermore, the distances of those onsets with clockwise nearest pulses will decrease by d^*, and those of onsets with counterclockwise nearest pulses will increase by d^*. Therefore, for rhythms with an odd number of onsets, such as the fume-fume, the sum of the increases cannot cancel out the sum of the decreases. This means that rotating in one of the two directions will cause the distance between the rhythm and its binarization to decrease, implying that a rhythm in general position cannot realize the minimum distance to its binarization. This means that to search for the nearest binarization to the fume-fume, it is sufficient to consider only anchored configurations. Consider, for example, the top-left rhythm in Figure 12.4. If the fume-fume is rotated clockwise by d^*, the distances will increase by d^* at pulses 16, 28, 36, and 0, and decrease by d^* at pulse 8, for an overall increase of $3d^*$. On the other hand, if the fume-fume is rotated in a counterclockwise direction by the same amount then the distances will increase by d^* at pulses 0, 36, and 8, and decrease by d^* at pulses 28 and 16, for an overall increase of d^*. Since rotating the rhythm in both directions increases the distance, we conclude that this binarization yields a local minimum in the distance function. Turning to the binarization in the upper right of Figure 12.4, where the fume-fume has one onset coincident with the lattice pulse at position 9, it may be observed that rotating the fume-fume either clockwise or counterclockwise by d^* causes in both directions a distance change of $3d^* - 2d^*$, for a net increase of d^*. Therefore, the clave son also realizes a local minimum in the distance function. The smallest of all the local minima obtained in this way is the global minimum that we are looking for, and in this case it does not correspond to the clave son.*

Therefore, we conclude that there exist only three different binarizations of the fume-fume, modulo rotation: [3-2-4-3-4], [3-3-4-2-4], and [2-3-4-3-4]. Furthermore, the first and third binarizations may be considered the same in the following sense. The first binarization starting at location zero has durations [3-2-4-3-4], but the third binarization has the same duration pattern [3-2-4-3-4] if it is started at position 18, and is played backwards, that is, in a counterclockwise direction. Therefore, the two binarizations obtained, aside from the clave son, are the same, modulo rotations and mirror image reflections. The topic of rhythm equivalence with respect to rotations and mirror image reflections will be explored further in the chapter on necklaces and bracelets.

Given that for the fume-fume rhythm the nearest-pulse snapping rule yields three binarizations (modulo rotations), it remains to explore geometrical explanations of how the clave son came to be chosen as the "authentic" binarization, as claimed by Rolando Pérez

* The general problem of finding such a matching between two sets of points on a line or circle is called a bijection in computer science, and has received a lot of attention in the field of operations research, and more recently music theory as well. Werman, M., Peleg, S., Melter, R., and Kong, T. Y. (1986) were the first to find the optimal solution described here, that is, that the rotation and matching that minimize the sum of the absolute values of the differences between all the pairs of matched points may be obtained by restricting the search to configurations in which pairs of points from both sets coincide. Chen, H. C. and Wong, A. K. C. (1983) actually used the same procedure a few years earlier, but believed it was a suboptimal approximation. There is a difference worth pointing out between the minimum matching sought by these authors and the problem in this chapter. In their problem, the set that is shifted never changes. However, in the binarization problem considered here, the set of points for which a matching is sought changes during the shift according to how the nearest pulses change. Therefore, their problem applies to those rotation intervals for which the nearest pulses remain fixed. For the application of these concepts to the theory of musical chords, and voice-leading, the reader is referred to Tymoczko, D. (2006) and the references therein.

Fernández. One compelling argument would be obtained by showing that of these three binarizations, the clave son is most similar to the fume-fume. This leads us to broach the vast field concerned with measuring the similarity between rhythms, a topic about which volumes has been written, and which we will revisit in more detail in a later chapter.[*] For now let us consider the simple measure suggested above: the sum of the absolute values of the differences between the location coordinates of the attacks of the fume-fume and the corresponding snapped attacks of its binarizations.

As pointed out above, this measure of distance does not realize its minimum value for the clave son, but rather for the two other binarizations, each of which has a distance of three from the fume-fume, whereas the distance to the son is four. One might suspect that the reason the hoped-for answer is not obtained is because this distance measure is not the "right" one for the task at hand. This distance measure is after all a special case of a large general family of measures called Minkowski metrics.[†] Consider two rhythms X and Y consisting of d attacks each. Let the circular arc coordinates of the attacks of rhythm X be $x_1, x_2, \ldots, x_d$, and of rhythm Y be $y_1, y_2, \ldots, y_d$. Then the Minkowski metric between rhythms X and Y is given by

$$d_p(X, Y) = \left(|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_d - y_d|^p\right)^{1/p}$$

where $1 \le p \le \infty$. In other words, the discrepancies between each corresponding pair of attack points are raised to the power of p, added together, and finally the p-th root of the resulting sum is taken. For the case when $p = 1$, and $d = 5$, the Minkowski metric reduces to the distance measure used above, that is, the sum of the absolute values of the five differences. When $p = 1$, the Minkowski metric is known by several popular names, including *Manhattan* metric, *city-block* distance, and *taxi-cab* distance.[‡] One might wonder if for a value of p different from one, the Minkowski measure of distance would favor the clave son binarization over the other two. The two most popular alternate values of p used in practice are 2 and ∞. When $p = 2$, the Minkowski metric becomes the ubiquitous Euclidean distance, and for $p = \infty$ it is called the *sup* metric[§] because it can be shown[¶] that as p becomes infinitely large

$$\left(|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_d - y_d|^p\right)^{1/p} = \max\left\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_d - y_d|\right\}.$$

[*] For a sampling of the literature on measuring rhythm similarity, refer to the following papers: Antonopoulos, I., Pikrakis, A., Theodoridis, S., Cornelis, O., Moelants, D., and Leman, M. (2007), Bello, J. P. (2011), Berenzweig, A., Logan, B., Ellis, D. P. W., and Whitman, B. (2004), Gabrielson, A. (1973a), Gabrielson, A. (1973b), Guastavino, C., Gómez, F., Toussaint, G. T., Marandola, F., and Gómez, E. (2009), Hofmann-Engl, L. (2002), Orpen, K. S. and Huron, D. (1992), Post, O. and Toussaint, G. T. (2011), Takeda, M. (2001), Toussaint, G. T. (2006b), Toussaint, G. T., Campbell, M., and Brown, N. (2011). Polansky, L. (1996) provides a survey of a plethora of similarity metrics for use in music. A general theory of similarity of chords based on submajorization, recently developed by Hall, R. W. and Tymoczko, D. (2012) may have interesting consequences for measuring rhythm similarity as well. Tversky, A. (1977) considers the broader issue of similarity as a psychological construct.

[†] Beckenbach, E. and Bellman, R. (1961), p. 103, Toussaint, G. T. (1970).

[‡] Krause, E. F. (1975), Reinhardt, C. (2005).

[§] Schönemann, P. H. (1983), p. 314.

[¶] Beckenbach and Bellman, *op. cit.*, p. 105.

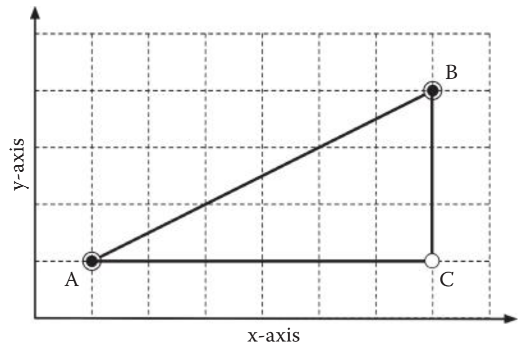

FIGURE 12.5 The Minkowski distances between points A and B with $p = 1$, 2, and ∞.

These three Minkowski metrics have the following natural geometric interpretations illustrated in Figure 12.5, where it is desired to measure the distance between two points A and B. The popular Euclidean distance between A and B is of course the length of the straight line from A to B. If A and B were two street corners in Manhattan, then with a helicopter one could travel such a straight line over the buildings. The city-block distance between A and B is the sum of the horizontal length AC, plus the vertical length CB. This corresponds to the minimum distance of a car would travel along two-way streets and avenues, to get from A to B. The sup metric distance with $p = \infty$ is the maximum of the horizontal length AC and the vertical length CB. In this case, the maximum is AC. There exist applications where this measure of distance is preferred over the others for different reasons. Examples include classification in data mining,* efficient warehousing in operations research, a class of problems similar to computing the shortest time for a mechanical plotter to draw a figure,† and optimizing geometric reconstruction problems in computer vision.‡

Some insight concerning the nature of the Minkowski metrics may be obtained by examining the shape of their unit "circles." Recall that a unit circle with center O is the locus of points all of which are at distance one from O. With the Euclidean distance, the unit circle is the round circle we know so well. Some unit "circles" for several Minkowski metrics are shown in Figure 12.6. The unit circle for $p = \infty$ is a square, for $p = 1$, it is a diamond, and for other values of $p > 1$, it is a convex curve that lies somewhere in between the square and diamond.

Tenney and Polansky (1980) experimented with the Euclidean and city-block distance measures for combining different features of music, and write that: "A definitive answer

* Anand, A., Wilkinson, L., and Tuan, D. N. (2009).

† Langevin, A. and Riopel, D., Eds. (2005), p. 96. Consider a computer plotter that has to make a large complicated technical drawing consisting of many lines consisting of start and end points. The ink-head must travel to all start points and trace the lines until the end points are reached. The ink-head is attached to motors on the sides of the table by cables. The motors pull the ink-head at a constant speed along both horizontal and vertical directions. Therefore the time taken for the ink-head to travel from point A to point B is determined by the maximum of the horizontal and vertical directions that the ink-head must travel.

‡ Hartley, R. I. and Schaffalitzky, F. (2004) compared the sup metric with the Euclidean distance to solve the computer vision problem of motion recovery from omnidirectional cameras. They found that the sup metric had the advantage of lower computational cost, but the drawback of being too sensitive to outliers in the data.

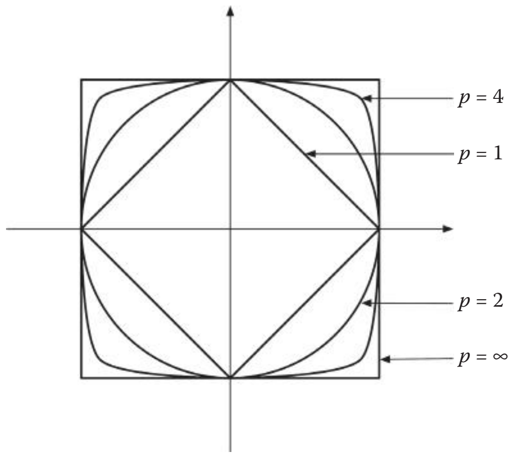

FIGURE 12.6 The Minkowski metric unit disks for a few values of p.

to the question as to which of these metrics is the most appropriate to our musical 'space' would depend on the results of psychoacoustic experiments."* However, in the absence of such experimental knowledge, they argue that the city-block distance is preferable since it treats all dimensions equally. Furthermore, the results they obtained with the city-block distance were superior to those observed with the Euclidean distance. Similar results favoring the city-block distance have been obtained in the visual domain.† Indeed, the "Householder–Landahl" hypothesis states that the city-block distance captures the dissimilarity judgments that human subjects make when comparing physical shapes.‡

So, how do these other Minkowski metrics compare in the case of our problem of finding which of the three anchored binarizations of fume-fume is closest to the clave son? From Figure 12.4, we can observe that the Euclidean distance between the fume-fume and the clave son is $[(0)^2 + (1)^2 + (1)^2 + (1)^2 + (1)^2]^{1/2} = [4]^{1/2} = 2$. On the other hand, the Euclidean distance between fume-fume and the other two binarizations is $[(0)^2 + (1)^2 + (1)^2 + (1)^2 + (0)^2]^{1/2} = [3]^{1/2} = 1.732$. So, here again the clave son loses out over the other binarizations.§

It remains to examine the Minkowski metric for $p = \infty$, the sup metric. Again, referring to Figure 12.4, we observe that now all the three binarizations are equally distant from the fume-fume, since $max\{0, 1, 1, 1, 1\} = max\{0, 1, 1, 1, 0\} = 1$. So for this metric, the three binarizations are tied, showing some promise. Although the son is not favored over the two other binarizations, is it also not excluded, and we can break the tie by comparing their rhythmic contours. The fume-fume with duration pattern [2-2-3-2-3], and the clave son

* Tenney, J. and Polansky, L. (1980), p. 213.

† Sinha, P. and Russell, R. (2011) tested perceptual image similarity judgments of human subjects and found a consistent preference for images matched with the city-block distance over the Euclidean distance, leading them to conclude that this "metric may better capture human notions of image similarity."

‡ Schönemann, *op. cit.*, p. 312.

§ Note that the Euclidean and sup metrics are calculated here with respect to the anchored binarizations that realize the local minima of the city-block distance function. It has not yet been determined if using the Euclidean and sup metrics in the quantization algorithm will generate binarizations of the fume-fume that are not generated by city-block distance quantization.

with duration pattern [3-3-4-2-4], both have the same rhythmic contour [0, +, −, +, −]. On the other hand, the two other binarizations of the fume-fume, [3-2-4-3-4] and [2-3-4-3-4] have rhythmic contours [−, +, −, +, −] and [+, +, −, +, −], respectively, both of which differ on their first symbol with the son and fume-fume. Furthermore, this first symbol is arguably the most important symbol since it determines the first two inter-onset intervals that we hear, and it is important that these two intervals have the same duration.

In attempting to explain the perceptual mechanism by which ternary rhythms could be converted to their binary counterparts, two different geometric models have been uncovered for converting the ternary fume-fume to the binary clave son. The first model (Figure 12.2) assumes that the ternary rhythm starts at its normal position, with the first onset at pulse zero, and snaps the remaining onsets, as they unfold, to their anticipated (clockwise) binary pulse positions, to obtain the clave son in a direct manner. The second model (Figure 12.4) assumes that the ternary rhythm starts a little later, and snaps the remaining onsets, as they unfold, to their nearest binary pulse positions, by using the sup metric as the distance, leading to an ambiguity that is resolved by comparing the durations of the first two inter-onset intervals. Which of these geometric models best fits the perceptual mechanism at work will, in the end, have to be determined by psychological experiments. However, if the rule of Occam's razor is invoked, then the simpler first model should be selected. The second model, in addition to being rather complex, has the awkward feature that to compute the maximum of the five onset discrepancies the listener has to wait until the entire rhythm is heard. By contrast, the first model allows the listener to project the anticipated binary locations on the fly.

Syncopated Rhythms

SYNCOPATION IS THE SPICE OF RHYTHM. It adds surprise to an otherwise bland rhythm. We can easily feel when a rhythm has syncopation, but translating that feeling to mathematical terms, my general goal with all musical properties explored in this book is easier said than done. A prerequisite for making progress in this direction is a precise constructive definition. Consider how some dictionaries explain what syncopation is. The *New Oxford American Dictionary* defines a *syncopated* rhythm as one in which the "beats or accents" are displaced "so that strong beats become weak beats and vice versa." From the mathematical point of view, this definition is not very satisfactory because the notions of "strong" and "weak" beats have not been defined. The *Oxford Grove Music Online Dictionary* defines syncopation as: "The regular shifting of each beat in a measured pattern by the same amount ahead of or behind its normal position in that pattern." This definition also lacks mathematical rigor because the notion of "normal" has not been specified. The *Harvard Dictionary of Music* defines syncopation as "A momentary contradiction of the prevailing meter."* This definition assumes we know what meter is, but more problematically, how do we interpret the words "momentary" and "prevailing?" As a final example, consider the lesser-known on-line *Virginia Tech Multimedia Music Dictionary*; it defines syncopation as the "deliberate upsetting of the meter or pulse of a composition by means of a temporary shifting of the accent to a weak beat or an off-beat." Does this mean that if the shifting of the weak beat is not deliberate, there is no syncopation? Furthermore, what is the difference between a weak beat and an off-beat?

There must be more than 50 traditional definitions of syncopation adorning the pages of dictionaries, books, and Internet sites. Like the definitions offered here, most have their own particularities. However, to a mathematician, they all have one thing in common: vagueness. Of course this is not surprising considering that we are trying to define with precise mathematical tools a slippery human perceptual skill. We can all read text without effort, which implies that we recognize characters such as A, B, C, and so on, without difficulty. However, we do not know how to define what an "A" or a "B" is. This is a major

* Randel, D. M., Ed. (2003).

problem in artificial intelligence.[*] Indeed, in the words of Michael Keith, "although syncopation in music is relatively easy to perceive, it is more than a little difficult to define precisely."[†] Some readers may believe that the desire for mathematical precision is completely inappropriate here, and that such demands lead inevitably to irrelevance regarding the psychological aspects of music. On the contrary, I concur with the philosopher Mario Bunge that we should mathematize everything we can, and the only way to know if a fuzzy concept can be successfully modeled mathematically is to try.[‡] This is the *sine qua non* and the hallmark of artificial intelligence, and it pushes the boundaries of the relevant psychology of music. In spite of the difficulties that such a task may pose, it is possible to construct unambiguous mathematical definitions of notions that may be used as useful models that replace the traditional concept of syncopation. Furthermore, in due time, as the scientific and technological approaches of the study of music continue to expand, some of these mathematical versions of syncopation may supplant the traditional notion. Syncopation is very much a Western concept, and for some types of music, new mathematical substitutes for syncopation, which are not culturally dependent, may be more appropriate and useful.[§] Referring to sub-Saharan music, Simha Arom states the case more bluntly: "terms such as … syncope … should be dispensed with as foreign to it."[¶] Jay Rahn reflects this evolving terminology by offering two definitions of syncopated, one descriptive of the Western culture, and another mathematically inspired.[**] His first definition of syncopated is "deviating from an oriental metrical organization in that one or both of the immediately adjacent presented moments to a given moment is not resolved." Here, the term *moment* is used to mean an "irreducible portion of time." His second definition of syncopated is "not commetric." The term "commetric" here is synonymous with regular, a well-defined mathematical notion. Thus, Rahn's second definition of a syncopated rhythm is one that is irregular.

In 1996, Fred Lerdahl and Ray Jackendoff published a book titled *A Generative Theory of Tonal Music*, in which they proposed a hierarchy of accents for musical rhythm inspired by research work in linguistics.[††] For a timeline of 16 pulses, their hierarchy of accents or metrical weights may be expressed using the graph shown in Figure 13.1. One way to construct this graph is as follows. First, starting at pulse zero, and proceeding from left to right, assign a weight of one to every pulse (shown as shaded boxes). Second, in a similar manner, increment by one the weight of every second pulse. Third, increment by one every fourth pulse. Next, increment by one every eighth pulse, and finally every 16th pulse. The resulting height of the column at any pulse gives the weight or degree of accent given to an onset that occurs at that pulse. In other words, the pulse emphasized most strongly is pulse

[*] Longuet-Higgins, H. C., Webber, B., Cameron, W., Bundy, A., Hudson, R., Hudson, L., Ziman, J., Sloman, A., Sharples, M., and Dennett, D. (1994).

[†] Keith, M. (1991), p. 133.

[‡] Mahner, M., Ed. (2001).

[§] Vurkaç, M. (2012), for example, finds the mathematical definition of *off-beatness* (to be explored later in the book) more useful than syncopation, for the analysis of rhythms in traditional contexts.

[¶] Arom, S. (1991), p. 183.

[**] Rahn, J. (1983), p. 248.

[††] Lerdahl, F. and Jackendoff, R. (1983).

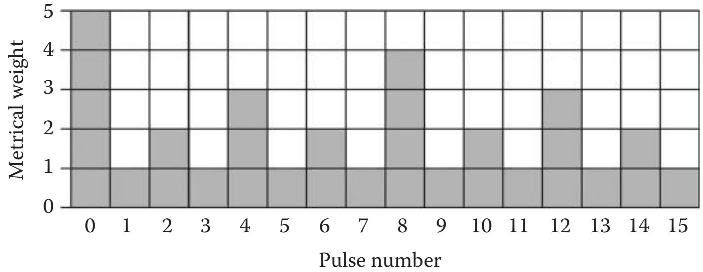

FIGURE 13.1 The metrical hierarchy of Lerdahl and Jackendoff.

zero with a weight of five. The next most salient pulse is number eight with a weight of four. Pulses 4 and 12 have a weight of three, pulses 2, 6, 10, and 14, have a weight of two, and all the remaining (odd-numbered) pulses have a weight of one.

This metrical hierarchy may be used to design a precise mathematical definition of syncopation, which we shall call *metrical complexity*, as follows.* Consider the clave son timeline shown in Figure 13.2 in box notation directly below the metrical hierarchy. The clave son consists of onsets at pulses 0, 3, 6, 10, and 12, with metrical weights equal to five, one, two, two, and three, respectively. These metrical weights express how normal or typical it is for a beat to occur at that pulse location according to the theory of Western music practice expressed by Lerdahl and Jackendoff. Therefore, the lower the weight for an onset, the more unexpected the onset, and thus the more syncopated it is as well. For the clave son, the onset with the lowest weight is the second onset occurring at pulse three that has a weight of one. Therefore, this onset is considered to be the most syncopated of the five onsets. Interestingly, in some Latin music such as *salsa*, a rhythm that accentuates this

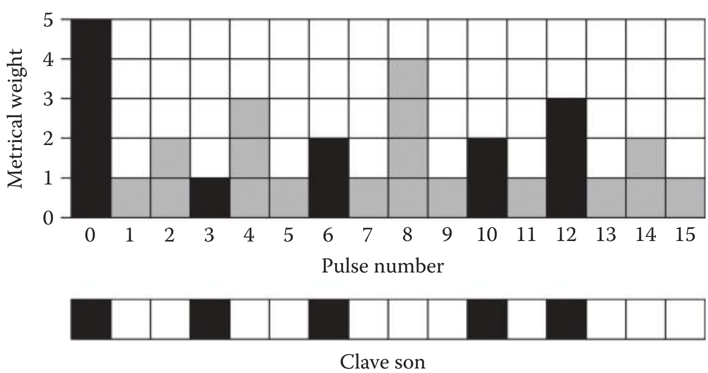

FIGURE 13.2 The metrical complexity of the clave son.

* This measure of syncopation was first proposed in Toussaint, G. T. (2002). In a subsequent study by Thul, E. and Toussaint, G. T. (2008a), it was compared with a large group of measures of rhythm complexity, irregularity, and syncopation, against human judgments of performance and perceptual complexity, and gave superlative performance. Flanagan, P. (2008) proposes a mathematical measure of syncopation that computes an average with respect to many possible underlying meters.

second onset of the clave son is called *bombó*, also the name of a bass drum used in the Afro-Cuban *comparsa* music that is played in carnivals.*

To measure the total metrical expectedness (or *simplicity*) of the rhythm, we may add the metrical weights of all its onsets. Thus, for the clave son, the metrical expectedness is equal to 13. To convert this measure to a measure of *metrical complexity* or *syncopation*, it suffices to subtract the metrical expectedness value of a given rhythm with *k* onsets and *n* pulses from the maximum possible value that any rhythm with *k* onsets and *n* pulses may have. For a rhythm with five onsets and 16 pulses, the maximum expectedness value is 17, obtained by summing the column heights at pulses 0, 8, 4, 12, and any one of 2, 6, 10, and 14. This value is realized by several rhythms, including the popular classical music ostinato rhythm [4-4-2-2-4] with onsets at pulses 0, 4, 8, 10, and 12. Thus, the metrical complexity of the clave son is 17 − 13 = 4. For comparison, the more syncopated clave rumba that has its third onset at pulse number seven has a metrical complexity equal to 17 − 12 = 5.

KEITH'S MEASURE OF SYNCOPATION

Michael Keith proposed a mathematical measure of syncopation in the context of sustained musical notes that is defined by onsets as well as offsets.† Recall that the onset is the point in time at which a note starts sounding, and an offset is the point at which it stops sounding. Keith's definition of syncopation is based on three types of events he calls *hesitation*, *anticipation*, and *syncopation*.‡ Although in the strict sense he reserves the term *syncopation* for the more restricted situation in which a note exhibits both hesitation and anticipation, his general measure of syncopation encompasses a weighted combination of all three properties. Figure 13.3 illustrates these events for the special case in which a rhythmic cycle (or measure) contains four fundamental beats at pulses zero, two, four, and six. The leftmost illustration shows an example without syncopation: the note starts and ends at two fundamental beats, in this case zero and two, respectively. The second diagram

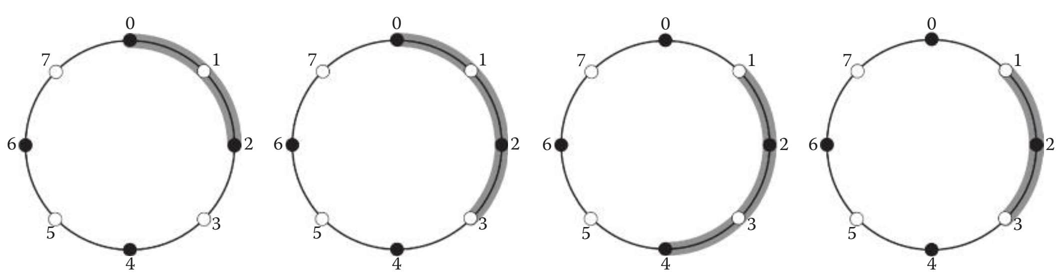

FIGURE 13.3 From left to right, no syncopation, *hesitation*, *anticipation*, and *syncopation*.

* Uribe, E. (1996), p. 49. Therefore, in this music as well as the *rumba* styles, it is not uncommon to consider that the rhythmic phrase starts on the second attack of the clave son, rather than the first. Morales, E. (2003), p. 174 quotes from I. Leymaire's book *Cuban Fire* that the *bombó* attack "falls on the second quarter note of the second bar." This would seem to contradict Uribe. Morales does not explicitly notate the rhyhms, but Leymaire is probably referring to the 2–3 version of the clave son in which the two bars are reversed as in [. . x . x . . . x . . **x** . . x .], where the *bombó* attack is shown in bold.

† Keith, *op. cit*., pp. 134–135.

‡ For additional types of syncopations and extensions, see Gatty, R. (1912), p. 370 and Smith, L. and Honing, H. (2006).

shows an example of *hesitation*: a note starts on a fundamental beat and ends *off the beat*, or in between two fundamental beats, here pulses zero and three, respectively. The third diagram, in which the note starts in between two beats (*off the beat*) and ends on a beat, here pulses one and four, respectively, is called *anticipation*. Finally, the rightmost diagram exhibits an example of genuine *syncopation*: both the start and end points of the note occur in between two beats, here at pulses one and three, respectively.

To construct his weighted general measure of syncopation, Keith assigns to *hesitation* a weight of one. He considers anticipation to be a stronger form of syncopation than hesitation, and therefore gives *anticipation* a weight of two. Finally, since syncopation combines both hesitation and anticipation, he adds these two weights together to obtain a weight of three for *syncopation*. It remains to define precisely what Keith means for an onset or offset to be "off the beat." To keep things simple, Keith restricts the definition of "off the beat" to hold only for cycles in which the total number of pulses n is a power of two, such as $n = 4 = 2^2$, $n = 8 = 2^3$, $n = 16 = 2^4$, and $n = 32 = 2^5$. This power is a parameter called d, and it is chosen so that 2^d is small enough to be able to identify the smallest inter-onset intervals necessary to specify the granularity (resolution) of the rhythm. For example, the tresillo has intervals [3-3-2], which makes $n = 8$ and $d = 3$. On the other hand, the clave son has intervals [3-3-4-2-4], making $n = 16$ and $d = 4$.

Denote by δ the duration of a note (in terms of the number of pulses that occur from onset to offset) as a multiple of $1/2^d$, and let S be the time coordinate at which the note starts (the onset). Furthermore, let D denotes the value of δ rounded down to the nearest power of two. Then the onset of the note is defined to be "off the beat" if S is not a multiple of D. Similarly, the offset of the note is defined to be "off the beat" if $(S + \delta)$ is not a multiple of D. The syncopation value for the ith note in the rhythmic pattern, denoted by s_i is defined as $s_i = 2$ (if the onset is off the beat) + 1 (if the offset is off the beat). Finally, the overall measure of syncopation of the rhythmic pattern is the sum of the syncopation values s_i summed over all i.

Keith's measure of syncopation is defined in the context of sustained notes that start and end at positions anywhere in the cycle. In this book, on the other hand, the rhythmic patterns consist of sounds that have extremely short durations, and that for all practical purposes, as well as theoretical analysis, are considered as attacks with zero duration. Therefore, it is assumed here that the offsets do not exist, and as a consequence no offset can be "off the beat." This implies that the second term in Keith's syncopation value for a "note" may be dropped altogether. Furthermore, the weight of two for the first term may now be changed to one, since it is no longer necessary to emphasize that an *onset* that is "off the beat" is twice as important as an *offset* that is "off the beat." This simplifies the computation of the syncopation value of a rhythmic onset: it is one if the onset is "off the beat," and zero otherwise. It is instructive as an example to walk through these computations for a couple of rhythms, a popular ostinato from classical music, which is considered to be unsyncopated, and the clave son, which is syncopated (refer to Figure 13.4).

The terms S, D, and δ in Figure 13.4 have already been defined. The rhythmic pattern is denoted by R, and K_i represents the syncopation value of the onset present at coordinate

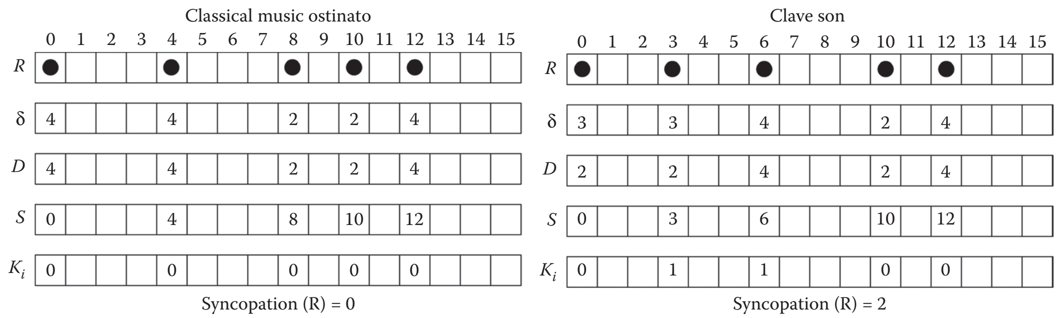

FIGURE 13.4 Illustrating the calculation of Keith's syncopation measure.

value i. First, consider the classical music ostinato on the left. Since for all five onsets the value of S is a multiple of D, by zero, one, four, five, and three, respectively, no onset is considered to be "off the beat." Therefore, the syncopation value for each onset is zero, and the overall syncopation value of the rhythm is also zero. On the other hand, in the case of the clave son on the right, for the second onset, three is not a multiple of two, and for the third onset, six is not a multiple of four. Therefore, both K_3 and K_6 take on a value of one, and the overall syncopation value of the clave son is two. Musicologists would unanimously agree, that in the context of a four-beat underlying meter, the clave son is more syncopated than the classical music ostinato, and thus in this particular case, Keith's measure correctly captures the human judgments.

Necklaces and Bracelets

Consider the *bembé* timeline played in the usual sub-Saharan African context of an underlying meter that places an accent at every third pulse starting with pulse zero. In Figure 14.1, these four metrically strong pulses {0, 3, 6, 9} are indicated by the vertical and horizontal lines (a four-beat measure). Note that the bembé (left) has attacks on the first and last metric accents at positions zero and nine. Examine what happens when this rhythm is rotated clockwise by one pulse so that the new rhythm starts on the last onset of the bembé, as shown in Figure 14.1 (right). The new rhythm contains attacks on the first, second, and third metrically strong pulses at positions zero, three, and six. This is a considerable change, and not surprisingly, if I play this rhythm on a bell with my hands, while playing a bass drum with my foot on pulses {0, 3, 6, 9}, the new rhythm sounds and feels quite different from the bembé. Indeed, it still feels considerably different even if I do not play the bass drum, and just mentally partition the cycle into a [3-3-3-3] metric subdivision. Therefore, from the point of view of music-making, we may consider that these two rhythms are different. However, it is obvious that the interval contents of these two durational patterns and their resulting histograms are identical, since the interval content of a rhythm is invariant to its rotations. Therefore, from certain analytical perspectives,

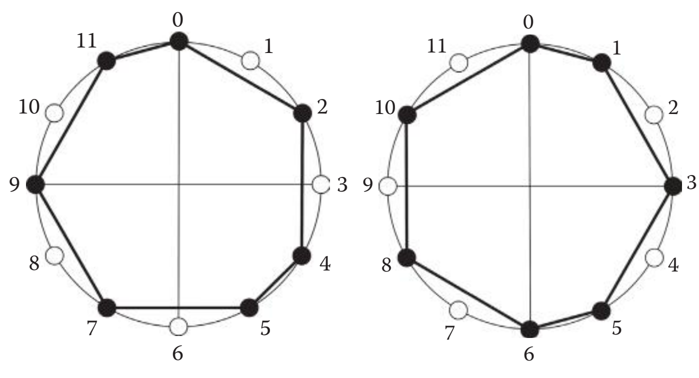

FIGURE 14.1 The bembé timeline (left) and its clockwise rotation by one pulse (right).

the two rhythms may be considered to be the same. In the mathematical field of combinatorics, the two rhythms in Figure 14.1 are said to be instances of the same *necklace*.* In the pitch domain in music theory, a necklace corresponds to a *chord type*.†

A necklace is a closed string of beads (or pearls) of different colors, such as one might wear around one's neck. We are interested here in *binary* necklaces, that is, necklaces with pearls of two colors: black and white. Two necklaces are considered to be the same if one can be *rotated* so that the colors of its beads correspond, one-to-one, with the colors of the beads of the other necklace. Figure 14.2 shows two more instances of identical necklaces. The rhythm on the right is obtained by rotating the one on the left clockwise by three pulses.

Obviously, it is possible to have two rhythms that are not instances of the same necklace and that still have the same interval content, namely, if one rhythm is the mirror image of the other. To include such cases, we use the mathematical term *bracelet*. In other words, two bracelets are considered to be the same if one of them can be *rotated* or *turned over* so that the colors of their beads are brought into one-to-one correspondence. Figure 14.3 shows two rhythms that are not the same necklace, but they are the same bracelet: the rhythm on the right is a mirror image reflection about a vertical axis through pulses zero and six, of the rhythm on the left. In the pitch domain in music theory, a necklace corresponds to a *chord*.‡ This chapter presents some of the most well-known rhythmic necklaces and bracelets used as timelines in music around the world.

One way to measure the robustness of the effectiveness of a necklace as a template for the design of rhythm timelines is by the number of its rotations that are actually used in practice. If many rotations are used, it suggests that the effectiveness of the rhythms it generates does not depend crucially on the starting onset, even though the result may sound

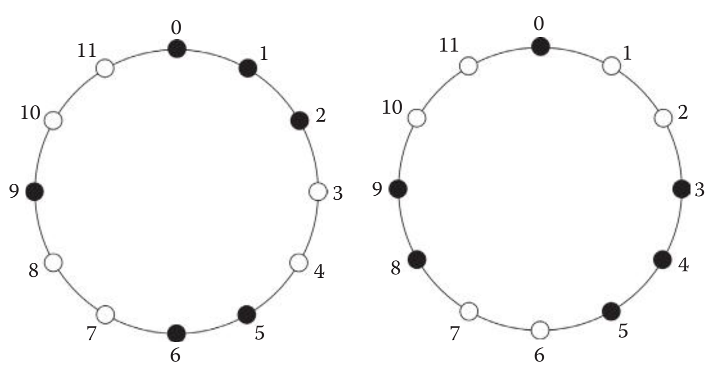

FIGURE 14.2 Two instances of the same *necklace*.

* Keith, M. (1991), p. 15. A useful algorithmic tool for studying rhythm necklaces is an algorithm for generating them. Ruskey, F. and Sawada, J. (1999) describe an efficient algorithm that, when given the number of pulses and attacks, generates all possible necklaces, in execution time proportional to the number of necklaces generated.

† Tymoczko, D. (2011), p. 38.

‡ *Ibid*, p. 39.

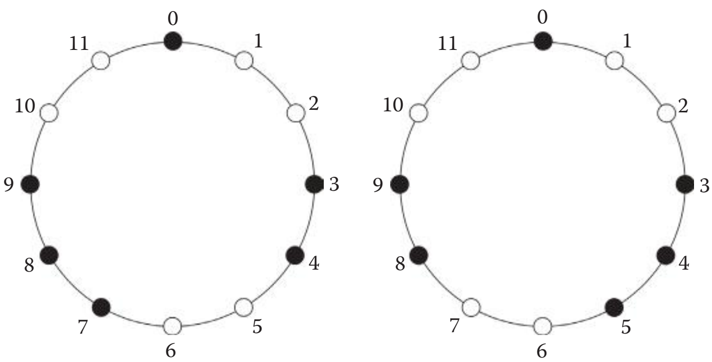

FIGURE 14.3 Two instances of the same *bracelet*.

quite different. A rhythm necklace that has the property that all its onset rotations are used as timelines in practice will be called a *robust* rhythm necklace.

The tresillo timeline, already introduced in Chapter 3, with durational pattern [3-3-2] shown in Figure 14.4 (Rotation 0 on the left) is one instance of a robust rhythm necklace. Note that Rotation 2 is also the mirror image reflection of Rotation 0 about a vertical line through pulses zero and four. The tresillo timeline is used so often in traditional music all over the world that it may be regarded as a *universal* rhythm or cultural *meme*.* In India, it is one of the *talas* used in Karnatic music; in Central Africa, it is played by the Aka Pygmies with blades of metal; and in the United States, it is a common rhythm played on the banjo in *bluegrass* music. Musicologists consider it to be a signature rhythm of Renaissance music. Historically, it can be traced back to at least classical Greece under the name of *dochmiac* rhythm. Its Rotation 1 in the center of Figure 14.4 is common in Bulgaria, Turkey, and Korea, and its Rotation 2 (right) is the Nandon Bawaa bell pattern of the Dagarti people of

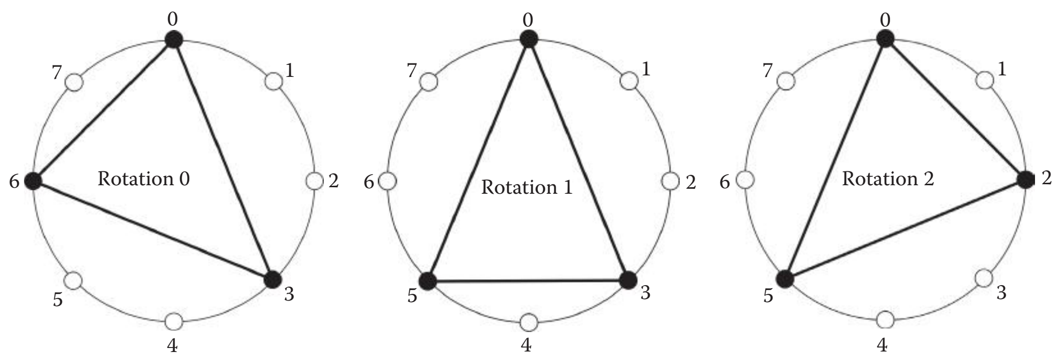

FIGURE 14.4 The *tresillo* timeline and its two onset rotations.

* Jan, S. (2007). Universal rhythms that exhibit symmetries such as mirror symmetry are instances of the principle of symmetry, a candidate for a more general *grand rhythmic universal*. Alexander Voloshinov (1996), p. 111, puts it this way: "Symmetry is a universal genetic constant, collectivizing each and every rhythm into the Rhythm par excellence."

Ghana. It is also found in other places such as Namibia in Southern Africa, and Bulgaria in Europe. By far, the most preferred of these rotations is Rotation 0. This can be explained by the fact that, unlike the other two, it creates in the listener an interesting broken expectation of a regular rhythm by starting with the pattern [3-3]. Although this necklace pattern is not as common in Southeast Asia as in the rest of the world, Rotations zero and two are found as drum patterns in Burma (Myanmar) and Cambodia.*

A robust rhythm necklace with four onsets among nine pulses is pictured in Figure 14.5. Rotation 0 is the Turkish *aksak* rhythm also found in Greece, Macedonia, and Bulgaria. Simha Arom made the surprising discovery, in one of his many excursions to Africa, that this rhythm is the timeline of a lullaby used in South-Western Zaïre. It is rather unusual to find nine-pulse rhythm timelines in sub-Saharan Africa. This traditional rhythm necklace has been incorporated into jazz as well as modern art music in the twentieth century. Dave Brubeck used this pattern as the meter in one of his best-selling compositions *Rondo a la Turk*.† Rotation 1 is used in Serbia as well as in Bulgaria. Rotation 2 is common in Bulgaria, Macedonia, and Greece. Rotation 3 is used in the traditional music of Turkey, and in modern Western music as the meter in *Strawberry Soup* composed by Don Ellis.

Although in the traditional music of Turkey all four of these rhythms are employed as timelines, there exists nevertheless a marked order of preference in terms of the frequency with which each pattern is used in practice that has been statistically observed and documented by ethnomusicologists. The most frequent of these is Rotation 0, followed in decreasing order by Rotation 3, Rotation 1, and Rotation 2. This preference may be explained in terms of Gestalt psychology principles. That Rotations 0 and 3 are preferred over the other two follows perhaps from the fact that rhythms are most easily perceived as starting or ending with the longest gap, in this case three pulses. Furthermore, Rotation 0 has a greater surprise value, or in technical terms, a more pronounced Gestalt *despatialization* effect,‡ due to the fact that the initial regular pattern [2-2-2] creates the expectation of the complete cycle [2-2-2-2], which is suddenly broken by the introduction of a three-pulse interval to yield the irregular rhythm [2-2-2-3]. This expectation of

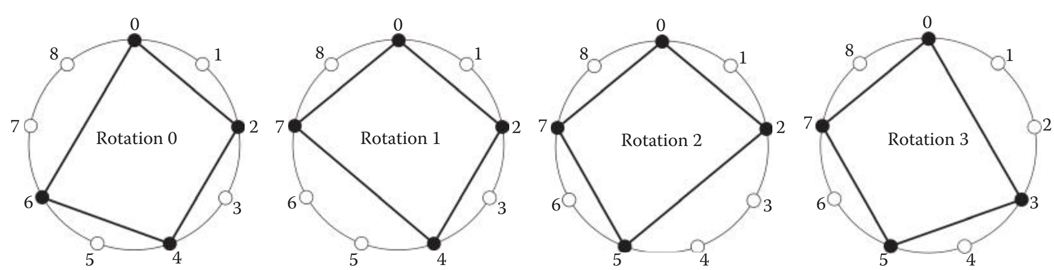

FIGURE 14.5 The *aksak* timeline and its three onset rotations.

* Becker, J. (1968), p. 186.
† London, J. (1995), p. 67.
‡ McLachlan, N. (2000).

a regular rhythm is not induced with Rotation 3, which starts with the irregular pattern [3-2]. This Gestalt despatialization effect also happens at the start of Rotation 1 with the pattern [2-2-3], but not with Rotation 2, thus perhaps explaining why the former is preferred over the latter.

Another well-known robust rhythm necklace is the five-onset, eight-pulse pattern shown in Figure 14.6. Rotation 0 is the well-known Cuban cinquillo pattern,* and like the tresillo, it is found in traditional music all over the world. Rotation 1 is a Middle Eastern popular rhythm, also called the *timini* in Senegal, the *adzogbo* in Benin, the *tango* in Spain, and the *maksum* in Egypt. Historically, it can be traced back to the thirteenth century Persia, where it went by the name *al-saghil-al-sani*. Rotation 2 is known as the *müsemmen* rhythm in Turkey. Rotation 3 is the *kromanti* timeline, popular in Surinam. Finally, Rotation 4 is the *lolo* timeline played in Guinea.

Figure 14.7 shows one of the most important families of ternary timelines used in sub-Saharan Africa. It consists of five onsets distributed among 12 pulses. Its most well-known representative is Rotation 0, common in West Africa but also used in the former Yugoslavia. In some places, it is called the *fume-fume*, and in others the *standard short pattern* or the African *signature* rhythm.† It is the ternary version of the clave son with inter-onset

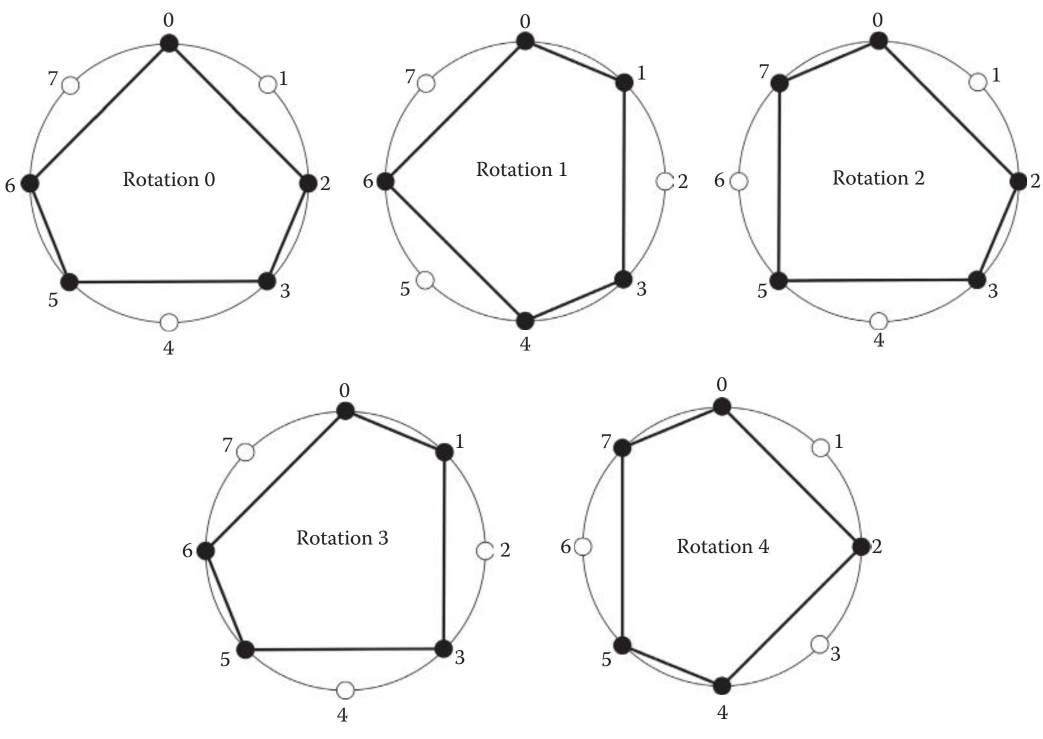

FIGURE 14.6 The cinquillo timeline (Rotation 0) and its four onset rotations.

* Manuel, P. with Bilby, K. and Largey, M. (2006), p. 40.

† Agawu, K. (2006).

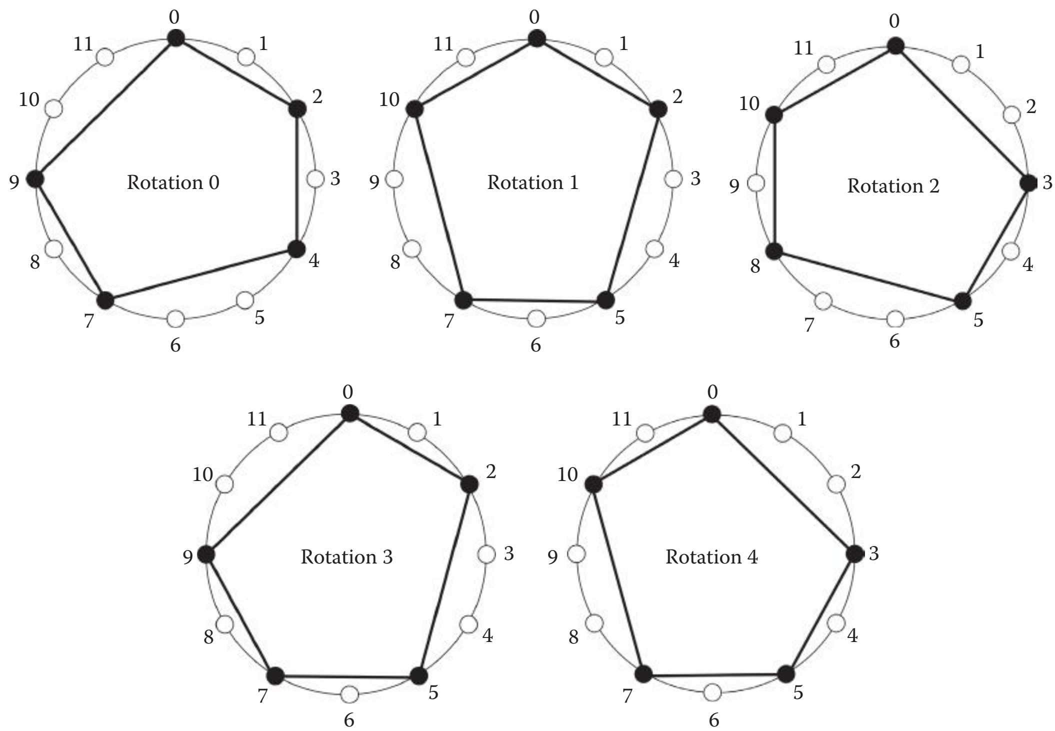

FIGURE 14.7 The ternary version of the clave son and its four onset rotations.

intervals [2-2-3-2-3]. Rotation 1 is a bell pattern used in the Dominican Republic as well as in Morocco. Rotation 2 (a vertical mirror image of Rotation 0) is used as a metal blade timeline by the Aka Pygmies of Central Africa. It is also a metric pattern used in Macedonian music, and employed as a hand-clapping rhythm in a children's song by the Venda people of South Africa. Rotation 3 is the *columbia* bell pattern popular in Cuba. It is also the *abakuá* timeline in West Africa,* as well as a timeline of the Swahili in Tanzania.† Finally, Rotation 4 (a vertical mirror image of Rotation 3) is a bell pattern used by the Bemba people of Zimbabwe as well as a Macedonian dance rhythm.

Of these five rhythms, the most written about in the literature are Rotations 0 and three.‡ This is probably because these are the only rotations that have their first and last onsets on the main duple beats at pulses zero and nine, which makes them more stable and easier to perform. Furthermore, of these two, Rotation 0 appears to be more popular. This may be again due to its surprise value caused by establishing the regular pattern [2-2] at the beginning, and then breaking it by introducing an interval of duration three.

Perhaps, the most important necklace in sub-Saharan Africa is the seven-onset, 12-pulse group of bell rhythms pictured in Figure 14.8. All seven of its onsets are used as starting

* Pérez Fernández, R. A. (2007), p. 7.
† Stone, R. M. (2005), p. 82.
‡ These rhythms are played in the Afro-Cuban religious *batá* drumming, Moore, R. and Sayre, E. (2006), p. 129.

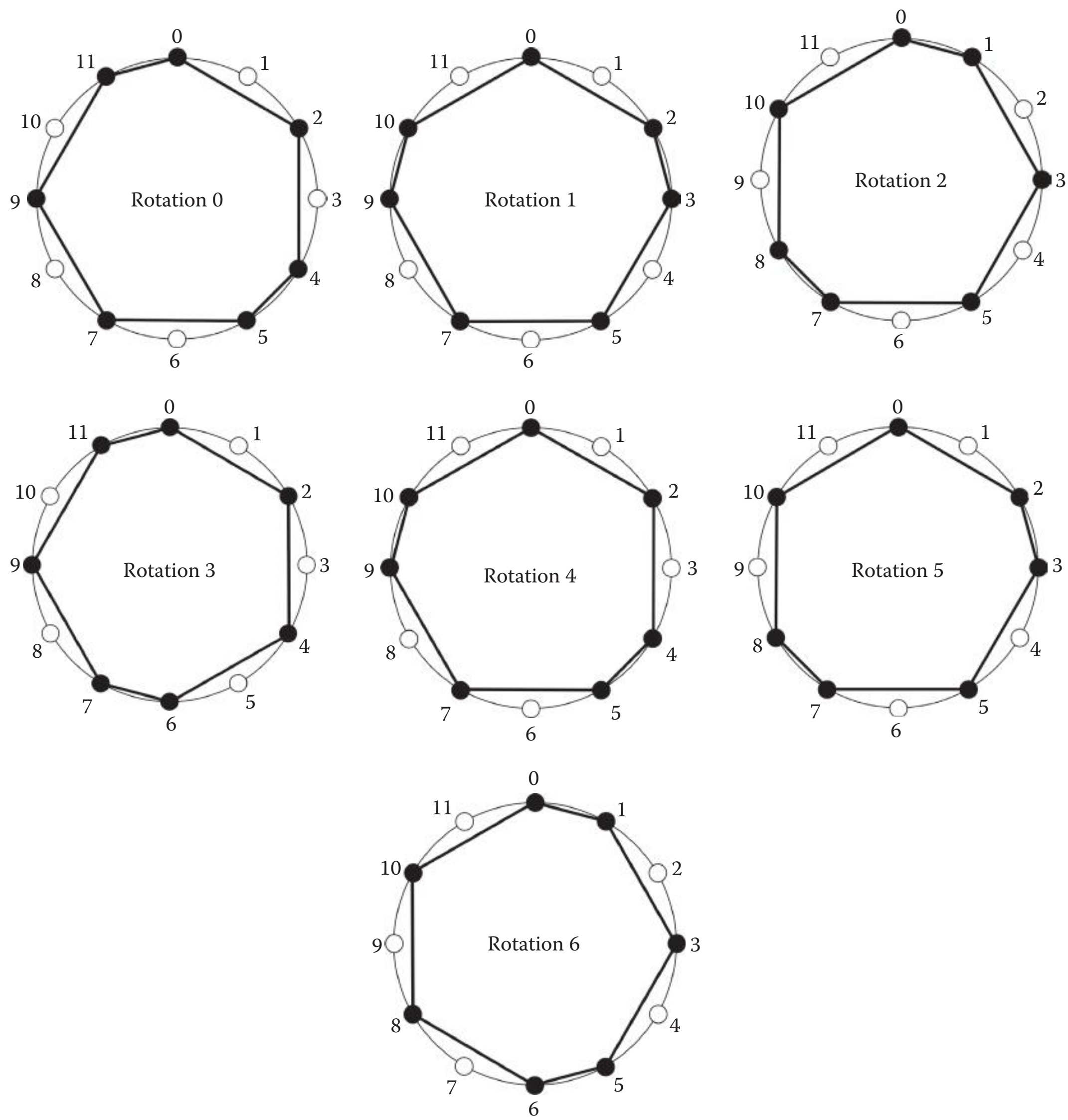

FIGURE 14.8 The standard seven-onset bell pattern and its six onset rotations.

points for timelines.* By far, the most important rhythm is Rotation 0, which corresponds to the major scale or the ionian mode of the diatonic scale. As we have already remarked in Chapters 10 through 12, this rhythm, denoted by [x . x . x x . x . x . x], is (internationally) the most well known of all the African timelines. Indeed, the master drummer Desmond K. Tai has dubbed it the *standard pattern*, and it also goes by the name African *signature tune*. In West Africa, it is found under various names among the Ewe and Yoruba peoples.

* Since this rhythm necklace in the time domain is isomorphic to the diatonic scale in the pitch domain, these seven starting points correspond to the seven modes of the diatonic scale. See Anku, W. (2007), p, 11, Ashton, A. (2007), p. 52, and Loy, G. (2006), p. 20. See also Leake, J. (2007), Pressing, J. (1997), and Rahn, J. (1996).

In Ghana, it is the timeline played in the *agbekor* dance rhythm found along the southern coast of Ghana,[*] and in the *agbadza*, as well as the *bintin* rhythms. Among the *Ewe* people, this rhythm is a bell pattern used in the *adzogbo* dance music. This standard pattern is one of the five Gamamla bell patterns played on the *gankogui*, with the first note played on the low-pitched bell, and the other six on the high-pitched bell. The same is done in the *sogba* and *sogo* rhythms. It is played in the *zebola* rhythm of the *Mongo* people of Congo, and in the *tiriba* and *liberté* rhythms of Guinea. It is equally widespread in America. In Cuba, it is the principal bell pattern played on the *guataca* or hoe blade, in the *batá* rhythms, such as the *columbia de La Habana*, the *bembé*, the *chango*, the *eleggua*, the *imbaloke*, and the *palo*.[†] The pattern is also used in the *guiro*, a Cuban folkloric rhythm. In Haiti, it is called the *ibo*. In Brazil, it goes by the name of *behavento*. In North America, this rhythm is sometimes called the *short* African bell pattern.[‡] Rotation 1 is a rhythm found in Northern Zimbabwe called the *bemba* (not to be confused with the *bembé* from Cuba), and played using axe blades. In Cuba, it is the bell pattern of the *sarabanda* rhythm associated with the Palo Monte cult. Rotation 2 is the *bondo* bell pattern played with metal strips by the *Aka* pygmies of Central Africa. Rotation 3 is a bell pattern found in several places in the Caribbean, including Curaçao, where it is used in a rhythm called the *tambú*, and where originally it was played with only two instruments: a drum and a metallophone called the *heru*.[§] Note that the word tambú sounds like *tambor*, the Spanish word for drum, and heru sounds like *hierro*, the Spanish word for iron. This bell pattern is also common in West Africa and Haiti. In Central Africa, it is called the *muselemeka* timeline, and in North America, it is sometimes called the *long* African bell pattern.[¶] Strangely enough, Changuito uses this pattern in what he calls the *bembé*, thus at odds with what everyone else calls *bembé*, namely the pattern [x . x . x x . x . x . x]. Rotation 4 is a *Yoruba* bell pattern of Nigeria, a *Babenzele* pattern of Central Africa, and a *Mende* pattern of Sierra Leone.[**] Among the Yoruba people, it is also called the *konkonkolo* or *kànàngó* pattern.[††] Rotation 5 is used in Ghana by the *Ashanti* people in several rhythms, and by the *Akan* people as a juvenile song rhythm. In Guinea, it is used in the *dunumba* rhythm. It is also a pattern used by the *Bemba* people of Northern Zimbabwe, where it is either a hand-clapping pattern, or played by chinking two axe blades together. Rotation 6 is a hand-clapping pattern used in Ghana, South Africa, and Tanzania. It is sometimes played on a secondary low-pitched bell in the Cuban *bembé* rhythm.

Note that Rotation 2 is a mirror image reflection of Rotation 0 about a vertical line through pulses zero and six. The same relation holds for Rotations 3 and 6, as well as Rotations 4 and 5. These rotations are equivalent to playing the rhythms backwards.

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[*] Chernoff, J. M. (1979), p. 119.

[†] The word *palo* in Spanish means *stick* but refers also to sugarcane. The rhythm acquired the name because it was played during the cutting of sugarcane.

[‡] Dworsky, A. and Sansby, B. (1999), p. 111.

[§] Jong, N. de (2010), p. 202, Rosalia, R. V. (2002).

[¶] Dworsky and Sansby, *op. cit.*, p. 111.

[**] Stone, R. M. (2005), p. 82.

[††] King, A. (1960), p. 52, considers this rhythm with duration pattern [2-2-1-2-2-1-2] to be a variant of the standard pattern [2-2-3-2-3].

In these examples, the numbers of onsets and pulses are relatively small. This appears to be a requirement for a timeline necklace to be robust. As these values become large, the number of rotations also grows, reducing the fraction of these that remain salient.

The rotations discussed earlier were called *onset* rotations because all rotations had an onset at pulse zero. However, there exist examples of timelines that do not have an onset at pulse zero. Indeed, one not uncommon property of African musical culture is the absence of an onset on the first pulse, or down beat. One example is the timeline from the highlife music* of West Africa, shown in the leftmost diagram of Figure 14.9. All three onset rotations of this pattern are popular in Afro-Cuban music. A second example is the rotation of the tresillo rhythm in a counterclockwise direction by two pulses, as shown in the second diagram in Figure 14.9. A third example is the rotation of the cinquillo pattern with durational pattern [4-2-4-2-4] in a counterclockwise direction by one pulse, as shown in the third diagram of Figure 14.9, which is a rhythm used in a Rumanian dance. The last example is the rotation of the bembé rhythm by six pulses as shown in the rightmost diagram of Figure 14.9. This is equivalent to a reflection of the bembé about the line through pulses 5 and 11. Note that this rhythm is the complementary rhythm of the five-onset fume-fume. This timeline is a *palitos* rhythm used in the *columbia* style of Cuban *rumba* dance music.

The clave son (Figure 14.10, left) is sometimes changed, to the silent first beat version (right), by rotating it by half a measure. These two claves are characterized as having a different *direction*.† Such a rotation is equivalent to two mirror image reflections, one about the vertical line through pulses zero and eight, and the other about a horizontal line through pulses 4 and 12. The result is that both versions retain mirror symmetry about the line through pulses 3 and 11. The left and right versions of the clave son are often called the *three-two* and *two-three* claves,‡ as well as the *forward* and *reverse* claves, respectively.§

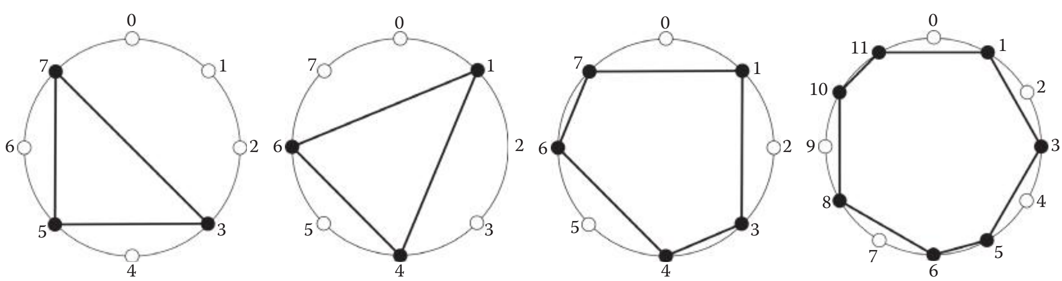

FIGURE 14.9 Some timelines that do not have an onset at pulse zero (anacrusis).

* Agawu, K. (1995), p. 129. This rhythm is also called the *sichi* rhythm (from Ghana) by Dworsky, A. and Sansby, B. (1999), p. 84.

† Vurkaç, M. (2011), p. 27.

‡ Mauleón, R. (1997), p. 24.

§ Traditionally, the contextual rhythmic analysis of the clave son is based on dividing the 16-pulse cycle into two eight-pulse half cycles corresponding to the *three-attack* and *two-attack* portions, and subjecting the two parts to further analysis based on syncopation. However, Vurkaç, M. (2012) finds it more useful to partition the 16-pulse cycle into an inner part flanked by two outer parts, and analyzing the parts by means of *off-beatness* rather than syncopation. We shall consider the notion of off-beatness in Chapter 16.

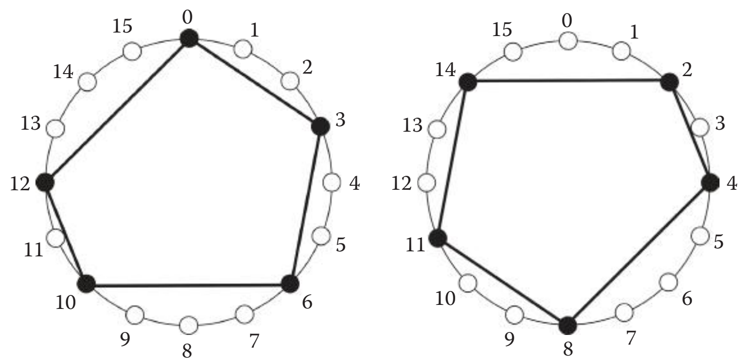

FIGURE 14.10 The clave son (left) and its rotation by eight pulses (right).

To close this chapter, let us return to the problem of listening to rhythms in the context of an underlying meter, whether sounded or internalized. At the beginning of the chapter, we saw an example in Figure 14.1 in which the bembé rhythm was rotated in a clockwise direction by one pulse, and the underlying four-beat meter remained constant at positions {0, 3, 6, 9}. The result was that the rotated rhythm sounded very different from the bembé. Perhaps, you were not too surprised by this effect since the rhythm was rotated. Therefore, consider now the more compelling case in which the bembé is not rotated, but rather is heard against two different underlying meters, the four-beat meter [3-3-3-3] and the three-beat meter [4-4-4], as illustrated in Figure 14.11. In Figure 14.11, the metric beats are highlighted with a circle, and connected with thin line segments with labels denoting their duration. In this situation, ignoring the meters, both rhythms are identical. However, if I now play the meters on a bass drum while playing the bembé rhythm on a bell, the difference in sound and feel between the two renditions of bembé is huge. Note that the resultant patterns obtained by taking the unions of the rhythm and meter attacks are very different. The resultant for the four-beat meter is [x . x x x x x x . x . x], whereas that

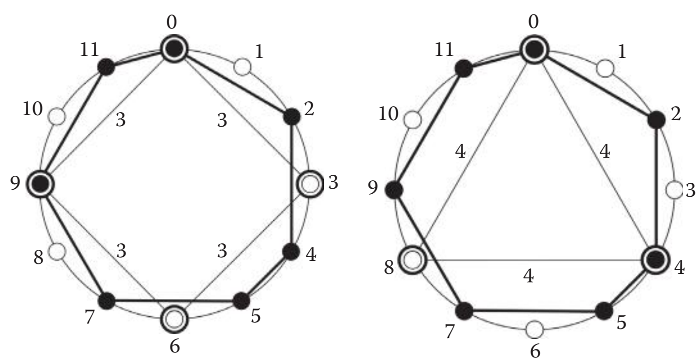

FIGURE 14.11 The bembé rhythm in articulated four-beat (left) and three-beat (right) meters.

for the three-beat meter is [x . x . x x . x x x . x]. The former has nine attacks whereas the latter has eight. Furthermore, the grouping is totally different. The former has one solitary attack, one group of two adjacent attacks, and one very large group of six adjacent attacks. On the other hand, the latter has four groups: one of size one, two of size two, and one of size three. Incidentally, this latter resultant pattern of the bembé and the [4-4-4] meter is the rhythmic necklace pattern used by Steve Reich in *Clapping Music*.

CHAPTER 15

Rhythmic Oddity

Do twentieth century, East London, acid jazz music and the ancient Aka Pygmy music of Central Africa have anything noteworthy in common? Yes, they do. There exist pieces of music in both domains that use rhythmic timelines that possess the *rhythmic oddity* property. But that is getting ahead of our story. First, we must backtrack half a century to 1963, when a 33-year-old horn player with the symphony orchestra of an Israeli radio station received an invitation to work on a project spearheaded by the Israeli Ministry of Foreign Affairs: the setting up of a youth orchestra in the Central African Republic. The horn player's name was Simha Arom, and although he was not overly enthusiastic about the project itself, he was excited by the possibility of discovering a world of music unknown to him. Besides, he was ready to break up the routine that had enveloped his life. When he first heard the music of the Aka Pygmies, he was instantly overwhelmed. He felt that their music not only had ancient roots, but that it also touched roots deep inside him.[*] The rest is history. Arom went on to develop original methods of musicological research, and new tools with which to collect data. He made multiple recordings of African traditional music, and created a museum of the Arts and popular traditions. He studied the music of the Aka Pygmies for decades, becoming one of the foremost systematic ethnomusicologists in the world.

While studying the music of the Aka Pygmies of Central Africa, Arom noticed that their music contained rhythmic timelines that exhibited a property that he christened *rhythmic oddity*. A rhythm with an *even* number of pulses in its cycle has this property if no two of its onsets divide the rhythmic cycle into two half-cycles, that is, two segments of equal duration.[†] This property is not defined for rhythms with an odd number of pulses since then it is impossible for two pulses to lie diametrically opposite each other on the rhythm circle (an odd number is not evenly divisible by two). It is quite easy in theory to construct examples of rhythms that have this property.[‡] Figure 15.1 shows two such

[*] Arom, S. (2009), p. 7.

[†] Chemillier, M. (2002), p. 176, Chemillier, M. and Truchet, C. (2003).

[‡] It is more difficult to enumerate *all* rhythms that have the rhythmic oddity property. Chemillier, M. (2004), p. 615, shows how this can be done using Lyndon words.

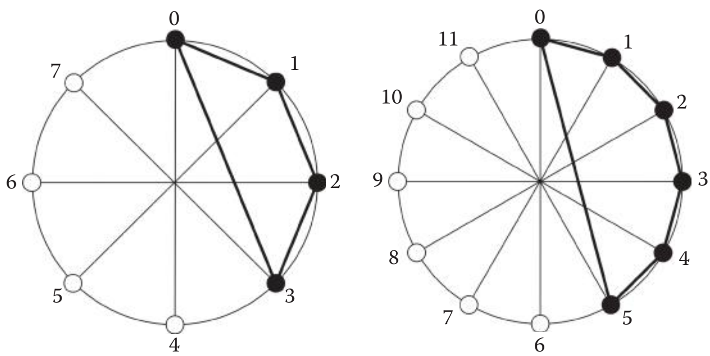

FIGURE 15.1 Two humdrum rhythm timelines that possess the *rhythmic oddity* property.

examples: the rhythm on the left has onsets on the first four of its eight pulses, and the one on the right has onsets on the first six of its 12 pulses. It is obvious that in general, for any even number n of pulses, a rhythm with less than $n/2$ consecutively adjacent onsets has the rhythmic oddity property. However, rhythms constructed in this way are not particularly interesting musically, and are not used as timelines in world music except when the number of pulses in the cycle is a small number such as $n = 4$ or $n = 6$, in which case we may obtain, for example, the two-onset and three-onset rhythms with inter-onset intervals [1-3] and [1-1-4], respectively, which can be heard as ostinatos in several musical traditions. Its drawbacks notwithstanding, we shall identify this procedure as the *Walk* Algorithm, since we can think of starting a walk at pulse zero, taking k short steps of length one pulse each, where k is less than $n/2$.

In spite of their easy construction from the mathematical point of view, in practice, timelines that possess the rhythmic oddity property are unusual in world music. For rhythms to be effective as timelines, they should in general not contain silent gaps longer than half of their cycle, and they should exhibit a certain degree of regularity or near-evenness. These two constraints are often enough to inadvertently prevent the rhythmic oddity property from being satisfied. Figure 15.2 illustrates four such examples of traditional rhythm timelines that satisfy these two conditions but lack the rhythmic oddity property. In the rhythm on the left with inter-onset intervals [1-1-2], the first and last onsets violate the rhythmic oddity property. This simple pattern is used almost universally. For example, it is the *baiaó*

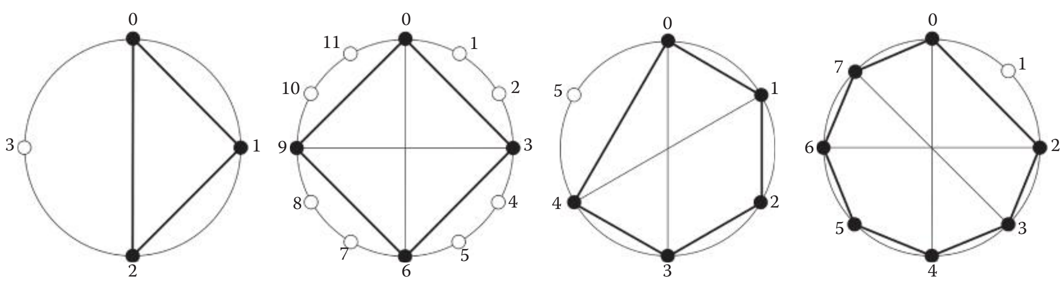

FIGURE 15.2 Some examples of rhythm timelines without the rhythmic oddity property.

rhythm of Brazil, as well as the *polos* rhythm of Bali. When it is started on the second onset, it turns into the *catarete* rhythm of the indigenous people of Brazil. Started on the third onset, it becomes an archetypal pattern of the Persian Gulf region,* the *cumbia* from Colombia, and the *calypso* from Trinidad. It is also a thirteenth century Persian rhythm called *khalif-e saghil*, as well as the *trochoid choreic* rhythmic pattern of ancient Greece. Starting it on the silent pulse (anacrusis) yields a popular flamenco hand-clapping pattern (also *compás*) used in the flamenco styles called the *taranto*, the *tiento*, the *tango*, and the *tanguillo*. It is also the *rumba* clapping pattern in flamenco, as well as another pattern used in the *baiaó* rhythm of Brazil.† In the second rhythm with inter-onset intervals [3-3-3-3], the rhythmic oddity property is violated twice, once with the first and third onsets, and again with the second and fourth onsets. This rhythm is the meter or compás of the *fandango* music of Spain. It is often accompanied by hand-clapping every pulse, but with loud claps at pulses zero, three, six, and nine. The third rhythm with inter-onset intervals [1-1-1-1-2] contains two violations of the rhythmic oddity property at pulses zero and three as well as one and four. It is the *york-samai* pattern, a popular Arabic rhythm, as well as a hand-clapping rhythm used in the *al-medemi* songs of Oman. Finally, the fourth rhythm with inter-onset intervals [2-1-1-1-1-1-1] contains three violations of the property at pulses zero and four, two and six, and three and seven. It is a typical rhythm played on the *bendir* (frame drum), and used for the accompaniment of women's songs of the Tuareg people of Libya.‡

Let us now turn to rhythms that contain the rhythmic oddity property and that satisfy the above constraints. Two examples are the 24-pulse timeline rhythms used by the Aka Pygmies pictured in Figure 15.3. The rhythm on the left has nine onsets with inter-onset intervals [3-3-3-2-3-3-2-3-2] whereas the one on the right has eleven onsets with inter-onset intervals [3-2-2-2-2-3-2-2-2-2-2].

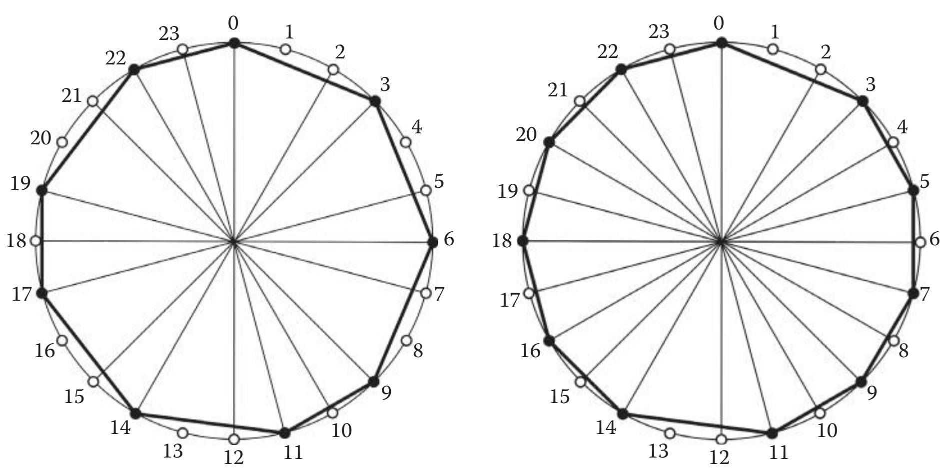

FIGURE 15.3 Two timelines with rhythmic oddity used by the Aka Pygmies.

* Olsen, P. R. (1967), p. 31.
† The song *Baião* by Luiz Gonzaga uses the rhythms [x . . x x . . .] and [. . x . . . x]. See Murphy, J. P. (2006), p. 97.
‡ Standifer, J. A. (1988), p. 50.

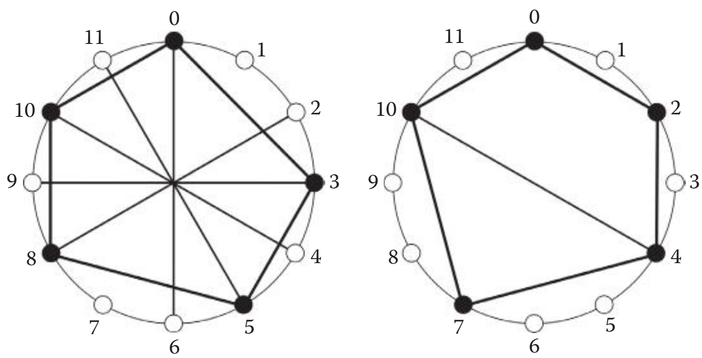

FIGURE 15.4 A 12-pulse timeline used by the Aka Pygmies (left) and the seguiriya *compás* of the flamenco music of southern Spain (right).

The Aka Pygmies also use the five-onset, 12-pulse timeline shown in Figure 15.4 (left). It has inter-onset intervals [3-2-3-2-2] and possesses the desired properties. As an aside, it is interesting to note that if the five intervals are permuted to yield [2-2-3-3-2], we obtain the hand-clapping pattern and meter (*compás*) used in the *seguiriya* style of the flamenco music of southern Spain shown on the right.* With the two inter-onset intervals of length three now adjacent to each other, the rhythmic oddity property is violated at pulses 4 and 10.

Given that the music of the Aka Pygmies is characterized by having rhythmic timelines that possess the rhythmic oddity property, a natural ethnomusicological question arises: to what extent does this property manifest itself in other cultures such as, for example, West Africa, South Africa, or Cuba? To try to answer this question, consider an archetype timeline structure used extensively in these three geographical regions, that consists of seven onsets in a cycle of 12 pulses, with the constraint that all the inter-onset intervals must be of only two distinct durations: either one unit or two units. Three examples of such bell-pattern timelines are the bembé, tonada, and sorsonet pictured in Figure 15.5. Note that none of them possess the rhythmic oddity property. However, before we dismiss the usefulness of this property altogether, it is worth noting that the bembé contains one violation, the tonada† contains two, and the sorsonet three. This information may be used to generalize the rhythmic oddity property as described in the following.

Arom defined the rhythmic oddity property in the form of a strict binary category, that is, a rhythm either has or does not have the rhythmic oddity property. This concept may be extended by means of a multivalued function that measures the *amount* of rhythmic oddity that a rhythm possesses. This function (rhythmic oddity) depends on the number of violations of the rhythmic oddity property present in a rhythm. Stated another way, a violation of the rhythmic oddity property yields a partition of the rhythmic cycle into two

* Fernández, L. (2004), p. 35.

† The tonada has one less onset than a popular traditional nineteenth century Cuban timeline called the *clave campesina* given by [x . x x . x x . x x x .] (see Mauleón, R. [1997], p. 10). With the additional onset in between the last two onsets of the tonada, the clave campesina has a third violation of rhythmic oddity at pulses three and nine.

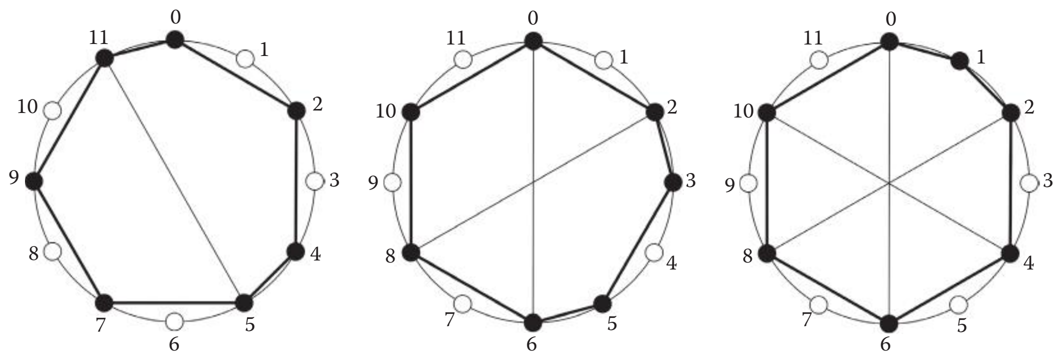

FIGURE 15.5 Three archetypal rhythmic timelines from West Africa.

half-cycles by pairs of its antipodal onsets. Let us call such a partition of the cycle an *equal bipartition*. Then, the fewer equal bipartitions a rhythm admits, the more rhythmic oddity it possesses. The timeline in Figure 15.4 (left) used by the Aka Pygmies contains no equal bipartitions, whereas the seguiriya *compás* of the flamenco music (right) contains one.

Let us return to the three archetypal timeline necklaces from West Africa shown in Figure 15.6. Recall that if we disregard the rotations of a rhythm, so that all its rotations form an equivalence class, we call such an object a necklace. The relevance of necklaces here comes from the fact that the rhythmic oddity function is independent of the rotations of a rhythm; it is a property of the necklace. Figure 15.6 depicts three distinct necklaces, and each necklace determines seven different rhythms depending on which onset of the rhythm is taken as pulse zero (not counting the rotations that yield rhythms with anacrusis that start on a silent pulse). As it turns out, if the inter-onset intervals are restricted to the values one and two, these three necklaces are the only mathematical possibilities. The two short intervals of length one may be separated by two, one, or zero long intervals of length two, in the bembé, the tonada, or the sorsonet, necklaces, respectively. Our original question concerning the postulated preference of timelines then becomes: which of the three necklaces in Figure 15.6 are preferred in West African music? This question is not easy to

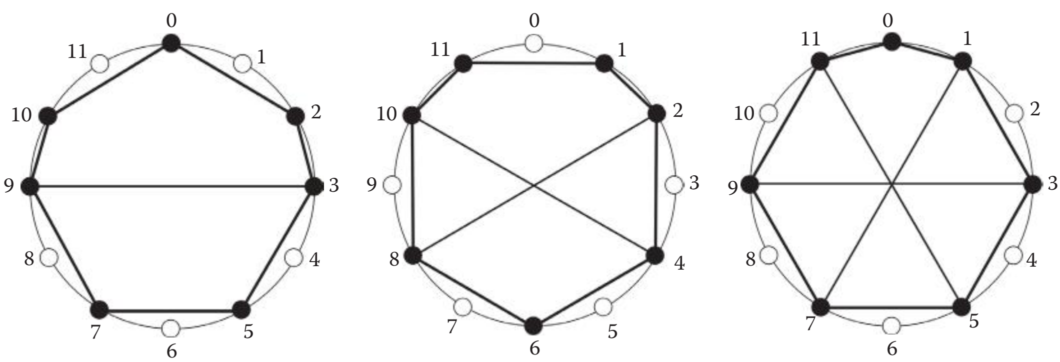

FIGURE 15.6 The three necklaces with two short and five long durations.

answer without first agreeing on the definition of "preference," and spending time in the field performing listening experiments. In the absence of all these requirements, we may attempt to answer this question as an arm-chair musicologist by counting, for each necklace, how many of its rotations are used in musical practice. This definition of preference still needs elaboration since it is conceivable that one necklace appears frequently, but in only one of its rotations, whereas the other necklaces appear infrequently, but in all their rotations. However, if only one rotation of a necklace is used, then it is that rhythm that is preferred, and not the necklace that gives rise to the rhythm. What is intended by preference here is precisely the necklace. Which necklace is preferred by nature; which has the greatest fecundity?

It turns out that in West African music, the sorsonet necklace is one of the least preferred of the three, yielding one timeline used in traditional music, the sorsonet rhythm of Figure 15.5 (right). However, the rotation with durational pattern [2-2-2-2-1-1-2] is the Persian rhythm *kitāb al-Adwār*,* and the rotation [2-1-1-2-2-2-2] is the rhythm of the Polish *polonaise*.† Rotations of the tonada necklace are encountered more frequently, yielding two West African rhythms, the tonada with intervals [2-1-2-1-2-2-2]‡ and the asaadua given by [2-2-2-1-2-1-2], and one Persian rhythm, the al-ramal with intervals [2-2-2-2-1-2-1]. The bembé necklace is overwhelmingly preferred over the other two necklaces. Indeed, all seven rhythms obtained by starting the cycle at every one of its seven onsets are heavily used. It is evident, then, that among this family of rhythms there may have been an evolutionary preference for those that admit as few as possible equal bipartitions, and thus a higher degree of rhythmic oddity. This is not to imply that there are no other mathematical properties that can produce the same preference ranking of these three necklaces. Perhaps, the most obvious one is the separation distance between the two short intervals in the cycle, which are separated either by a minimum of two, one, or zero long intervals. This implies that the same preference ranking may also be obtained by measuring how evenly the seven attacks are spaced out in the circle. Yet another method for obtaining the same preference ranking is by calculating the minimum number of elementary mutations required for each of the necklaces to become a regular hexagon, which is another measure of evenness of these necklaces. An elementary mutation here either deletes an attack or inserts an attack. In the leftmost necklace of Figure 15.6, deleting the three attacks at pulses 10, 0, and 2, and inserting two attacks at pulses 11 and 1 does the job, yielding a total of five mutations. The necklace in the middle may be transformed into a regular hexagon by deleting the attacks at pulses 11 and 1, and inserting an attack at pulse zero, for a total of three mutations. Finally, the necklace on the right requires only the deletion of one attack at pulse zero. We will return to such mutation operations in more depth later in the book. All these mathematical methods are in effect theoretical explanations that fit the data. Whether any of these methods actually guided the evolutionary selection process is another matter altogether. It would be interesting to test experimentally which of these properties has the most perceptual reality. Under what

* Wright, O. (1995).

† Dahlig-Turek, E. (2009), p. 127.

‡ Kwabena Nketia, J. H. (1962), p. 85, lists this rhythm as a hand-clapping pattern of the Akan people of Ghana. However, the term *tonada* is used in Cuba to describe this rhythm.

circumstances, if any, is the degree of rhythmic oddity possessed by a rhythm, more easily perceived by humans than the amount of evenness? A dancing culture might have selected a timeline on the basis of rhythmic oddity, in as much as this property has a marked effect on the order of the upbeats and downbeats of the feet, thus rendering evenness as a by-product.

In the pitch domain, the three necklaces in Figure 15.6 are the three well-known scales called (from left to right) the diatonic scale, the ascending melodic minor scale, and the Neapolitan major scale.* Michael Keith proposes measuring the evenness of scales by a suitable distance function between each note and the ideal note. The ideal notes are located at multiples of 12/7 on the circle, yielding the coordinate values along the circle: 0.0, 1.714, 3.428, 5.142, 6.856, 8.570, and 10.284. His measure called the *scale-idealness* also ranks the three necklaces of Figure 15.6 in decreasing order from left to right.† If we compute the sum of the absolute values of the differences between these coordinates, and those of the attacks of the bembé, tonada, and sorsonet rhythms of Figure 15.5, we obtain the distances: bembé = 2.290, tonada = 2.566, and sorsonet = 4.994. Thus, the bembé is slightly more even than the tonada, and both are much more even than the sorsonet.

Let us return to the topic of generating rhythms that exhibit the rhythmic oddity property. At the beginning of this chapter, the *Walk* algorithm was presented that constructs rhythms that have the rhythmic oddity property but place all the onsets within a total duration region that spanned less than one half-cycle, thus producing not the best of timelines. We close this chapter with a demonstration of a modification of the procedure that yields timelines that satisfy the rhythmic oddity property, and such that every half-cycle contains at least one onset. Furthermore, the timelines obtained in this way turn out to be better. This algorithm will be called the *Hop-and-Jump* algorithm. It falls in the general category of algorithms for obtaining the so-called generated rhythms, and later in the book we shall see its relation to other generative methods for producing *deep* rhythms. Thus, one application of the rhythmic oddity property is to the algorithmic generation of "good" rhythms.

Let us assume that we want to generate a rhythm with five onsets in a cycle of 12 pulses. The algorithm is illustrated with the five clock diagrams (left to right) in Figure 15.7. The first onset is placed at pulse zero. This implies that the diametrically opposite pulse six is now unavailable for placing an onset, since we want the rhythmic oddity property satisfied. To place the next onset, we *hop* to pulse two, making pulse eight unavailable. This process is continued, always advancing by hopping a distance of two units if this is possible. When this is not possible, as is the case when we want to hop to onset number four at pulse six (which is unavailable), we try the next pulse (here pulse seven). If it is available (as it is in this example), we take it. Otherwise, we continue skipping pulses until an available pulse is found. Since in this case we advanced by a distance of more than two pulses, we call this a *jump*. Following a jump, we continue as before making hops of distance two if possible (or jumps otherwise), here yielding the fifth onset at pulse nine.

The *Hop-and-Jump* algorithm is obviously guaranteed to yield rhythms with the rhythmic oddity property since it never places an onset on an unavailable pulse location.

* Rappaport, D. (2007), p. 322 and p. 323.

† Keith, M. (1991), p. 97. See also Tymoczko, D. (2011).

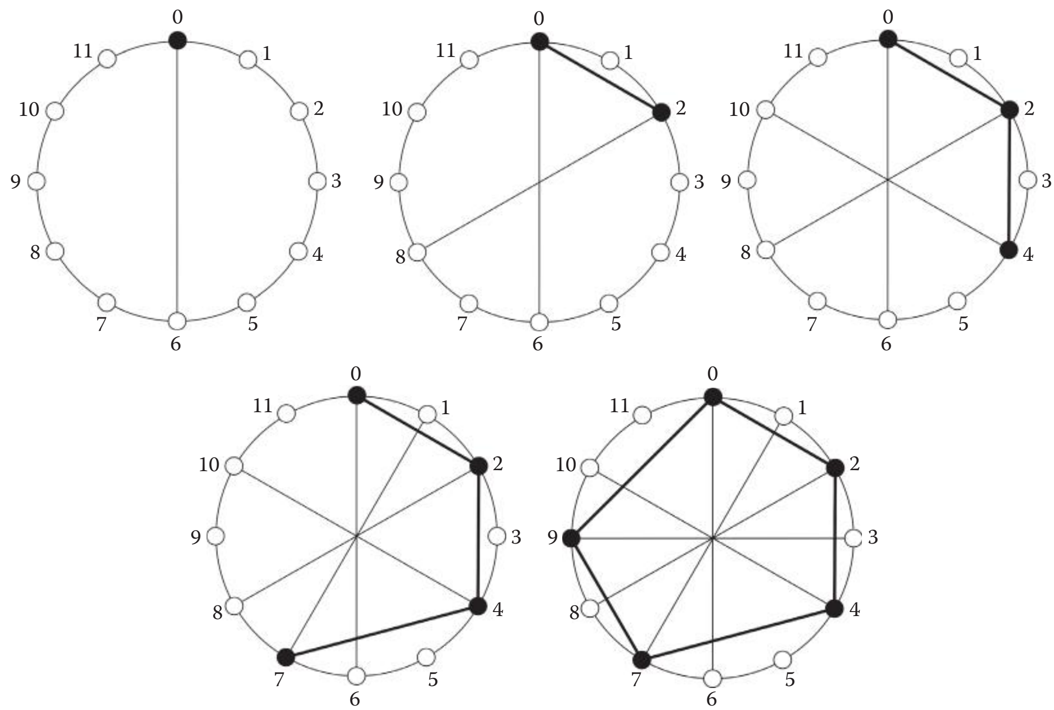

FIGURE 15.7 The *Hop-and-Jump* algorithm for generating good rhythms that have the rhythmic oddity property: five onsets among 12 pulses.

Furthermore, by choosing the number of onsets and *hop* distance appropriately, we may guarantee that there are no silent gaps longer than a half-cycle. Finally, note that the resulting five-onset rhythm obtained in Figure 15.7 is the *fume-fume* bell pattern (also the *standard* pattern) widely used in West Africa, and is the same as the Aka Pygmie rhythm of Figure 15.4 either played backwards or rotated in a counterclockwise direction by four pulses.

Let us consider a few more examples with different numbers of onsets and pulses, and different sizes of hops, to substantiate our claim that the *Hop-and-Jump* algorithm is successful at generating good timelines.

For three onsets out of eight pulses, and hop-size two, the algorithm generates the rhythm with inter-onset intervals [2-3-3] as shown in Figure 15.8. Recall that this rhythm is the *nandon bawaa* bell pattern of the Dagarti people of northwest Ghana, and is also found in Namibia and Bulgaria.* It is a rotation of the rhythm with inter-onset intervals [3-3-2] (the Cuban *tresillo*), which, as pointed out earlier, is the most important traditional bluegrass banjo rhythm, as well as a metal-blade pattern of the Aka Pygmies. The latter is common in West Africa and many other parts of the world such as Greece and Northern Sudan. Some scholars consider it to be one of the most important rhythms in Renaissance

* Kwabena Nketia, J. H. (1962), p. 123, includes this rhythm as hand-clapping pattern used in *Nayalamu*, a recreational maiden song of the Gonja people of Ghana.

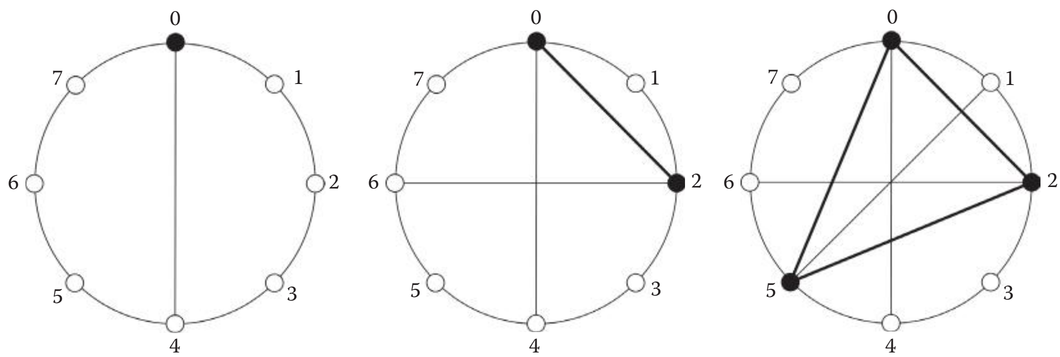

FIGURE 15.8 The *Hop-and-Jump* algorithm with three onsets among eight pulses.

music. Indeed, the pattern [3-3-2] dates back to the Ancient Greeks who called it the *dochmiac* pattern. In India, it is one of the talas of Karnatic music. The rotation with intervals [3-2-3] is a drum pattern used in Korean instrumental music, and is also found in Bulgaria and Turkey.

Figure 15.9 illustrates the algorithm at work with five onsets out of 16 pulses, and hop-size three. It generates the rhythm with inter-onset intervals [3-3-3-3-4] that is a rotation of the bossa-nova clave rhythm of Brazil. The actual bossa-nova rhythm usually starts on the third onset. It is also a maximally even rhythm since this is the most even manner in which one may distribute five onsets among 16 pulses.

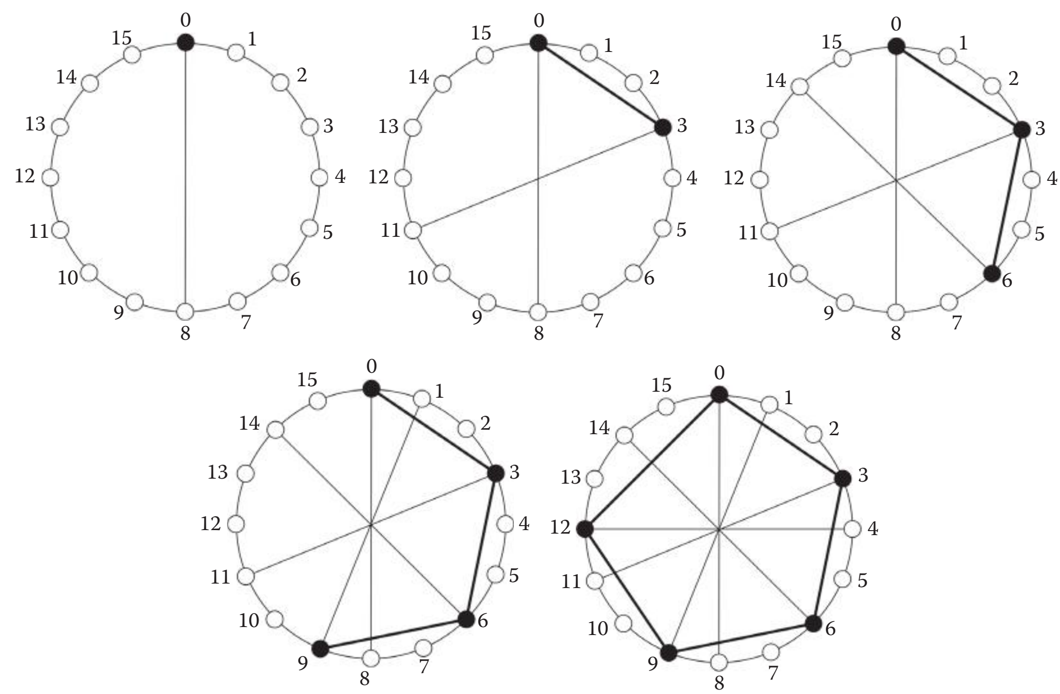

FIGURE 15.9 The *Hop-and-Jump* algorithm with five onsets among 16 pulses.

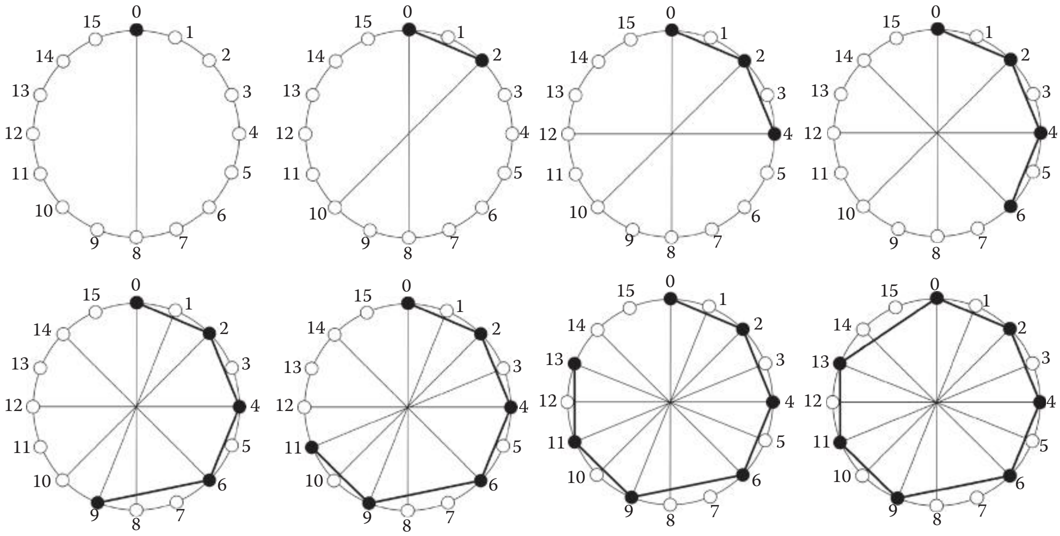

FIGURE 15.10 The *Hop-and-Jump* algorithm with seven onsets among 16 pulses.

The last example in Figure 15.10 shows the generation of a rhythm with seven onsets among 16 pulses using a hop-size of two pulses. It has inter-onset intervals [2-2-2-3-2-2-3], and is a rotation of a Samba rhythm from Brazil. The actual Samba rhythm starts on pulse four, and coincides with a Macedonian rhythm. Other rotations of this rhythm are found in the music of Ghana as well as the former Yugoslavia.

All the examples of timelines containing the rhythmic oddity property discussed in the preceding emerge from the traditional drumming music of West and Central Africa as well as the African diaspora. With the modern and commercial preoccupation of twentieth century music, and the ubiquitous "square" divisive timelines that dominate much of pop music, one may wonder if there are any contemporary newly composed timelines out there that exhibit the rhythmic oddity property. A fascinating example of one such timeline that uses a highly syncopated 10-onset, 32-pulse cycle may be found in the song *Cosmic Girl* released in the United Kingdom in 1996 by the acid jazz band Jamiroquai.* Acid jazz is both an East London recording company as well as a music genre, and there is an ongoing debate about which of the two came first and influenced the other. As a genre, acid jazz appears to combine elements of hip-hop, funk, and jazz with a strong rhythmic element that uses rhythm timelines, or *looped beats*, as they are called in the electronic music world. As the lyrics of *Cosmic Girl* testify, Jamiroquai wanted to create a feeling of outer space, of distance, of strangeness and science fiction, by using words such as "hyperspace," "galaxy," "quasar," "teleport," and the lightness of "zero gravity." To create these feelings, the band composed a timeline that does the job quite well. The electronic timbre is no doubt appropriate, but what really made this psychedelic song blast off to number six on the U.K. music charts may have been its unique timeline. The length of this rather

* I am indebted to mathematician Ben Green of the University of Cambridge for bringing this music to my attention.

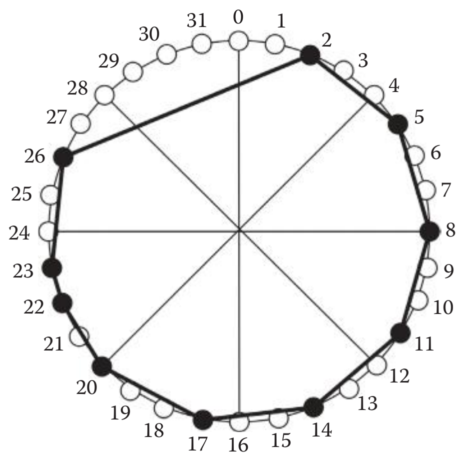

FIGURE 15.11 The opening timeline of acid jazz band Jamiroquai's *Cosmic Girl*.

long 32-pulse timeline gives the cosmic feeling time to sink in, but the timeline's main power comes from its long sequence of three-pulse inter-onset intervals that twist and turn around the regular four-pulse underlying beats shown with thin solid lines in Figure 15.11, and the property that even with as many as 10 onsets, the timeline still manages to exhibit the rhythmic oddity property.

A straightforward application of the *Hop-and-Jump* algorithm with nine onsets and 32 pulses yields the rhythm in Figure 15.12 (left). If Jamiroquai experimented with this version before adopting their final one, it is easy to see why they would have abandoned it. With three of its onsets coinciding with downbeats at pulses 0, 12, and 24, and two of these downbeats being the most important downbeats, namely the first and last of the sequence, it lacks the element of surprise, and provides little energy in the margins. Furthermore,

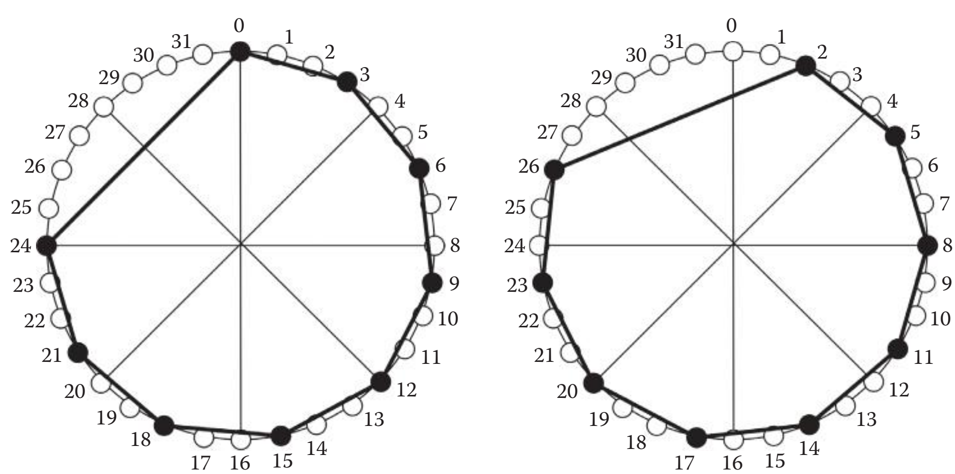

FIGURE 15.12 Rhythm obtained with the *Hop-and-Jump* algorithm (left), and its forward rotation by two pulses (right).

the onset on the downbeat at pulse 12 falls squarely halfway between those at pulses 0 and 24, providing too much symmetry. Consider what results when this rhythm is rotated in a clockwise direction by two pulses to yield the rhythm on the right. Now there are onsets on only two downbeats, and they are not the first or last in the sequence, but rather the third on pulse eight and the sixth on pulse 20, more unexpected locations to be sure. Furthermore, since *Cosmic Girl* starts with only this timeline for a while, we are totally surprised when the beats start coming in with the remaining percussion instruments. Finally, add a 10th onset on pulse number 22, as in Figure 15.11, and now the timeline also has closure just before the end of the cycle; it is the icing on the cake. As an aside, it is interesting to note that the first four attacks of this timeline are the same as the repeating rhythmic riff in George Gershwin's "I Got Rhythm," given by [. . x . . x . . x . . x . . . .], the source for the most common chord-progression in jazz.[*]

The methods described in this chapter are radically different from those used for generating rhythms automatically, which are described in the artificial intelligence and music information retrieval literatures. The latter approaches are inspired either by models of biological processes, such as neural networks that learn from experience, by genetic programming methods that model the evolutionary laws of natural selection, or by statistical models such as Markov processes.[†] Many of these techniques are based on guided random search of the space of all possible rhythms. Typically, genetic methods first define a measure of rhythmic "goodness" generally termed a *fitness function*, and then use simple rules for transforming a given collection of rhythms in such a way as to improve their fitness. These rules are usually described in general terms as *reproduction*, *crossover*, and *mutation*, and applied in this order. Reproduction selects a pair of rhythms, say A and B, at random from the collection. Crossover involves creating new offspring rhythms of A and B by swapping some elements from A to B and vice versa. Mutation involves changing one of the elements of a new offspring of A or B at random, and usually with low probability. Finally, the algorithm stops (or is stopped) when the fitness function has (or seems to have) reached a maximum value.[‡]

M. Gibson and J. Byrne (1991), incorporate a neural network in their genetic approach. First they use humans to label a collection of training rhythms as either "good" or "bad." Then they use the trained neural network to classify new rhythms generated by the genetic algorithm as either "good" or "bad," thus serving as the fitness function.[§] D. Horowitz (1994) describes an interactive approach that allows the user to "simply execute fitness functions (i.e., to choose which rhythms or features of rhythms the user likes) without necessarily understanding the details or parameters of these functions."[¶] This "ostrich-head-in-the-sand" approach may be attractive and useful to those composers and other users who are satisfied with only the end product. By contrast, the methods proposed in this chapter and the book in general for generating "good" rhythms are *structural* in nature,

[*] Crawford, R. (2004), p. 163. I am indebted to Dmitri Tymoczko for pointing out this connection.
[†] Paiement, J.-F., Bengio, S., Grandvalet, Y., and Eck, D. (2008).
[‡] Burton, A. R. and Vladimirova, T. (1999).
[§] Gibson, M. and Byrne, J. (1991).
[¶] Horowitz, D. (1994).

and guided by musicological and empirical knowledge of rhythms that humanity has come to cherish over thousands, if not millions, of years of evolution. The crux in these methods is precisely the understanding of the details, and the elimination of the parameters in neural networks that must be tweaked to obtain good rhythms. The methods proposed here are closer in spirit to computational music theory, and represent an attempt to understand the temporal structures that make a rhythm "good." Furthermore, if desired, the properties discussed here may also be incorporated into fitness functions for use in genetic algorithms.

Off-Beat Rhythms

THE MUSIC OF THE SAN PEOPLE, a group of hunters and gatherers that live in the Southern African countries of Angola, Botswana, and Namibia, is characterized by the use of an instrument called the *musical bow*, illustrated in Figure 16.1. This instrument consists of a bow, such as one might use for hunting, with a tight string attached to the two ends. In addition, a gourd attached to the bow is used to create a resonating cavity to produce a particular tone and timbre. One rather unique tradition of these hunter-gatherers, called *kambulumbumba*, involves three individuals playing one bow simultaneously.*

One player, while securing the bow with his feet and mouth, plays the leftmost rhythm in Figure 16.2 by striking the string with a stick. This rhythm sets up an isochronous steady regular rhythm [3-3-3-3] with four beats per cycle. Another musician plays the rhythm in the center, also with a stick, but on the upper end of the bow. This rhythm has five onsets with intervals [3-3-2-2-2]. The first three onsets of the latter rhythm coincide with the first three onsets of the former rhythm. However, the last two onsets at pulses 8 and 10 fall in

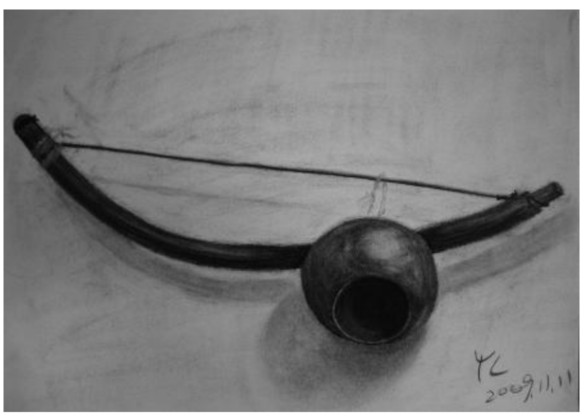

FIGURE 16.1 Musical bow. (Courtesy of Yang Liu.)

* Kaemmer, J. E. (2000), p. 314. See Kubik, G. (1975–1976) for an account of musical bows in Angola.

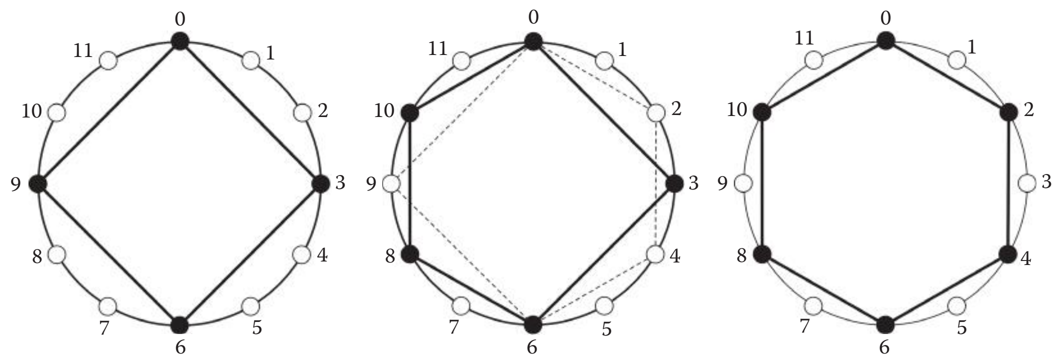

FIGURE 16.2 The three musical bow rhythms employed by the *San* people.

between the regular beat, and are therefore *off-beat*. The third performer plays the right-most rhythm in Figure 16.2 also with a stick, in between the other two players.

The third rhythm, also perfectly regular with intervals [2-2-2-2-2-2], has four onsets that coincide with the second rhythm at pulses 0, 6, 8, and 10. However, it has two onsets at pulses two and four that are off-beat with respect to the first regular rhythm. The diagram in the middle shows how the two regular rhythms (dotted lines) interact with the irregular rhythm. Interestingly enough, this irregular rhythm is also the meter of the guajira style of the flamenco music of southern Spain.

Although the multivalued measure of the quantity of rhythmic oddity, discussed in Chapter 15, is more successful than the binary-valued rhythmic oddity property, at discriminating rhythmic preference in West Africa, it still has its limitations. For instance, among the seven rhythms determined by the rotations of the bembé necklace, some are preferred over others. In fact, one of these, the bembé itself, with intervals [2-2-1-2-2-2-1] is by far the most favored of the seven, as it is considered to be the African signature bell pattern. Afro-Cuban music has escorted it across the planet, and it is used frequently on the ride cymbal in jazz. Since all seven rhythms belonging to this necklace obviously have exactly one equal bipartition, even the multivalued rhythmic oddity measure does not discriminate among these seven, and thus does not favor the bembé rhythm over its six other rotations. To resolve this conundrum, we recruit another mathematical measure of syncopation or irregularity termed *off-beatness*. To illustrate how this measure works, consider a cycle of 12 pulses. Such a cycle may be evenly divided (without remainder) by the integers two, three, four, and six, to yield the four regular rhythms with inter-onset intervals [6-6], [4-4-4], [3-3-3-3], and [2-2-2-2-2-2], respectively, pictured in Figure 16.3. If a piece of music uses a particular regular meter that has strong beats at say pulses zero, three, six, and nine, as in the third diagram from the left, then notes that are played on the other eight pulses are considered to be off-beat relative to such a meter. Sub-Saharan African drum ensemble music is polyrhythmic and most music that uses a 12-pulse cycle incorporates most if not all four rhythms shown in Figure 16.3, played either on different types of drums, other percussion instruments, or clapping.* If we superimpose all four rhythms on one circle, we

* Kwabena Nketia, J. H. (1962), p. 83.

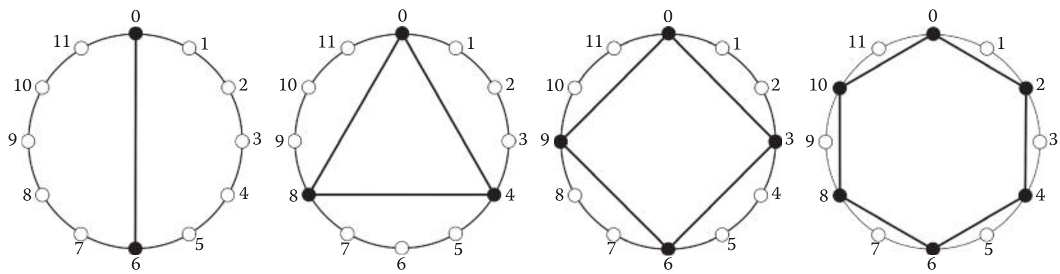

FIGURE 16.3 The four divisors of 12 other than 1 and 12.

obtain the diagram in Figure 16.4 that reveals four pulses that remain without any onsets; these occur at positions 1, 5, 7, and 11. If beats were played at any of these four positions in this context, they would be considered as being *strongly* off the beat, in the sense that they are off-beat relative to all possible regular meters. Therefore, we define the *off-beatness* measure of a rhythm as the number of onsets that the rhythm contains at these four distinguished locations.*

The off-beatness measure is the converse of Stephen Handel's measure of *metrical strength*, determined by the number of co-occurrences of the onsets of the rhythm with the metrically strong beats, which in a 16-pulse cycle as in Figure 16.6 are {0, 4, 8, 12}.†

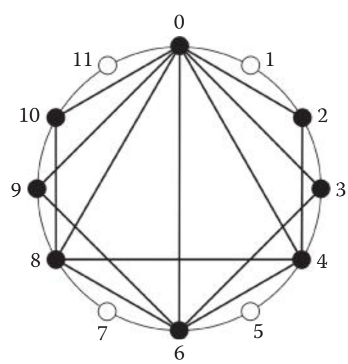

FIGURE 16.4 In a 12-pulse clock the *off-beat* onset positions are {1, 5, 7, 11}.

* The off-beatness measure is a precise objective mathematical measure. Syncopation, on the other hand, has many definitions in the literature, most of which are subjective. Ruth Stone (1985), p. 140, however, equates off-beat with syncopation. Locke, D. (1982) provides a discussion on how important the principle of off-beat timing is in much of the sub-Saharan music. Vurkaç, M. (2011, 2012) uses the off-beatness measure to analyze the directionality of timelines in a variety of Afro-Latin musics.

† Handel, S. (1992). As a measure of "irregularity," the off-beatness measure as defined here attributes a lot of weight on only four metric positions 1, 11, 5, 7. However, these are the most important positions in a polymetric context, and it is interesting to determine how useful such a streamlined measure can be. The measure can be generalized to a weighted version that does not put so much weight on these four positions. One way to do this is to create an off-beatness weight for every pulse in the cycle that depends on how many meters render the pulse off-beat. Thus, for a 12-pulse cycle with meters [12], [6-6], [4-4-4], [3-3-3-3], and [2-2-2-2-2-2], we would obtain, for pulses 0 to 11, the weights {0, 5, 4, 4, 3, 5, 2, 5, 3, 4, 4, 5}. The generalized off-beatness value would then be calculated by summing these weights for all pulses that have an onset, and normalizing by dividing the sum by the number of attacks in the rhythm. It would be interesting to compare such a weighted off-beatness measure with the unweighted version.

Armed with this new measure of irregularity, let us reconsider the *bembé*, *tonada*, and *sorsonet* rhythm necklaces shown in Figure 15.5. It is noteworthy that both the tonada and sorsonet rhythms take on off-beatness values equal to 1 due their onsets at positions five and one, respectively, whereas the bembé has an off-beatness value of 3 due to its onsets at positions 5, 7, and 11. Indeed, all the rotations of the three necklaces used in practice have off-beatness values equal to 1 and 2, except for the bembé. Therefore, the off-beatness measure may provide a mathematical formula for rating the preference of the bembé timeline among this family of timelines.

The set of four off-beat pulse positions {1, 5, 7, 11} has an interesting mathematical interpretation as well. These numbers are the numbers between 0 and 12 that generate (visit) all 12 pulses when we travel along the circle starting at zero and advance in steps of size equal to the numbers. Assume, for example, that we travel in a clockwise direction starting at pulse zero, and refer to Figure 16.5. If the steps are of size 1, the sequence (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) that determines a convex polygon is generated. If the steps are of size 5, the sequence (0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7) determining a star-polygon is obtained. Steps of size 7 realize the sequence (7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5, 0), the same star-polygon. Finally, steps of size 11 produce the sequence (0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1), the same as the previous convex polygon.

The off-beatness measure is easily generalized to other even values of the number of pulses. For 16-pulse cycles, the off-beat onset positions are {1, 3, 5, 7, 9, 11, 13, 15} as illustrated in Figure 16.6,* and for 24-pulse cycles, the off-beat onset positions are {1, 5, 7, 11, 13, 17, 19, 23} as shown in Figure 16.7.

The off-beatness property provides a tool for categorizing rhythms, as well as for illuminating musicological discourse, as the following examples illustrate. For the first example, consider the clave son and the clave rumba illustrated in Figure 16.8. In Cuba, the clave son is associated with secular music and dance heavily influenced by Spanish Christian sensibilities, whereas the rumba is considered to be less commercial and closer to traditional folkloric African religious roots. Whereas the Christian church has had a

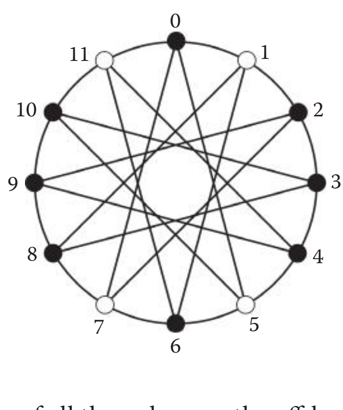

FIGURE 16.5 The four generators of all the pulses are the off-beat pulse numbers.

* Reinhard Flatischler (1992), p. 120, calls these pulse positions *double-time off-beat*, and reserves the term *off-beat* for pulses {2, 6, 10, 14}.

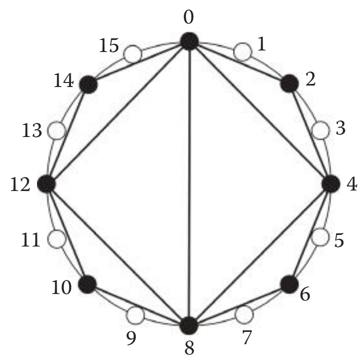

FIGURE 16.6 In a 16-pulse clock, the *off-beat* positions are {1, 3, 5, 7, 9, 11, 13, 15}.

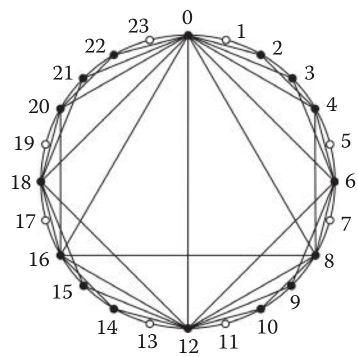

FIGURE 16.7 In a 24-pulse clock, the *off-beat* positions are {1, 5, 7, 11, 13, 17, 19, 23}.

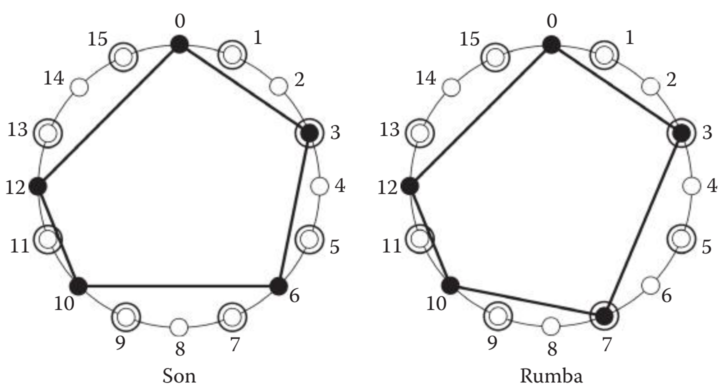

FIGURE 16.8 The *off-beatness* of the clave son and the clave rumba.

long history of vilifying syncopated music, the African religions venerated it. As a consequence, one might expect the more traditional rumba to be more syncopated than the son. From Figure 16.8, we see that the off-beatness value of the son is 1 since it has only one onset at pulses {1, 3, 5, 7, 9, 11, 13, 15}. On the other hand, the rumba has an off-beatness value of 2 due to the onsets at pulses three and seven. Since off-beatness measures a type of mathematical syncopation, it confirms our expectation. In contrast to the off-beatness measure, Handel's metrical strength yields a value of 2 for both rhythms determined by pulses {0,12}, and thus does not discriminate between the son and the rumba.

For a second example, consider the five ternary meters (compás) used in more than 70 styles of flamenco music of southern Spain, and refer to Figure 16.9. First, if we compare the off-beatness measure with the rhythmic oddity property, it is interesting to note that of the five meters, the bulería is the only one that has the rhythmic oddity property, and thus it is not a very discriminating property. While it is true that the bulería is the only rhythm among these five that contains intervals of lengths 1, 2, 3, and 4 (the other rhythms have intervals of lengths 2 and 3 only), it would be nice to be able to discriminate between the remaining rhythms based on some measure of syncopation. The off-beatness value goes further in this direction. The fandango and guajira are the only rhythms with an off-beatness value of 0. The seguiriya has an off-beatness value of 1, the buleria an off-beatness value of 2, and the soleá has the highest value of 3. It is worth noting that the soleá is considered to be one of the most paradigmatic and genuine styles of flamenco music. In the words of Nan Mercader, *la soleá es uno de los palos más jondos del flamenco*. Might this be explained by the fact that it has such a high off-beatness value?

As a final example, consider the three rhythms belonging to the [3-3-2] necklace that start on the three onsets (refer to Figure 16.10). The rhythms on the leftmost and the

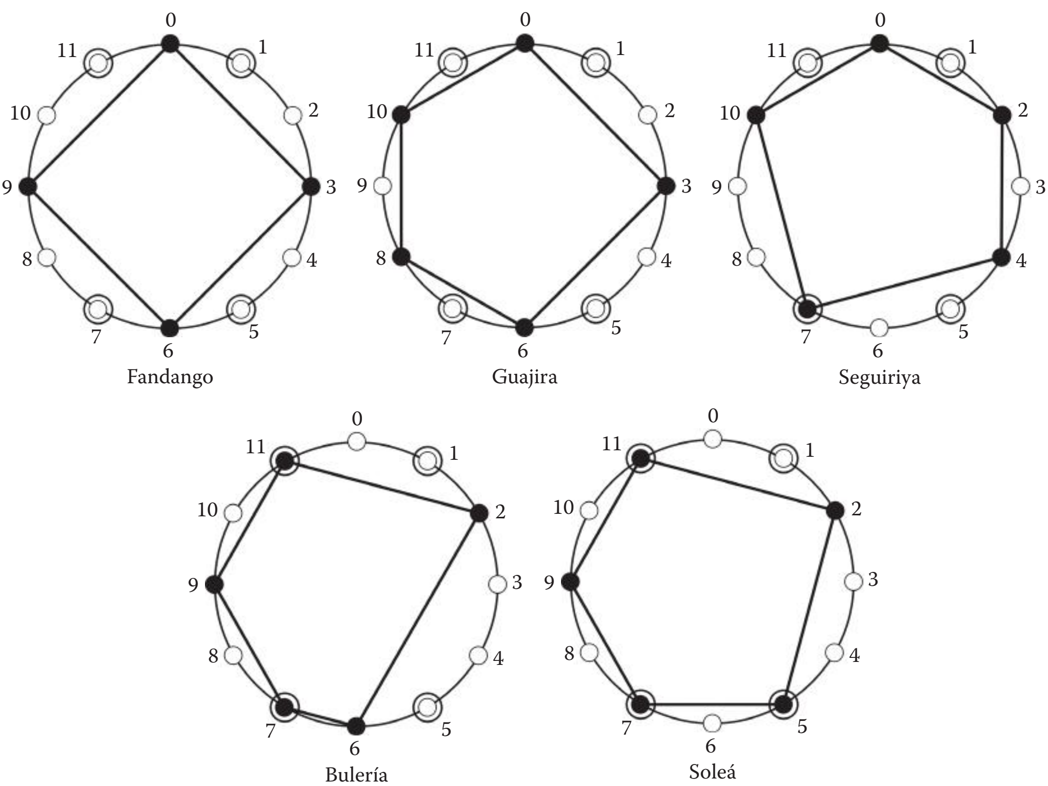

FIGURE 16.9 Calculation of the *off-beatness* values of the five flamenco ternary meters.

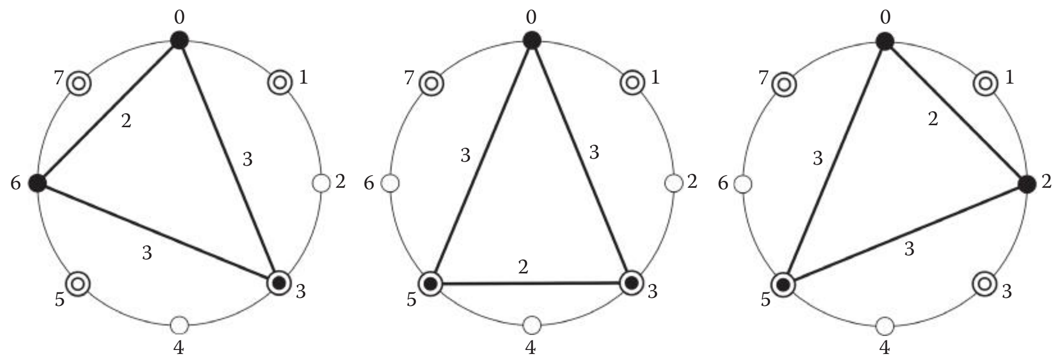

FIGURE 16.10 The off-beatness of the three most common rhythms of the [3-3-2] necklace.

rightmost diagrams have an off-beatness value of 1, whereas the one in the center has a value of 2. Of these three rhythms, the leftmost rhythm [3-3-2] is preferred over the other two, in the sense that it is encountered in the musical practice of more cultures around the world. Note that only this rhythm has two of its onsets on the first and last fundamental beats at pulses zero and six, thus introducing closure. Furthermore, only this rhythm engenders a cognitive surprise due to the fact that the first two inter-onset intervals are equal. The listener expects a regular rhythm with inter-onset interval durations of three pulses, which is broken by the third interval of two pulses. Therefore, in this example, these properties appear to override the possible desirability of a higher off-beatness value of the rhythm in the middle.

CHAPTER 17

Rhythm Complexity

ARE WEST-AFRICAN TRADITIONAL RHYTHM *TIMELINES* more complex than North-Indian *talas*? Can the choice of the ostinato rhythmic pattern in Steve Reich's *Clapping Music* be informed in terms of rhythm complexity? Has the evolution of the popular rhythms of the world favored an increase in their complexity? Can the difficulty of learning to perform a rhythm be predicted with a simple and elegant mathematical formula? How similar is the rhythmic oddity property prevalent in the Aka Pygmy music to the Western concept of syncopation? How powerful is rhythm complexity as a feature for music genre classification and music information retrieval? An introduction to the search for answers to these questions is the focus of this chapter.

Rhythm is arguably the most fundamental aspect of music,* and complexity is one of its most salient features.† Musicologists routinely comment on the complexity of rhythm present in music from different cultures. In his analysis of African rhythmic systems, Simha Arom writes that they are "the most complex of all those which are known all over the world."‡ According to the Reverend Arthur Morris Jones: "No European musician could clap and sing any but the more simple examples of African music."§ Yet, the formal investigation of the complexity of rhythm has been largely overlooked in the literature. A musical concept closely related to rhythm complexity is syncopation, a topic already explored in Chapter 13. However, as we saw there, formal definitions of syncopation are lacking. A typical definition of syncopation is the one found in the *Collins English Dictionary*:

* Although most musicologists argue for the supremacy of rhythm over other features of music, the thesis is not without its detractors. The composer Olivier Messiaen, for example, writes: "The melody is the point of departure. May it remain sovereign! And whatever may be the complexities of our rhythms and our harmonies, they shall not draw it along in their wake, but, on the contrary, shall obey it as faithful servants." See Messiaen, O. (1956), p. 13.

† Gabrielson, A. (1973a,b) used factor analysis and multidimensional scaling to uncover 15 perceptual features of rhythm, and complexity stood out among them. Conley, J. K. (1981), p. 69, experimented with 10 physical features of music complexity calculated from Beethoven's *Eroica* Variations, Op. 35, and found that the rate of rhythmic activity (in terms of the number of rhythmic events) was the most powerful measure of complexity. Interestingly, Wang, H-M., Lin, S.-H., Huang, Y.-C., Chen, I.-C., Chou, L.-C., Lai, Y.-L., Chen, Y.-F., Huang, S.-C., and Jan, M.-Y. (2009) showed that the complexity of rhythms can modify the interbeat duration patterns of the heart of the listener.

‡ Arom, S. (1984), p. 51.

§ Jones, A. M. (1949), p. 295.

"The displacement of the usual rhythmical accent away from a strong beat onto a weak beat." A mathematician would not only demand formal definitions of "strong" and "weak" beats, but would be baffled by how to interpret the term "usual." On the other hand, many formal (mathematical) definitions of complexity do exist, most from domains other than music, but some from music itself. A typical example of the former is the Lempel–Ziv complexity of a binary sequence,[*] and two representatives of the latter are the *rhythmic oddity* property[†] and the *off-beatness* discussed in the previous chapter.[‡]

Concerning the rhythms of India, the journalist and producer Joachim-Ernst Berendt writes: "It is necessary … to say a few words about the mysteries of Indian music. Its talas, its rhythmic sequences—incomprehensible for Western listeners—can be as long as 108 beats; yet the Indian ear is constantly aware of where the sam falls."[§] Kofi Agawu reviews a plethora of published claims about the purported complexity of African rhythms.[¶] Comparing African and Indian music with European music, Benjamin I. Gilman writes: "Hindu and African music is notably distinguished from our own by the greater complication of its rhythms. This often defies notation."[**] Concerning the measurement of rhythmic complexity, Martin Clayton writes: "I can think of no objective criteria for judging the relative complexity or sophistication of rhythm in, for example, Indian rag music, Western tonal art music, and that of African drum ensembles."[††]

The concept of complexity is extremely fluid. Its definition depends to a great extent on the context[‡‡] and the purpose to which it is put.[§§] In an information theory context, a metronomic pulsation is least complex, and random noise is most complex. However, in a musical context, random noise is not complex at all. The most complex musical rhythms exhibit a degree of complexity that lies somewhere between complete order and complete disorder.[¶¶] However, determining the exact point within this continuum is easier said than done. As a consequence, numerous definitions of complexity have been proposed. Complexity is also multidimensional, and there are many ways of measuring and combining these dimensions.[***] Ilya Schmulevitch and Dirk-Jan Povel distinguish between three broad categories of complexity measures for musical rhythms: *hierarchical*, *dynamic*, and *generative*.[†††] Hierarchical measures refer to structure at several levels simultaneously, dynamic measures refer to the nonstationarity of the input over time, and generative measures depend on the

[*] Lempel, A. and Ziv, J. (1976). See Coons, E. and Kraehenbuehl, D. (1958) for some early work on the application of information theory to the analysis of musical structure.

[†] Chemillier, M. (2002), Chemillier, M. and Truchet, C. (2003).

[‡] Toussaint, G. T. (2005b).

[§] Berendt, J.-E. (1987), p. 202.

[¶] Agawu, K. (1995).

[**] Gilman, B. I. (1909), p. 534.

[††] Clayton, M. (2000), p. 6.

[‡‡] Repp, B. H., Windsor, W. L., and Desain, P. (2002). Toussaint, G. T. (1978) provides a tutorial survey on the dependence between the perception and recognition of patterns in spatial (visual) and temporary (auditory) modalities, and the context in which those patterns are perceived.

[§§] Wolpert, D. H. and Macready, W. (2007). See Crofts, A. R. (2007), p. 25, for the relevance of complexity to evolution.

[¶¶] Eglash, R. (2005), p. 154.

[***] Sioros, G. and Guedes, C. (2011), p. 385, propose a complexity measure that combines the density of events in a rhythm with its syncopation by means of the formula: complexity = $\{\text{density}^2 + \text{syncopation}^2\}^{1/2}$. See also Essens, P. (1995).

[†††] Shmulevich, I. and Povel, D.-J. (1998, 2000a,b).

amount of effort required to generate the rhythms. Rhythm complexity may also be measured with respect to perception and performance (also called production[*]). Furthermore, these complexities depend on additional factors such as tempo and the underlying meter.[†] Experiments by D. J. Povel demonstrated that "changing the tempo of temporal sequences may cause dramatic changes in the perceived rhythmical characteristics."[‡] In some contexts such as music transcription, it is desirable to determine the notation of a rhythm that minimizes the performance complexity while affecting the perceptual complexity as little as possible.[§] In this chapter, several definitions and measures of rhythm complexity are compared.[¶] Some of these are better than others, and the reader may wonder: why not just describe the best one? The answer is that there is no best. The usefulness of a measure depends on its intended application. Furthermore, it is hoped that some readers may be inspired by these concepts to invent new measures that may perhaps combine features of the measures described here. It has been my experience during many years of teaching at universities that just presenting the best correct algorithms is not necessarily the best way to teach. It is sometimes better to teach incorrect algorithms first. Even better is to teach incorrect algorithms that students believe to be correct. Then, after seeing counter-examples, the students have the opportunity to learn the reasons for their failure, and to attempt to fix them. Following such an experience, students have greater appreciation for the correct solutions. The best learning does not always happen when knowledge is served on a plate, but when the learner has to construct that knowledge.

OBJECTIVE, COGNITIVE, AND PERFORMANCE COMPLEXITIES

Everyone can understand the principle behind juggling three balls, as well as the written instructions in a book on how to juggle. On the other hand, picking up three balls and juggling them is another matter altogether. In other words, we all know very well that *perceptual* or *cognitive* complexity is not the same as *performance* complexity. It is easier to recognize a favorite song than to sing it. In the words of Marvin Minsky: "Learning to recognize is not the same as memorizing. A mind might build an agent that can sense a certain stimulus, yet build no agent that can reproduce it."[**] In the same way, there is no logical *a priori* reason why a formal mathematical measure of complexity should agree

[*] Fitch, W. T. (2005), p. 31.

[†] Vinke, L. N. (2010), p. 41, Scheirer, E. D., Watson, R. B., and Vercoe, B. L. (2000). Palmer, C. and Krumhansl, C. L. (1990) have shown experimentally that rhythm perception (and therefore rhythm complexity) is influenced by the underlying meter. Rhythm perception also depends on rhythmic grouping. Music theorists such as Lerdahl, F. and Jackendoff, R. (1983) have argued that meter and figural grouping are independent. However, psychologists have obtained experimental evidence that they are not only dependent, but that figural grouping may be even more important than meter in judging rhythm similarity. See Handel, S. (1992, 1998). Furthermore, in the field of music information retrieval, Chew, E., Volk, A., and Lee, C.-Y. (2005) have shown that a type of meter extracted from onset grouping is quite successful at classifying certain types of music. They call the measure-based definition of meter used in traditional Western music, the *outer meter*, and a meter extracted from the grouping of the note onsets (ignoring the measures and bar lines), the *inner meter*. This approach to solving music problems is referred to as *inner metric analysis*.

[‡] Povel, D. J. (1984), p. 330.

[§] Nauert, P. (1994), p. 227.

[¶] See Thul, E. and Toussaint, G. T. (2008a) for a comparison of many more measures of rhythm complexity, and Thul, E. and Toussaint, G. T. (2008b,c) for a comparison of the complexity between African timelines and North Indian talas.

[**] Minsky, M. (1981), p. 30.

with either cognitive or performance complexities. The structures inherent in cognitive or performance complexities may not be adequately captured by a simple mathematical formula. In this section, several measures of the complexity of rhythm are compared by means of illustration with respect to the popular five-onset, 16-pulse clave rhythms highlighted in the preceding chapters.

One of the oldest measures of complexity used in music analysis is defined in terms of the predictability of the outcomes of a random process. To illustrate this idea, assume for the sake of a simple gedanken experiment that in the music from a fictitious planet called Alpha, songs use inter-onset interval durations of either one pulse or two pulses, and that each of these two types of songs occurs with the same frequency, that is, if we select at random a rhythm from a song from planet Alpha, it will have durations of one pulse with probability 0.5 and durations of two pulses with probability 0.5. The probability distribution characterizing this scenario is pictured in Figure 17.1 (left). Assume further that in another planet Zeta, the inhabitants use only durations of two pulses. Then, the probability distribution characterizing the songs from planet Zeta is given in Figure 17.1 (right). The difference between these two extreme distributions implies that in planet Alpha, one cannot predict with certainty the durations used in a song chosen at random, whereas in planet Zeta, one is certain that a song selected at random will have durations of two pulses. In this context, predictability implies that there is no information obtained by selecting a song from planet Zeta. On the other hand, the nonpredictability of the outcome in planet Alpha implies that a maximum amount of information about the duration used is gained by selecting a song. Translating these ideas into the language of complexity, we obtain that predictability implies simplicity, whereas nonpredictability or randomness implies complexity.

From the geometrical point of view, nonpredictability, randomness, and therefore complexity may be characterized by the flatness of the underlying probability distribution. In this sense, a flatter distribution implies greater complexity, and thus the music in planet Alpha is more complex than the music in planet Zeta. Thus, the problem of measuring the complexity of a process has been converted to measuring the *flatness* of a distribution.

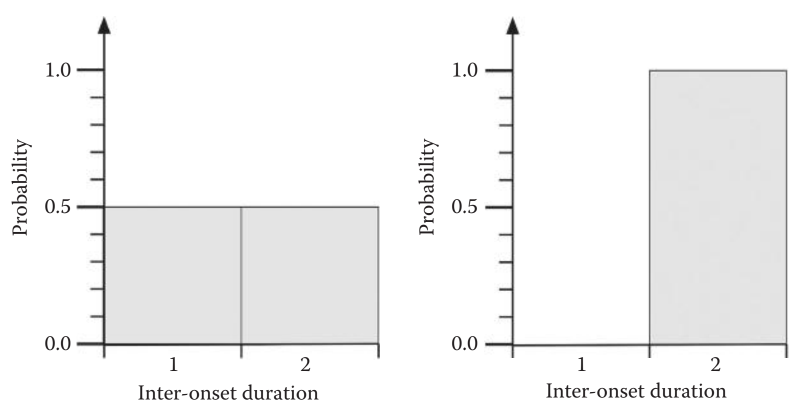

FIGURE 17.1 Two probability distributions: perfectly flat (left) and perfectly peaked (right).

There are an uncountable number of ways to measure the flatness of a probability distribution, histogram, or by analogy, a geographical terrain. One measure is the smallness of the maximum height of the distribution. In Figure 17.1, the maximum height on the left is 0.5 and on the right 1.0. Since 0.5 is smaller than 1.0, we would conclude that the distribution on the left is flatter than the one on the right. In general, a smaller maximum height implies a flatter distribution, but this is not necessarily so when the random variable can take on more than two values.

One very popular measure of the flatness of a distribution is the *entropy*.* With respect to the probability distribution in Figure 17.1, let p_1 and p_2 denote the probabilities of observing inter-onset durations of one and two pulses, respectively. Let $\mathrm{P} = (p_1, p_2)$ denotes the probability distribution. It follows that $p_1 + p_2 = 1$. Then the entropy, usually denoted by H(P), is given by the negative of the quantity $(p_1 \log p_1 + p_2 \log p_2)$. This quantity takes a maximum value when the distribution is flat, that is, when all the probabilities are equal, in this case when $p_1 = p_2$. It takes on its minimal value when one probability is equal to one and the other zero. In this case for the distribution on the left with $p_1 = p_2 = 0.5$, the entropy is one, and for the distribution on the right with $p_1 = 0$ and $p_2 = 1$, the entropy is zero (note that by convention $0 \log 0 = 0$).

In the more general case in which the random variable takes on N different values where $N > 2$, we have a probability distribution given by $\mathrm{P} = (p_1, p_2, \ldots, p_N)$, and the entropy is then given by

$$\mathrm{H(P)} = -\sum p_i \log p_i$$

where the summation is over all $i = 1, 2, \ldots, N$.

One natural way to use the entropy as a measure of the flatness in the case of rhythms is to apply it to the histogram of all the inter-onset intervals contained in a rhythm. Strictly speaking, such a histogram is not a probability distribution that describes the behavior of a random variable, but rather a frequency or multiplicity count of the number of inter-onset durations of any given length that are present in the rhythm. Nevertheless, we may normalize the histogram so that its area is equal to one, and pretend that it is a probability distribution. The important point is not the faithfulness of the inter-onset interval histogram to probabilistic reality, but rather the entropy's ability to measure the flatness of any histogram, no matter what its origin. Consider the African bell pattern shown in

* Kulp, C. W. and Schlingmann, D. (2009), Cohen, J. E. (2007), p. 139, Gregory, B. (2005), p. 11, Streich, S. (2006), p. 20. In his PhD thesis, Streich proposes algorithms to compute estimates of a variety of features of music complexity (based on rhythm, tonality, and timbre) from musical audio signals. Don, G. W., Muir, K. K., Volk, G. B., and Walker, J. S. (2010), p. 44, quantify the complexity of musical rhythms (represented as binary sequences) with the entropy function. In an early influential book, Abraham Moles (1966) applies Shannon's entropy-based theory of information (uncertainty) to the analysis of expectancy and originality in music. Other information measures that have also been applied extensively to music (when *two* probability distributions are involved) include the *discrimination information* (also called the Kullback–Liebler distance). For example, Farbood, M. M. and Schoner, B. (2009) apply the discrimination information to determine the salience of several features of the acoustic signal for the perception of musical tension. That the discrimination information is appropriate for measuring the distance between two probability distributions is due to the fact that the measure is closely related to the Bayes error probability (or *variation*); see Toussaint, G. T. (1975). Scheirer, E. D. (2000), p. 98, uses the variance of the inter-onset durations as a measure of rhythm complexity.

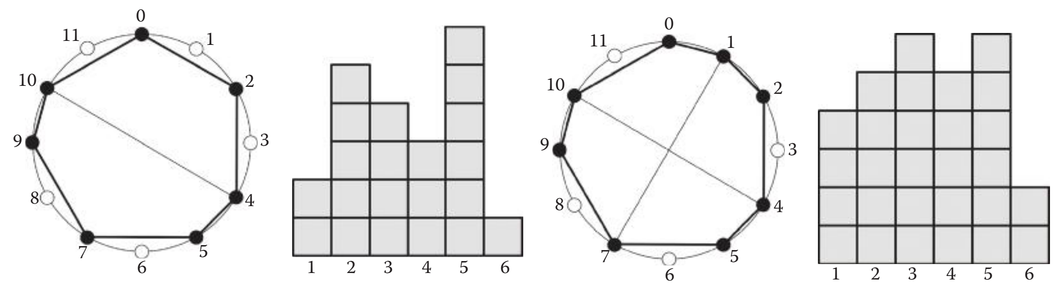

FIGURE 17.2 Entropy (left) = 2.398 and (right) = 2.513.

Figure 17.2 (left). This is a rotation of the bembé timeline, and it is a deep rhythm, as can be seen from its histogram laid out next to it. Deep rhythms have relatively peaked histograms since they resemble one-sided Maya pyramids when the histogram bin-heights are sorted by increasing height. The rhythm on the right in Figure 17.2 is the clapping pattern used by Steve Reich in his minimalist piece *Clapping Music*. Reich inserted one additional onset at pulse one to the African bell pattern. By doing so, he introduced an additional antipodal pair with distance six between pulses one and seven, to the pair (4, 10) already present. However, as may be observed from the figure, this change also makes the histogram flatter. This is reflected by the increase in entropy from 2.398 to 2.513. In this sense, Reich's choice transformed the African bell pattern into a more complex rhythm by adding that onset. Alternately, consider the nine-onset *shekere* rhythm used in the *bembé* music of Cuba pictured in Figure 17.3. Removing the onset at pulse 11 converts this rhythm to a rotation of Reich's *Clapping Music* pattern (when started at pulse five). From this point of view, Reich transformed the asymmetric *bembé* shekere rhythm to one that contains mirror symmetry.

In addition to the wooden claves and metal bells described in Chapters 4 and 5, a *shekere*, such as the one illustrated in Figure 17.4, is another widely used instrument for playing rhythmic timelines in African and Afro-Cuban traditional music. It is made from a hollowed-out gourd, around which is enveloped a fishnet that holds a large quantity of

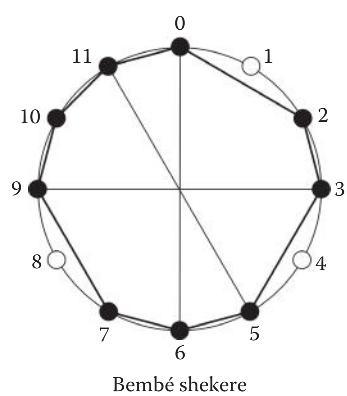

FIGURE 17.3 A *shekere* rhythm used in the *bembé* rhythm ensemble.

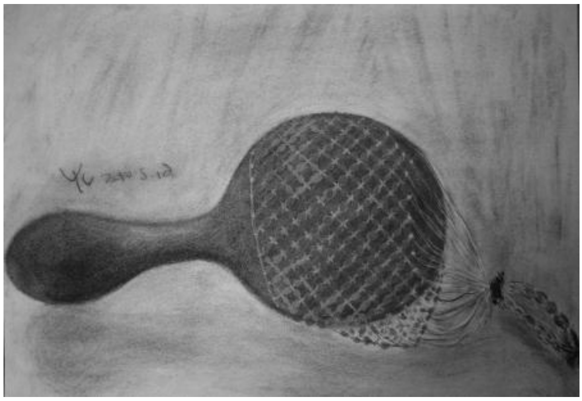

FIGURE 17.4 A *shekere*. (Courtesy of Yang Liu.)

beads (or seeds) around the gourd. Rhythms are typically played by either pulling on the fishnet or bouncing the gourd between one's thigh and the free hand.

Steve Reich intended *Clapping Music* to be performed by two people clapping hands. Both performers clap the same rhythm shown in Figure 17.2 (right). One performer repeats the sequence continually throughout the piece, while the second player shifts the pattern by one time unit every time the pattern has been repeated 12 times. The piece ends when both performers play in unison again.

There has been speculation about how Reich might have come to adopt this particular rhythmic pattern [x x x . x x . x . x x .] for this composition. One combinatorial analysis by Joel Haak proceeds by eliminating candidates while respecting several mathematical constraints.* His argument proceeds as follows. There are eight claps per cycle of 12 pulses in *Clapping Music*. The number of candidates or ways one can select 8 out of 12 pulses in which to clap is (12!)/(8!)(4!) = 495. His first constraint is that a pattern should begin with a clap rather than a silent pulse. His second constraint is that the silent intervals between two consecutive claps should be short, and therefore two consecutive pauses (silent pulses) are not permitted. With these two constraints, the original 12 units, composed of eight claps and four rests, are reduced to eight units made up of four clap-rest patterns [x .] and four solitary claps [.]. In this setting, there are now only eight two-valued elements taken four at a time, and thus the formula for the total number of possible patterns becomes 8!/((4!)(4!)) = 70. Among these 70 patterns, there are some that are rotations of each other, and therefore are redundant, since they would yield the same composition from a different starting point. This observation leads to Haack's third constraint: the patterns should not be cyclic permutations of each other. With the addition of this third constraint, the 70 possible patterns are reduced to only 10 patterns. His fourth constraint is that during the execution of the entire piece, the combined 12-pulse clapping patterns made by both performers should not repeat themselves before the ending of the piece. His fifth and last constraint is that consecutive repetitions of the number of claps between consecutive pauses are not allowed. In other words, patterns such as

* Haak, J. K. (1991, 1998).

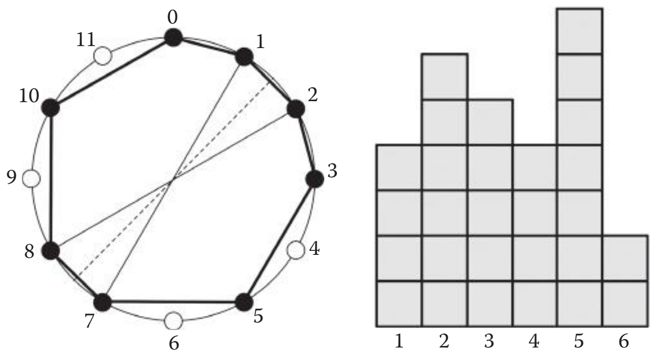

FIGURE 17.5 The second rhythm found by Joel Haak (left), and its interval histogram (right) with entropy = 2.49.

[x x x . x x . x x . x .] and [x x x x . x x . x . x .] are not permitted because of the presence of the repetitions [x x . x x .] = [x x .] [x x .] and [x . x .] = [x .] [x .], respectively. With these five constraints, only two of the 495 patterns remain. One is the pattern Reich chose in Figure 17.2 (right), and the other is the pattern [x x x x . x . x x . x .] shown in polygon notation in Figure 17.5.

Haak does not speculate on the criteria that might be employed for choosing between the two finalist candidates that remain after the five rounds of constraint-satisfaction eliminations have been applied. Indeed, there are several arguments that emerge from musicology, geometry, and information theory that may be enlisted to come to the rescue here. One obvious solution is to pick the rhythm that minimizes the number of consecutive claps without gaps of silent pulses. Then, we end up with Reich's pattern that starts with a group of three rather than four claps.

Other possible criteria for selecting Reich's pattern become evident by comparing the polygonal representations of these rhythms in Figures 17.5 and 17.2 (right). For one, although both patterns exhibit mirror symmetry, the mirror symmetry in Reich's pattern is with respect to a line that is incident to two antipodal onsets at pulses one and seven. Haak's rhythm is not symmetric about a pair of onsets but rather about a line that falls midway between the pairs (1, 2) and (7, 8). Whether this mathematical property has musicological capital is yet to be investigated. Another difference between the two rhythms with respect to their antipodal pairs of onsets is that although both rhythms contain two such pairs, the pairs in Haak's rhythm given by (1, 7) and (2, 8) are adjacent to each other, whereas the pairs in Reich's rhythm given by (1, 7) and (4, 10) are orthogonal to each other, and thus form a regular four-beat underlying structure. This difference between the two candidates probably carries greater musicological weight, although the nature of this weight is also a topic that needs investigation.

Alternately, we may resort to the criterion of rhythmic evenness of the two patterns to arrive at Reich's pattern. Figure 17.6 shows the two rhythms with their onsets plotted in a two-dimensional *onset-pulse* plane in which the x-axis is the pulse number (time) and the y-axis is the onset number. The onsets are connected together to form a polygon (shaded). The longest edge of this polygon, from pulse 0 to pulse 12 (zero), represents the locations

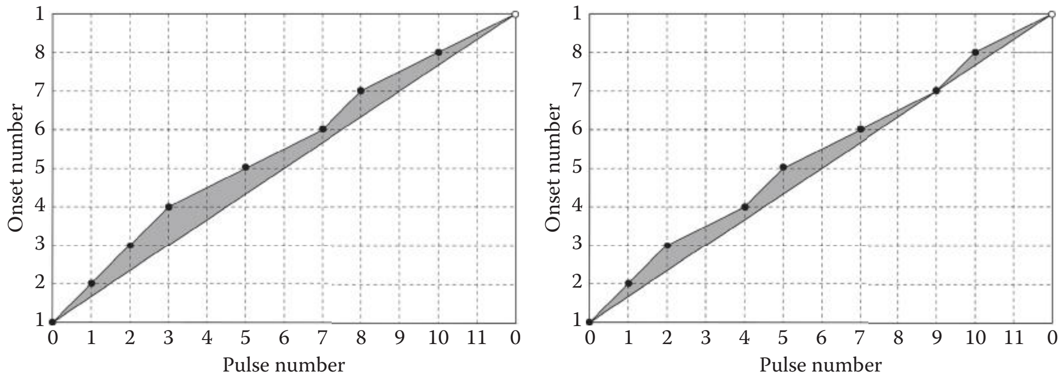

FIGURE 17.6 Deviation of Haak's (left) and Reich's (right) rhythms from a perfectly even rhythm.

Entropy	Bossa-Nova	Shiko	Son	Rumba	Gahu	Soukous
Full interval	1.84	1.84	2.24	2.44	2.72	2.72
Adjacent interval	0.72	0.97	1.52	1.52	1.52	1.92

FIGURE 17.7 The full and adjacent interval entropies of the six distinguished timelines.

of the onsets of perfectly even rhythms. Thus, the area of the shaded polygon is a measure of the unevenness of the rhythm. A smaller area implies a more even rhythm.* Comparing the polygon on the left from Haak's second rhythm with the polygon on the right from Reich's rhythm, it is clear that Reich's rhythm is more even. Indeed, the reduction in area is the result of the movement of the fourth and seventh onsets, by one pulse each, closer to the diagonal baseline of the polygon.

Finally, one could use the entropy of the interval content histograms to select Reich's pattern over Haak's second rhythm. Haak's second rhythm has entropy equal to 2.49, whereas Reich's pattern has entropy 2.513. The difference between the two is not large, but Reich's pattern still comes out ahead. Reich considered his pattern to be so successful that he went on to use it in several other compositions such as *Music for Eighteen Musicians*.†

The entropy may also be used as a global feature for comparing and classifying rhythms. As an example, consider the six distinguished five-onset, 16-pulse timelines of Figure 17.1. The entropies of their full and adjacent interval histograms shown in Figure 8.3 are listed in the following table in Figure 17.7, in increasing order from left to right. One of the weaknesses of the entropy for measuring rhythm complexity is immediately evident from the table. Even though the bossa-nova is arguably more complex than the shiko, and their full interval histograms look quite different, they have the same entropy value of 1.84. This is because the entropy function depends only on the heights of the histogram bins, and not on their location within the histogram. Since both histograms have four occupied bins of heights one, two, three, and four, their entropies are equal. Similarly, the entropy cannot distinguish between the full interval histograms of the

* This is but one measure of rhythmic evenness, a topic to be explored deeper in Chapters 18 through 21.
† Potter, P. (2000), p. 225. See Cohn, R. (1992a) for an analysis of some of Steve Reich's other phase-shifting music.

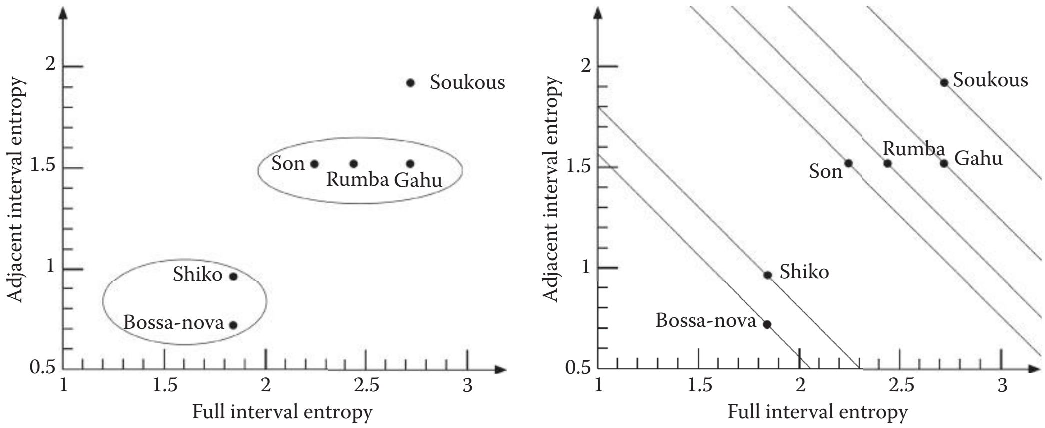

FIGURE 17.8 Clustering and ordering entropies of adjacent and full interval histograms.

gahu and soukous. Interestingly enough, in these two cases, the entropies of the adjacent interval histograms disambiguate the shiko from the bossa-nova, and the gahu from the soukous, even though they cannot by themselves distinguish between the son, rumba, and gahu.

Figure 17.8 (left) shows a plot of the full interval histogram entropy along the abcissa and the adjacent interval histogram entropy along the ordinate. The six timelines naturally fall into three clusters: one cluster is made up of shiko and bossa-nova that share the same value of full interval entropy, another cluster consists of the son, rumba, and gahu, which have the same value of adjacent interval entropy, and soukous is off by itself.

Although either of the two entropies is unable to distinguish between all six rhythms, if a new measure is defined as the sum of both entropies, then a perfect distinguishing ordering is possible, as illustrated in Figure 17.8 (right). This diagram shows the diagonal lines that are loci of constant sum of the ordinate and abcissa values. This measure produces the ordering: bossa-nova, shiko, son, rumba, gahu, and soukous.

In Chapter 9, we described several popular methods used in music information retrieval to classify rhythms automatically. Here, we take this opportunity to revisit the topic by introducing another widely used approach to classification that constructs decision trees by means of *space partitioning*.* The method is illustrated in Figure 17.9 with the toy example of the six distinguished timelines used previously. The idea is to partition the space with vertical and horizontal lines in an alternating fashion (if possible) so that we can easily produce a decision tree afterwards. First, the vertical line A is inserted at a coordinate value of 2.0. This produces two half-spaces that are partitioned next. Accordingly, horizontal line B is inserted on the left at coordinate 0.85, and horizontal line C on the right at coordinate 1.7. Next, vertical lines are inserted at coordinates 2.6 and 2.35 to separate the son, rumba, and gahu rhythms.

The binary space partition of the six rhythms shown in Figure 17.9 yields the binary decision tree shown in Figure 17.10.

* Safavian, S. R. and Landgrebe, D. (1991).

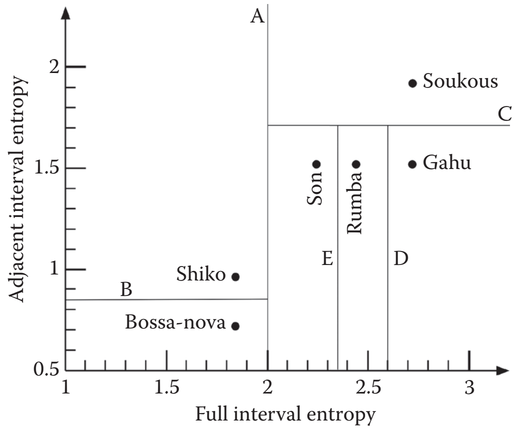

FIGURE 17.9 A binary space partition of the rhythms based on the two entropies.

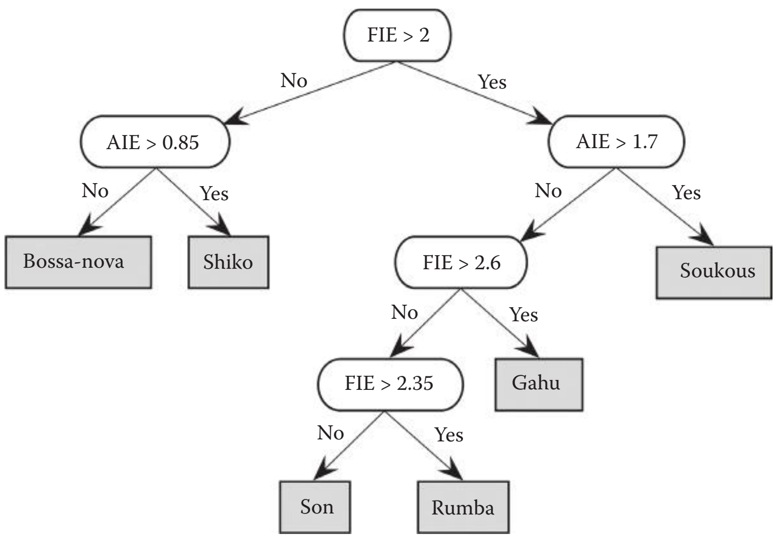

FIGURE 17.10 A binary decision tree based on the space partitioning in Figure 17.9.

LEMPEL–ZIV COMPLEXITY

In 1976, Abraham Lempel and Jacob Ziv proposed an empirical information-theoretic measure of the complexity of a finite-length sequence of symbols, in the context of data compression.* Their goal was to store a sequence of symbols in such a way as to use as little memory as possible. Their novel approach in fact yields a measure of the complexity of a given finite-length sequence by scanning it from left to right, looking for the shortest subsequences (patterns) that have not yet been encountered during the scan. Every time such a pattern is found, it is inserted in a growing dictionary of patterns. When the scan is completed, the *size* of this dictionary is the measure of the complexity of the sequence. For the

* Lempel, A. and Ziv, J. (1976).

special case of cyclic sequences such as the rhythms considered here, a concatenation of two instances of the rhythm cycle is scanned. The application of this measure of sequence complexity to musical rhythm was explored by I. Shmulevich and D.-J. Povel.*

Since the publication of the original data compression algorithm of Lempel and Ziv, many variations on their theme have been proposed. To illustrate just one simple variant by means of an example, consider the clave son timeline shown in binary box notation in Figure 17.11. Concatenating two copies of this rhythm yields the 32-pulse pattern shown at the top along with the pulse numbers. Figure 17.11 also shows each new subsequence found during the scan. The arrows underneath the sequence indicate the positions at which a new subsequence is discovered. The brackets with numbers underneath the arrows indicate the first and last pulses of each new subsequence discovered. A subsequence is considered newly discovered if it does not occur to the left of the previous arrow. Let us step through the algorithm to clarify the process. The scan is initialized at pulse zero, and of course this pattern is the first newly discovered sequence. The scan advances to pulse one, discovering another new sequence consisting of pulse number one. The next newly discovered pattern consists of pulses two and three. The fourth pattern consists of pulses four through seven, the fifth is made up of pulses 8 through 12, and the sixth of pulses 13 through 18. Starting at pulse 19, no new patterns are found because the sequence from pulse 19 through to the last pulse 31 with intervals [3-4-2-4] already occurs between pulses 3 and 16. All the different subsequences discovered in this way are listed in a dictionary at the lower left, and labeled in the order in which they are discovered. For the clave son, rhythm six subsequences are generated by this procedure, and therefore, its complexity is equal to six. This measure is relatively simple to compute, and it is, like the entropy, completely objective in the sense that it is defined in pure mathematical terms without any dependencies on psychological principles of perception.

The Lempel–Ziv measure has been compared experimentally with the complexity perceived by human subjects. The experiments used rhythms with a 16-beat measure typical of those found in Western music, and yielded negative results. The comparison of this

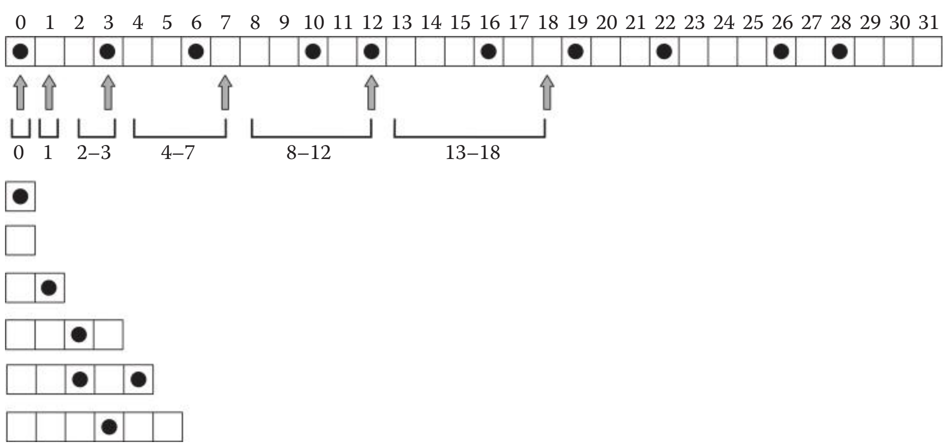

FIGURE 17.11 Illustrating the computation of the Lempel–Ziv complexity of the clave son.

* Shmulevich, I. and Povel, D.-J., *op. cit.*

measure with the other measures of rhythm complexity discussed here, with respect to the six African, Cuban, and Brazilian clave patterns, indicates that this measure is also inferior for non-Western rhythms. Furthermore, looking at the scores obtained for the six rhythms in Figure 17.13 shows that this measure is deficient for other reasons as well. There is almost no variance in the scores: all values are six except for the bossa-nova that receives a five. So, the measure does not discriminate well between short sequences such as these. Also, the scores do not make sense to anyone experienced in teaching or playing these rhythms. For example, shiko is the simplest of the six rhythms, and gahu more complex, both to recognize and to play, yet the Lempel–Ziv complexities are six for both of these rhythms. In conclusion, at present, it appears that information-theoretic measures per se are not able to capture well the human perceptual, cognitive, or performance complexities of short musical rhythms such as timelines. It is quite probable that the Lempel–Ziv measure may perform better for much longer rhythms or entire musical compositions.[*]

COGNITIVE COMPLEXITY OF RHYTHMS

In contrast to the information-theoretic measures of complexity, Jeff Pressing proposed a measure of the cognitive complexity of musical rhythms based on psychological properties of perception as well as musicological principles such as the amount of syncopation present in the rhythm.[†] The cognitive complexities of the 10 four-pulse patterns containing one or two onsets computed with Pressing's measure are given in Figure 17.12. One simple way to obtain a measure of cognitive complexity for longer rhythms such as the 16-pulse rhythms considered here is to first partition these rhythms into four units of four pulses each, then compute the complexities for each unit, and finally add these four complexity values. For example, the shiko pattern consists of the concatenation of the patterns [x…], [x. x.], [. . x.], and [x. . .]. Referring to Figure 17.12, we find the corresponding complexity values 0, 1, 5, and 0 for a total of 6. On the other hand, the rumba yields a Pressing cognitive complexity of 4.5 + 7.5 + 5 + 0 = 17. Examining the Pressing cognitive complexities of all six clave rhythms in the table of Figure 17.13 reveals more information than the Lempel–Ziv

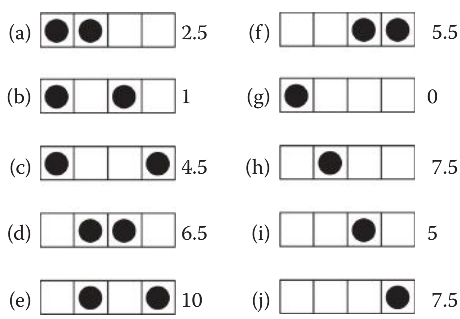

FIGURE 17.12 Pressing's cognitive complexities of 10 basic four-pulse rhythmic units (listed to the right of each unit).

[*] See also Chaitin, G. J. (1974) for information measures based on the shortest possible description of a rhythm.
[†] Pressing, J. (1997).

	Pressing	Lempel–Ziv	Entropy Adjacent	Entropy Full	Metric	Distinct Distances	nPVI
Shiko	6	6	0.97	1.84	2	4	66.7
Son	14.5	6	1.52	2.24	4	5	40.5
Soukous	15	6	1.92	2.72	6	7	70.5
Rumba	17	6	1.52	2.44	5	6	41.0
Gahu	19.5	6	1.52	2.72	5	7	23.8
Bossa	22	5	0.72	1.84	6	4	14.3

FIGURE 17.13 A comparison of seven measures of rhythm complexity.

complexity. For one, all the scores are different and the variance is quite large ranging from six for the shiko to 22 for the bossa-nova. The scores are also in good agreement with my personal teaching and performing experience. Shiko is easy, rumba is more difficult than son, and bossa-nova is the most difficult to recognize and perform.

IRREGULARITY AND THE NORMALIZED PAIRWISE VARIABILITY INDEX

The complexity of a rhythm may be characterized by the irregularity of the durations of its inter-onset intervals. There are many possible ways to measure irregularity. One approach is to measure the distance between the given rhythm and a perfectly regular one.* A widely used measure of irregularity in statistics is the classical standard deviation, which has been applied frequently to the analysis of rhythm in speech and language. However, in these applications, as in musical rhythm, the order relationships between adjacent intervals are important, and the standard deviation disregards them. To take order information into account, a measure should be sensitive to local change. The "normalized Pairwise Variability Index" (nPVI) is a measure that attempts to capture this notion of change. The nPVI for a rhythm is defined as

$$\text{nPVI} = \left(\frac{100}{m-1}\right)\sum_{k=1}^{m-1}\left|\frac{d_k - d_{k+1}}{\left(d_k + d_{k+1}\right)/2}\right|$$

where m is the number of adjacent inter-onset intervals and d_k is the duration of the kth interval. Although the nPVI has a long history of application to language, its ramifications in the music domain are just beginning to be explored.†

The values of the nPVI in Figure 17.13 show that irregularity does not necessarily translate monotonically to complexity. Shiko is clearly a much less complex rhythm than bossa-nova, as Pressing's complexity underscores. However, the nPVI is much greater for Shiko (66.7) than for bossa-nova (14.3). These results serve to highlight the fact that measuring rhythmic complexity is a complex problem, and much work still remains to be done.

* Toussaint, G. T. (2012b).

† Toussaint, G. T. (2012c).

CHAPTER 18

Dispersion Problems and Maximally Even Rhythms

Assume that a satellite company wants to install six electronic communications transmission towers in the city where you live, and that the city government has given the company permission to build them in any of 15 possible locations approved by its citizens, as indicated in the schematic map shown in Figure 18.1 (left). The company desires to select the 6 out of 15 locations that minimize the total signal interference between all pairs of transmission towers. Clearly, to satisfy this criterion, the locations selected should be as far away from each other as possible. But how should we measure the concept: *as far away from each other as possible*? There is a profusion of ways to measure this notion. However, in an idealized situation, and in the absence of additional knowledge that may affect our choice, it is reasonable to pick a natural and easily understood criterion: select the six locations that maximize the *average distance* between pairs of towers. Figure 18.1 (right) shows six candidate locations along with the distance between every pair indicated by a straight edge. Since, the average distance between a set of locations is the sum of all the pairwise distances divided by the number of distances, and the number of distances is fixed no matter which locations we select (in this case 6(5)/2 = 15), this criterion is equivalent to

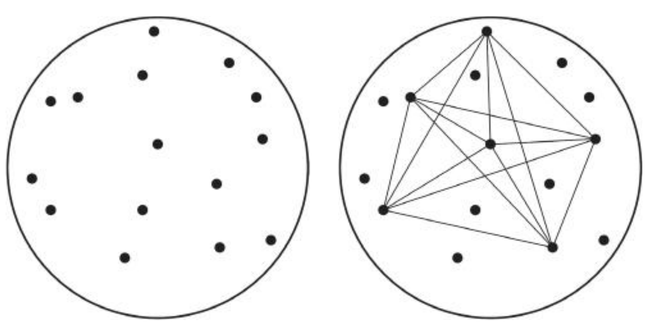

FIGURE 18.1 Selecting a subset of maximally dispersed locations in a region.

choosing the six locations that maximize the *sum* of their pairwise distances. This type of problem is called a *dispersion problem** in the field of *obnoxious facility location*, within the broader research area of *operations research*.

One of my main goals in writing this book is to build bridges between problems in music and in other areas of science, mathematics, and engineering. It is hoped that such bridges will help both areas that are connected by the bridge. To obtain some insight into the maximum dispersion problem, and how it relates to the theory of musical rhythm, consider first the equivalent problem in a simpler situation where the locations are restricted to lie on a perfectly straight highway such as one might find in Kansas, Saskatchewan, or Abu Dhabi. For concreteness, and as a start, consider the example of selecting 2 of 10 possible locations illustrated in the top diagram of Figure 18.2. Obviously, the two extreme locations at A and B, in the second diagram, maximize the distance between them. Consider now selecting the best three locations. We must select a third location C from among the eight remaining locations. However, in this situation, it does not matter which location is selected (refer to the third diagram), because no matter where C is located in between A and B, the total sum of distances increases by AC + CB = AB, and hence the total distance sum equals 2AB. This condition changes radically, however, when four locations are selected (refer to the fourth diagram). Having selected C anywhere between A and B, consider selecting the fourth location D anywhere between C and B. Clearly, no matter where D is located, its sum of distances to A and B remains fixed and equal to AD + DB. However, its distance to C varies depending on where D is located. Hence, D should be chosen to be as far as possible from C, that is, in the location adjacent to B. Furthermore, by symmetry, given D's position, the total sum of distances can be increased further by selecting C to be as far as possible from D, that is, in a location adjacent to A. This argument generalizes to any even number of locations selected from any number of allowable positions. In other words, the solution, in general, implies that half of the selected locations should be chosen from the leftmost positions, and the other half from the rightmost positions. The fourth diagram in Figure 18.2 shows the optimal solution for selecting four locations.

Since rhythm timelines are cyclic, let us turn our attention to the analogous dispersion problem of locating k points on a circle so as to maximize the sum of their pairwise distances. Since the highway is now a *beltway* (circle), the straight-line distances of Figure 18.2 become arc-lengths, or geodesic distances, as illustrated in Figure 18.3. The distance between A and

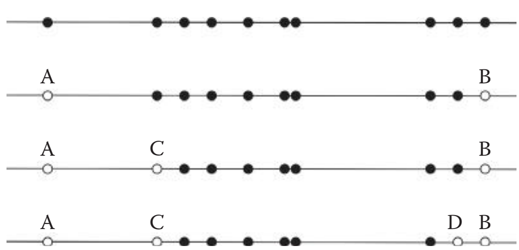

FIGURE 18.2 Selecting a subset of maximally dispersed locations on a line.

* Ravi, S. S., Rosenkrants, D. J., and Tayi, G. K. (1994) and Tamir, A. (1998).

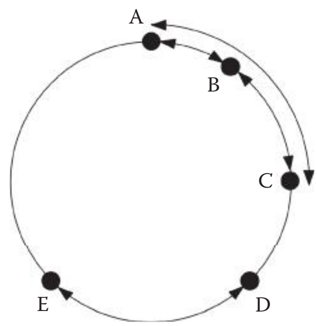

FIGURE 18.3 The distance between two points on a circle is the geodesic distance or shortest arc-length between them.

B, for example, is the clockwise arc-length starting at A and ending at B, or the counterclockwise arc-length starting at B and ending at A, but not the clockwise arc-length starting at B and ending at A, nor the counterclockwise arc-length starting at A and ending at B.

The question now becomes: how should a set of *k* points on a circle be arranged so that the sum of their pairwise arc-lengths is maximized? It turns out that if the points are located such that they form the vertices of a regular *k*-sided polygon (all sides are of equal length), then the resulting sum is a maximum. Thus, the structure of this solution is completely different from the case in which the points fall on a straight line, and might raise hopes that the sum of pairwise arc-lengths might be used either as a criterion for generating regular rhythms or perhaps *maximally even* rhythms, or for measuring how regular they are for the purpose of their automatic classification. Unfortunately, there is little cause for celebration here because it turns out that the converse is not true; rhythms other than maximally even or regular, and even highly irregular rhythms, can also maximize the sum of pairwise arc-lengths. A rhythm is maximally even if its attack points are distributed in time as evenly as possible.*

Before proceeding further, however, the concept of *maximally even* rhythms should be made precise, and before that, it is instructive to present their simpler offspring, *perfectly even* rhythms, and their opponents, *perfectly uneven* rhythms. Figure 18.4 (top) shows four regular rhythms. The first has two onsets in a cycle of four pulses, and the other three have four onsets in cycles of 8, 12, and 16 pulses, respectively. Regular rhythms are examples of perfectly even rhythms. The four rhythms at the bottom of Figure 18.4 are the perfectly uneven counterparts of the rhythms at the top: their onsets are clustered together as tightly as the underlying discrete pulse structure permits.

To emphasize the point that the sum of pairwise arc-lengths between the onsets of rhythms is not an acceptable measure of rhythm evenness, consider the six distinguished Afro-Cuban timelines in Figure 18.5. Each rhythm is shown with straight-line segments joining every pair of onsets to clearly identify all the arc-lengths. First, note that for all the rhythms the arc-lengths corresponding to the heavy edges that make up the convex polygon

* Clough, J. and J. Douthett, J. (1991) in their seminal award-winning paper, originally developed the concept of maximally even sets in the context of pitch class sets. See also Block, S. and Douthett, J. (1994).

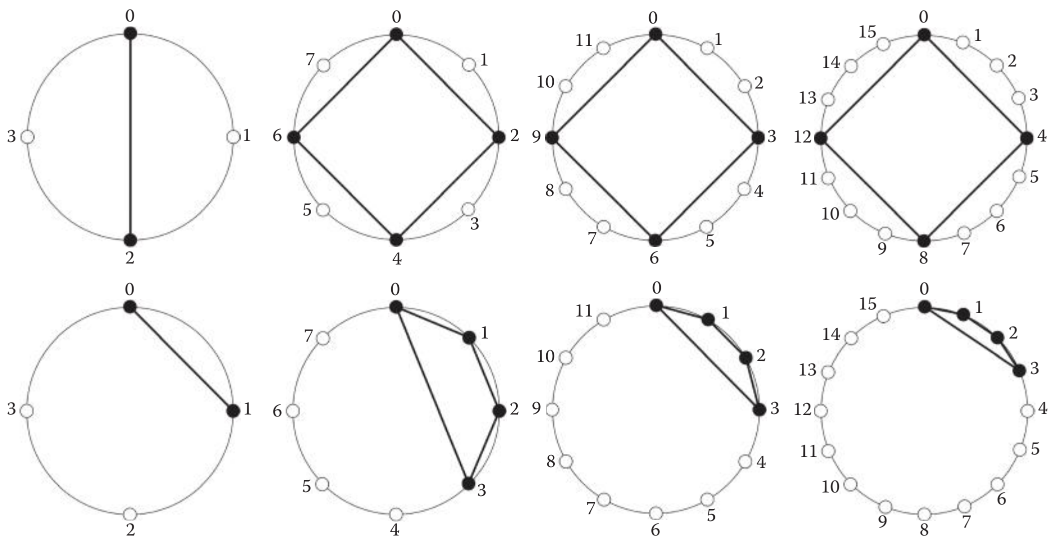

FIGURE 18.4 Four perfectly even rhythms (top) and their perfectly uneven counterparts (bottom).

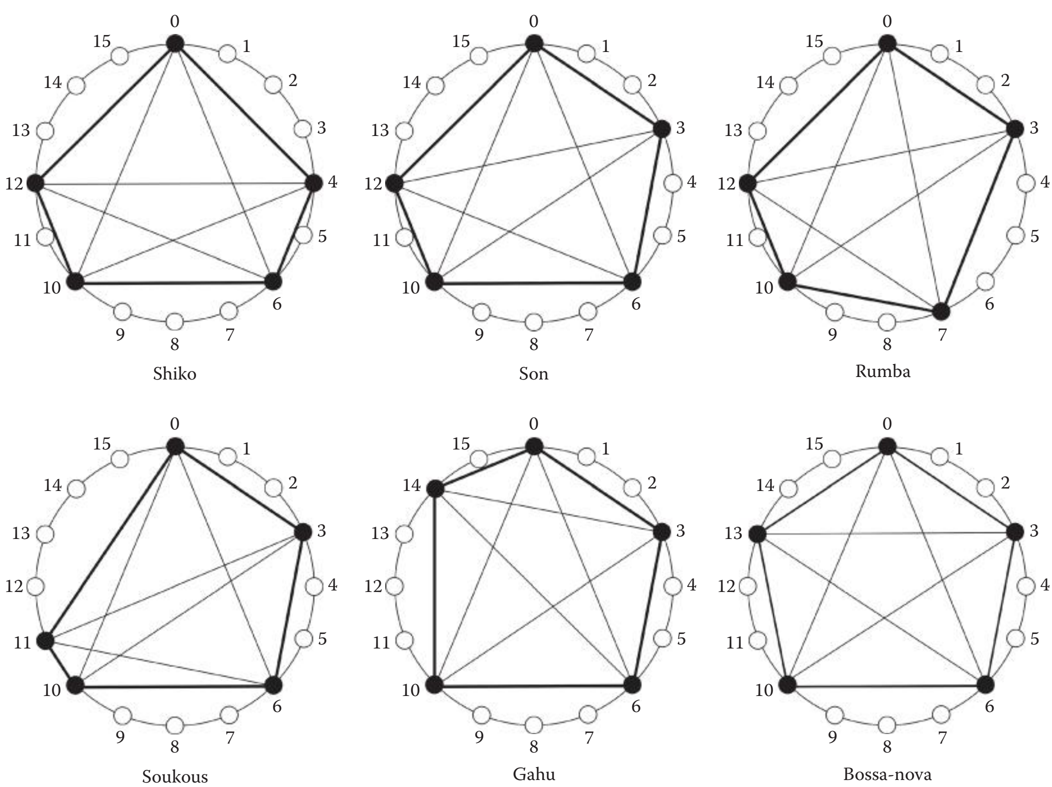

FIGURE 18.5 The six distinguished timelines all have the same sum of pairwise arc-lengths.

add up to 2π, since they determine one entire circumference. This follows from the fact that no arc-length is greater than a half-circle, and so all the arc-lengths are traversed in a clockwise order around the circle. For example, the clave son (second diagram in Figure 18.5) is traversed in the pulse sequence (0, 3, 6, 10, 12, 0). Next, consider the five-sided star-polygons drawn with thin lines that start at pulse zero and connect all the remaining onsets by skipping an onset at each step. For the clave son, this traversal consists of the pulse sequence (0, 6, 12, 3, 10, 0). None of these arc-lengths is greater than a half-circle either, and, as we traverse the edges of the star-polygons in clockwise order until we return to pulse zero, we have gone around the circle twice in every case, so that the sum of the arc-lengths corresponding to the light edges is 4π. This means that for all six rhythms, the sum of all pair-wise arc-lengths between the five onsets is 6π. This implies that this distance is completely ineffective for discriminating between these six rhythms, and thus for measuring the evenness of rhythms. Clearly, the bossa-nova is more even than the son, which is in turn more even than the soukous, but this measure fails to capture these differences.

To understand the behavior of the sum of arc-lengths measure, the concept of a *balanced* rhythm is convenient. Consider a rhythm such as the clave son in Figure 18.6 (left) represented by its onsets as points on the circle. Draw any straight line through the center of the circle such as line *a*. Now imagine rotating this line about this center through a full revolution until it returns to its starting orientation. If for every position of the line such that it is not incident on an onset, the number of onsets on one side of the line differs by at most one from the number of onsets on the other side, then the rhythm is called *balanced*. In 2008, Minghui Jiang proved that the sum of arc-lengths for a rhythm is a maximum if, and only if, the rhythm is balanced.* Examination of the six rhythms in Figure 18.5 reveals that they are all balanced, and hence they all yield a maximum value for this sum. This characterization of the rhythms that maximize the sum of pair-wise arc-lengths also clarifies why this is not an effective measure of the evenness of a rhythm. As the diagram on the right in Figure 18.6 makes clear, a rhythm can be extremely uneven and still be balanced.

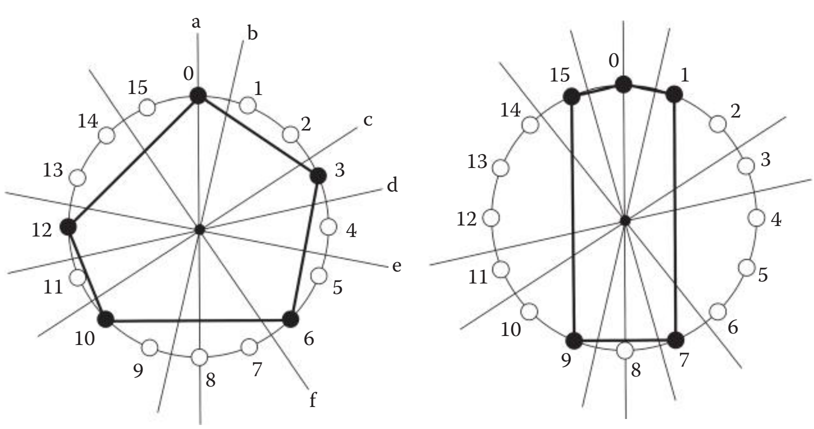

FIGURE 18.6 The clave son (left) is a *balanced* rhythm, but so is the uneven rhythm (right).

* Jiang, M. (2008).

Fortunately, there is another method to measure the evenness of the distribution of points on a circle that works extremely well for the present application. Rather than calculating the distance between two onsets using arc-length (which corresponds to duration time), use instead the length of the straight line or chord of the circle that connects the two points. This seemingly innocent change between the straight-line segments connecting a pair of points on a circle, and the arcs they subtend, makes all the difference in the world, in spite of signaling anathema to some music theorists. One might argue intuitively that the rhythms displayed on the circle of time are one-dimensional objects that are not embedded in the two-dimensional plane, and that the durations should correspond to arc-lengths. However, from the mathematical point of view, there is nothing wrong with such a mapping, if it serves some purpose, and here it does. Whereas the arc-lengths fail to measure maximal evenness, the straight-line distances between the vertices of the two-dimensional polygons work admirably well.

In 1956, László Fejes Tóth proved that given k points in a circle the sum of all the pairwise Euclidean (straight-line) distances between the points is maximized if, and only if, the points form the vertices of a regular k-sided polygon inscribed in the circle.* This result immediately provides a definition of *perfectly even rhythms* as those consisting of k onsets placed so that they correspond to regular k-sided polygons on the continuous circle of time, as illustrated in Figure 18.7 for $k = 3, 4, 5$, and 6.

The regular rhythms notated in Figure 18.7 without the specification of the number of pulses in the cycle are fine for certain types of music such as electronic music or music that contains no meter. However, most music around the world, and especially dance music, depends heavily on an underlying cycle consisting of n underlying pulses, and furthermore, the onsets of the rhythm are required to fall on a subset of these pulses. If k divides evenly into n, then perfectly even rhythms corresponding to regular polygons are trivial to construct. For example, the rhythms in Figure 18.7 with three, four, and six onsets may be realized in a cycle of 12 pulses, but not the third rhythm with five onsets. The five-onset rhythm exists in a cycle of 10 pulses, but not the other three, and all four rhythms may

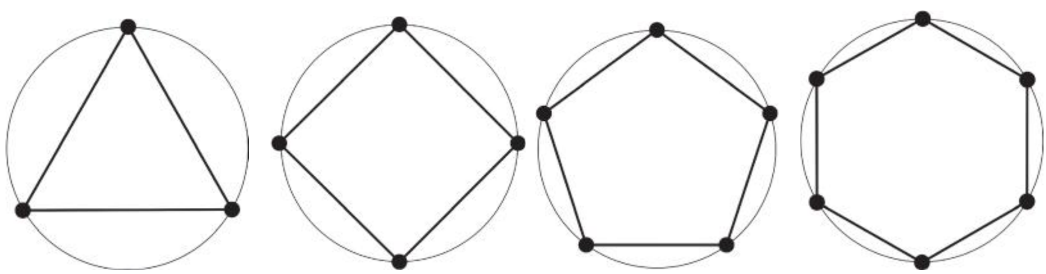

FIGURE 18.7 Regular polygons with $k = 3, 4, 5, 6$ inscribed on the *continuous* circle of time.

* Tóth, L. F. (1956, 1959). While it is true that almost anyone's intuition would dictate that the sum of pairwise (straight) distances of points on a continuous circle should be realized only by points that are equally spaced, it is a different matter altogether to prove it mathematically, and dangerous to take it as a postulate before a proof is established. After all, almost anyone's intuition would dictate that the sum of pairwise (arc) distances of points on a continuous circle should also be realized only by points that are equally spaced, but this is very far from the truth.

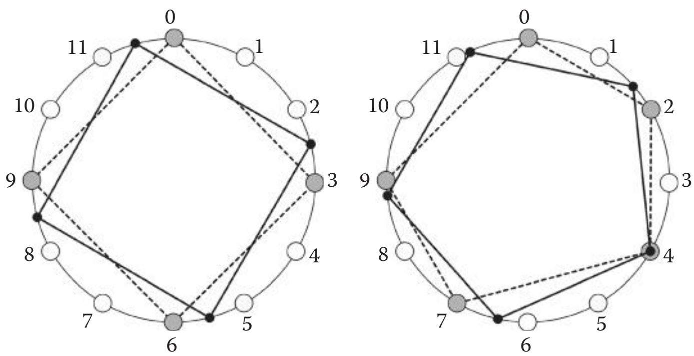

FIGURE 18.8 Maximally even rhythms via *snapping* regular polygons to lattice points.

be embedded in a cycle of 60 pulses. Therefore, what is to be done with regular rhythms that have onsets with cardinalities such as five and seven, if the cycle of interest has a total of 8, 12, or 16 pulses? The answer to this conundrum is to relax the notion of *perfectly even* rhythms that correspond to *regular* polygons to that of *maximally even* rhythms that correspond to polygons that are *almost regular*. There are several ways to proceed in this direction. One natural solution is to apply the same snapping (quantization) process that was used in an earlier chapter to solve the problem of binarization of ternary rhythms, as illustrated in Figure 18.8 using a cycle of 12 pulses. If a maximally even rhythm with k onsets is desired, the idea is to first place a regular k-sided regular polygon anywhere on the circle, and then to snap each vertex to its nearest pulse, or its nearest pulse in a clockwise direction, or its nearest pulse in a counterclockwise direction. It is to be understood that if a vertex of the regular polygon lies on a pulse, then it stays where it is. Alternately, the regular polygon may be positioned so that its vertices do not lie on pulse positions or halfway between two pulse positions. In any case, the snapped positions then become the onsets of the maximally even rhythm.* For example, the left diagram of Figure 18.8 shows an arbitrarily placed square (solid lines with small black vertices) with its vertices snapped to its nearest clockwise pulses (dotted lines with gray onsets). A more interesting example is the pentagon on the right. Here, the initial placement has a vertex on pulse four, which stays there. The other four vertices snap to the pulses seven, nine, zero, and two, yielding the well-known African standard ternary bell pattern. Note that if the vertices are rounded to their nearest pulses, the same necklace pattern is obtained at pulses 2, 4, 6, 9, and 11.

The concept of elements in a set being distributed close to evenly, such as the maximally even rhythms considered here, is also important in the pitch domain because it turns out that chords that are highly consonant tend to divide the octave nearly evenly.†

* For an alternate definitions of maximally even sets, see Douthett, J. and Krantz, K. (2007) and Johnson, T. A. (2003), p. 27. Amiot, E. (2007) gives a short history of maximally even sets. Amiot, E. (2009) presents new proofs of several theorems concerning maximally even sets.

† Tymoczko, D. (2011), p. 63.

CHAPTER 19

Euclidean Rhythms

In the year 300 BC, the city of Alexandria in present-day Egypt was endowed with a magnificent library similar in spirit to the modern institution we call a university. In this Royal Library, which contained reading rooms and a large quantity of books in the form of papyrus scrolls, scholars from diverse parts of the neighboring world, financially supported by the government in Alexandria, gathered together to carry out research and write about a wide variety of topics, including astronomy, geometry, and music.[*] One scholar in particular, named Euclid, wrote a book that became one of the best sellers of all time for more than two thousand years. This book titled *The Elements* contains a wonderful compilation of algorithms (recipes) that were known at that time for solving an extensive variety of *geometric* problems that many of us have studied in secondary school.[†] Aside from geometry, Euclid also described a compelling algorithm for solving a fundamental problem concerned with the arithmetic of *numbers*. This algorithm has found many mathematical and computational applications since then, and is still actively investigated today.[‡] Little did Euclid know that 2300 years later this numerical algorithm would be shown to generate traditional musical rhythms used throughout the world. Although some readers may be surprised to learn that numbers and music are intimately related, the German philosopher and mathematician Gottfried Leibniz once wrote: *The pleasure we obtain from music comes from counting, but counting unconsciously. Music is nothing but unconscious arithmetic.*[§]

The algorithm in question is one of the oldest and well-known algorithms, described in Propositions 1 and 2 of *Book VII* of *The Elements*. Today, it is referred to as the *Euclidean algorithm*. This algorithm solves the problem of computing the greatest common divisor

[*] Beckman, P. (1971).

[†] Toussaint, G. T. (1993) traces the history of the second proposition of Book I of *The Elements*, which in effect states that any problem that can be solved with a straight edge and the *modern* compass can also be calculated with a straight edge and the *collapsing* compass. The distinction between the two compasses is that with the modern compass, one is permitted to transfer a distance from one location on the paper to another, whereas this is not allowed with the collapsing compass. Not surprisingly, algorithms that utilize the collapsing compass require more steps than those that employ the modern compass.

[‡] Bach, E. and Shallit, J. (1996), p. 67.

[§] Sacks, O. (1998).

of two given natural numbers. The computer scientist Donald Knuth calls it the "granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day." The idea is captivatingly simple. Repeatedly replace the larger of the two numbers by their difference until both are equal. This last number is then the greatest common divisor. Consider as an example the pair of numbers (3, 8). First, eight minus three equals five, so we obtain the new pair (3, 5); then, five minus three equals two giving (3, 2); next, three minus two equals one, which yields (1, 2); and lastly, two minus one equals one, which results in (1, 1). Therefore, the greatest common divisor of three and eight is one. This procedure admits a compelling visualization illustrated in Figure 19.1 for the pair of numbers (3, 8). First, construct a 3×8 rectangle of unit squares as shown in the upper left. Repeatedly subtract a 3×3 square from this rectangle until it is impossible to do so. This leaves a 2×3 rectangle remaining (upper right). Now proceed to remove 2×2 squares from this 2×3 rectangle until it cannot be done. This yields the 2×1 rectangle remaining (lower left). Finally, remove a 1×1 square from the 2×1 rectangle to obtain a 1×1 square remaining.

How do we make the Euclidean algorithm generate musical rhythms? The key is to shift our attention from the answer given by the algorithm to the history (or sequence of calculations) of the algorithm, as it makes its way toward the answer. This procedure is illustrated with the preceding pair of numbers (3, 8). For this purpose, the smaller number three is associated with the number of onsets that we want the rhythm to have, and the larger number eight, with the total number of pulses that determine the rhythmic span or cycle. The algorithm works as illustrated in Figure 19.2 using box notation. First, write a rhythm of eight pulses and three onsets in which all the onsets are completely on the left at pulses one, two, and three as in (a). Next, take (subtract) three silent pulses from the right (pulses six, seven, and eight) and place them below the others, flush to the left as in (b). Now, there is a remainder of two silent pulses. Move these below the rest, also flush to the left, as in (c). Now that we have a single column remaining at pulse position three, the repeated subtraction phase of the algorithm is finished. The concatenation phase of

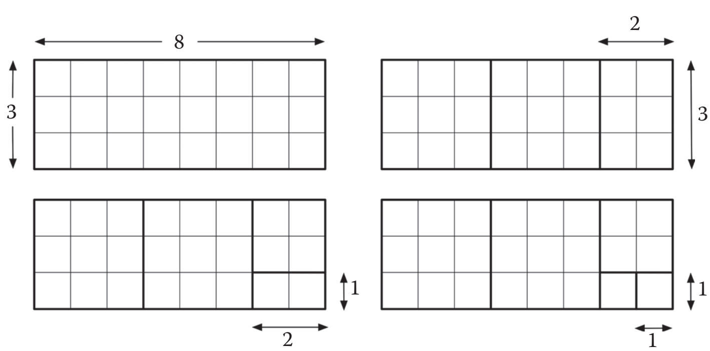

FIGURE 19.1 Visualization of the *Euclidean* algorithm for computing the greatest common divisor of the integers 3 and 8 by repeated subtraction.

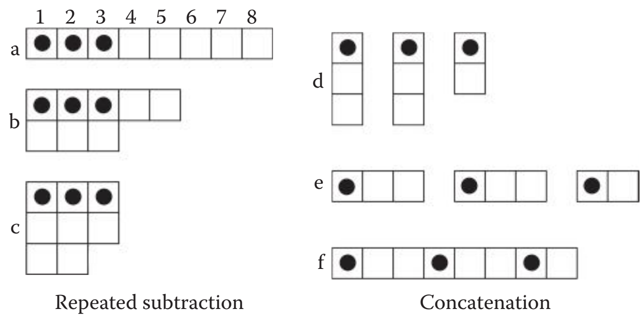

FIGURE 19.2 Generation of the Cuban *tresillo* timeline.

the algorithm follows next. First, separate the three columns as in (d). Next, rotate each column to become a row as in (e), and lastly, concatenate the three rows to form the generated rhythm in (f). Note that the rhythm generated by this procedure is none other than the [3-3-2] pattern found all over the world. In particular, it is the Cuban *tresillo* as well as the first half of the clave *son* that we have encountered repeatedly throughout the book.

This procedure attempts to distribute the three onsets among the eight pulses as evenly as possible. It is this property, to a large extent, that is most responsible for obtaining rhythms that are popular throughout the world. The family of rhythms generated in this way are called *Euclidean* rhythms because they are generated using the structure of the Euclidean algorithm, and will be denoted by E(k,n), where k is the number of onsets and n is the number of pulses in the cycle. Thus, the Cuban tresillo is denoted by E(3,8).

When the number of onsets is greater than the number of silent pulses, all the silent pulses are moved in the first step of the algorithm. The remainder of the procedure stays the same. This may be seen with the numbers (5, 8) illustrated in Figure 19.3. First, the three silent pulses (6, 7, 8) are moved, as in (b). Now, there are two single-onset columns remaining at pulse positions 4 and 5. These are moved next, as in (c). The concatenation phase in (d), (e), and (f) is the same as before. Note that the resulting Euclidean rhythm E(5,8) is the Cuban cinquillo timeline [2-1-2-1-2].

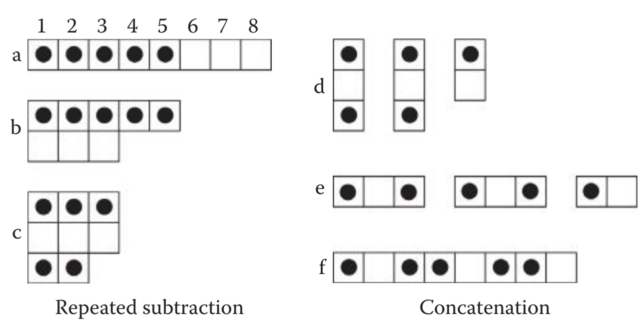

FIGURE 19.3 Generation of the Cuban *cinquillo* timeline.

If we apply this algorithm with the numbers 5 and 12, illustrated in Figure 19.4, we obtain the rhythm E(5,12) = [3-2-3-2-2], which is a timeline played on a metal bell by the Aka Pygmies of Central Africa.*

Applying the algorithm to the pair of number (7, 12), as shown in Figure 19.5, results in the rhythm E(7,12) = [2-1-2-2-1-2-2], a popular West African bell pattern used in Ghana and Guinea.

Substituting the pair of numbers (5, 16) into the algorithm, illustrated in Figure 19.6, yields the rhythm E(5,16) = [3-3-3-3-4], a popular rhythmic pattern used in modern electronic dance music, which is in fact a rotation of the bossa-nova timeline.

Many other Euclidean rhythms that are used in music throughout the world may be generated in this way by suitably picking the number n of pulses and the number k of onsets. A list of some examples of Euclidean rhythms used in traditional music practice, for values of n and k, such that k does not divide evenly into n, are listed in the following, where the notation normally used here for a durational pattern is shortened from, for example, [a-b-c] to (abc), in order to accommodate the long rhythms at the end of the list into single lines of text

E(2,3) = [x x .] = (12)

E(2,5) = [x . x . .] = (23)

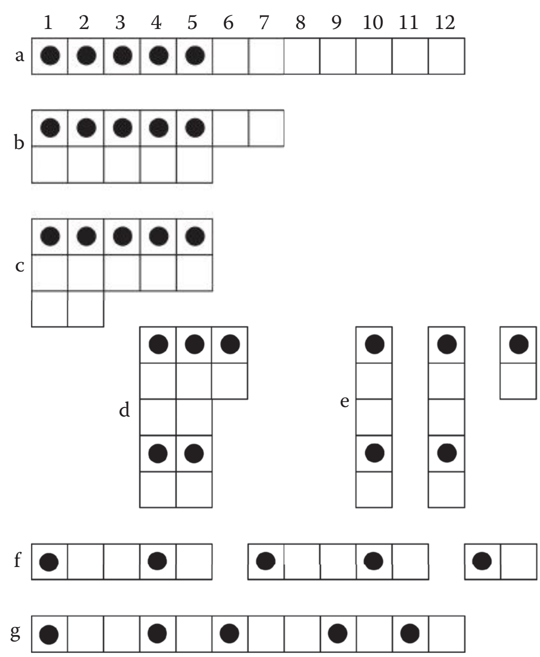

FIGURE 19.4 Generation of an Aka Pygmy timeline.

* Chemillier, M. (2002), p. 175. The Aka Pygmies of Central Africa use several additional rhythms with a similar pattern of using two intervals of duration three followed by two groups of elements of duration two, such that the cardinality of the two groups differs by one. Such rhythms include [3-2-2-3-2-2-2] and [3-2-2-2-2-3-2-2-2-2-2].

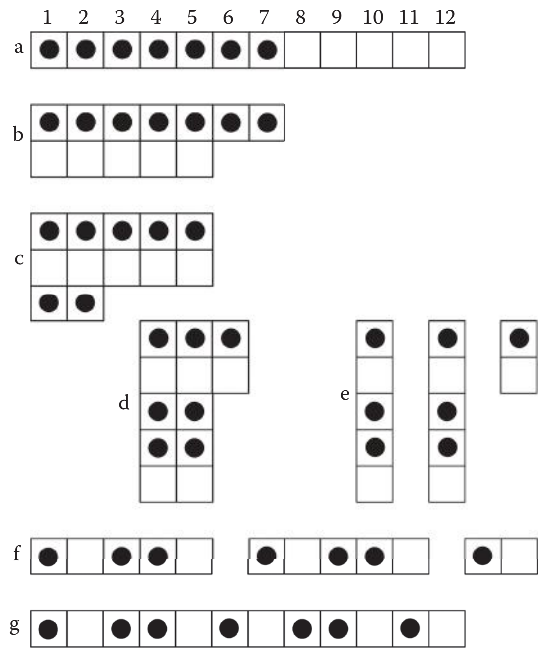

FIGURE 19.5 Generation of a ternary West African timeline.

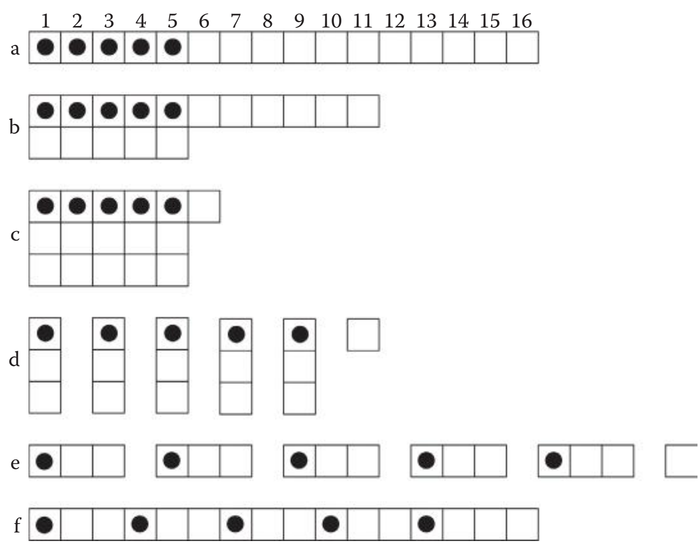

FIGURE 19.6 Generation of a binary West African timeline.

E(2,7) = [x . . x . . .] = (34)

E(3,4) = [x x x .] = (112)

E(3,5) = [x . x . x] = (221)

E(3,7) = [x . x . x . .] = (223)

E(3,8) = [x . . x . . x .] = (332)

E(3,10) = [x . . x . . x . . .] = (334)

E(3,11) = [x . . . x . . . x . .] = (443)

E(3,14) = [x . . . . x . . . . x . . .] = (554)

E(4,5) = [x x x x .] = (1112)

E(4,7) = [x . x . x . x] = (2221)

E(4,9) = [x . x . x . x . .] = (2223)

E(4,11) = [x . . x . . x . . x .] = (3332)

E(4,15) = [x . . . x . . . x . . . x . .] = (4443)

E(5,6) = [x x x x x .] = (11112)

E(5,7) = [x . x x . x x] = (21211)

E(5,8) = [x . x x . x x .] = (21212)

E(5,9) = [x . x . x . x . x] = (22221)

E(5,11) = [x . x . x . x . x . .] = (22223)

E(5,12) = [x . . x . x . . x . x .] = (32322)

E(5,13) = [x . . x . x . . x . x . .] = (32323)

E(5,16) = [x . . x . . x . . x . . x . . .] = (33334)

E(6,7) = [x x x x x x .] = (111112)

E(6,13) = [x . x . x . x . x . x . .] = (222223)

E(7,8) = [x x x x x x x .] = (1111112)

E(7,9) = [x. x x x . x x x] = (2112111)

E(7,10) = [x . x x . x x . x x] = (2121211)

E(7,12) = [x . x x . x . x x . x .] = (2122122)

E(7,15) = [x . x . x . x . x . x . x . .] = (2222223)

E(7,16) = [x . . x . x . x . . x . x . x .] = (3223222)

E(7,17) = [x . . x . x . . x . x . . x . x .] = (3232322)

E(7,18) = [x . . x . x . . x . x . . x . x . .] = (3232323)

E(8,17) = [x . x . x . x . x . x . x . x . .] = (22222223)

E(8,19) = [x . . x . x . x . . x . x . x . . x .] = (32232232)

E(9,13) = [x . x x . x x . x x . x x] = (212121211)

E(9,14) = [x . x x . x x . x x . x x .] = (212121212)

E(9,16) = [x . x x . x . x . x x . x . x .] = (212221222)

E(9,20) = [x . . x . x . x . x . . x . x . x . x .] = (322232222)

E(9,22) = [x . . x . x . . x . x . . x . x . . x . x .] = (323232322)

E(9,23) = [x . . x . x . . x . x . . x . x . . x . x . .] = (323232323)

E(11,12) = [x x x x x x x x x x x .] = (11111111112)

E(11,20) = [x . x x . x . x . x . x x . x . x . x .] = (21222212222)

E(11,24) = [x . . x . x . x . x . x . . x . x . x . x . x .] = (32222322222)

E(13,24) = [x . x x . x . x . x . x . x x . x . x . x . x .] = (2122222122222)

E(15,34) = [x . . x . x . x . x . . x . x . x . x . . x . x . x . x . . x . x .] = (322232223222322)

The durational patterns in the above list should be viewed more generally as necklaces rather than mere rhythms, since in many cases, rotations of these patterns yield other rhythms that are also used in traditional music around the globe. For example, the rhythm E(2,3) = [x x .] when started on the second onset becomes [x . x], which is a rhythm of a *Drum Dance* song of the Slavey Indians of Northern Canada,* as well as the hallmark rhythm of the *Lenjengo* recreational dance of the Mandinka people of West Africa.† Starting the rhythm E(3,4) = [x x x .] on the third onset yields [x . x x], which is the Arabic rhythm *wahdah sāyirah*.‡ Similarly, starting E(5,16) = [x . . x . . x . . x . . x . . .] = [3-3-3-3-4] on the third onset yields the bossa-nova clave [3-3-4-3-3].§

Furthermore, the patterns in this list are also expressed in their shortest form. Linear stretching of these rhythms will produce other rhythms. For example, when E(2,3) = [x x .] = [1-2] is multiplied by two it becomes [x . x . . .] = [2-4], a variant of the Mexican *son* rhythm.¶

Euclidean rhythms are closely related to a similar concept studied in theoretical computer science: *Euclidean strings*.** A string of integers such as an interval vector denoted by V = (v_0, v_1, …, v_{n-1}), is a *Euclidean string* if increasing v_0 by one, and decreasing v_{n-1} by one, yields a new string that is a rotation of V. For example, if the operation is applied to the

* Asch, M. I. (1975), p. 249.
† Knight, R. (1974), p. 28.
‡ Touma, H. H. (1996), p. 50.
§ Butler, M. J. (2006), p. 147, points out that [3-3-3-3-4] is a rhythm often used in electronic dance music.
¶ Stanford, E. T. (1972), p. 79.
** Ellis et al. (2003). Euclidean strings are a topic that falls under the more general *theory of words*. Domínguez et al. (2009) provide a translation bridge between the theory of words and music theory.

Euclidean rhythm E(4,9) = [2-2-2-3], one obtains [3-2-2-2], and since this is a rotation of [2-2-2-3], it follows that E(4,9) is a Euclidean string.*

In closing this chapter, it should be mentioned that the Euclidean algorithm has also been applied to music in a completely different context. Viggo Brun used it in the construction of stringed musical instruments, in which there are constraints on the lengths of the strings, as well as on the ratios of these lengths.†

* See Toussaint, G. T. (2005c) for a more detailed comparison of Euclidean rhythms with Euclidean strings in the context of Balkan *aksak* rhythms and sub-Saharan African timelines.

† Brun, V. (1964), p. 128.

CHAPTER 20

Leap Years

The Rhythm of the Stars

For many thousands of years, human beings all over the planet have spent countless hours gazing at the stars, the Sun, and the moon. If they lived far from the equator, they often experienced vast differences between four seasons: a cold winter with snow and ice; a hot summer; a spring when new leaves are born, bears come out from hibernation, and colorful flowers bloom; and an autumn when the leaves turn to a variety of shades of brown, yellow, and red, before detaching themselves from their branches, and coasting to the ground. Some countries experience only two seasons: the rainy season and the dry season. In other countries such as Egypt, these types of seasons are nonexistent. Indeed, in ancient Egypt, the changes that people observed in the desert had to do mainly with the level of water in the river Nile: was it constant, rising, or falling. The ancient Egyptians had three seasons of four months each, where each month lasted 30 days, for a total of 360 days. In addition, they concluded from their celestial observations that the number of days spanning the three seasons was 365. Therefore, at the start of each year, they added five extra days of festivities, thus making their year last a total of 365 days.*

Astronomers also counted how many moons and days were observed in the cycles of their seasons. Such measurements inspired disparate cultures to design calendars in different ways. Let T_s denotes the time duration of one revolution of the Earth around the Sun, more commonly known as a year. Let T_e denotes the time duration of one complete rotation of the Earth, more commonly known as a day. The exact values of T_s and T_e are of course constantly varying, since the orbits of the stars and planets in the universe are themselves continually changing. Every galaxy exerts gravity on every other galaxy, and all the stars and planets in each galaxy exert gravity on each other. However, from the mea-

* Winlock, H. E. (1940), p. 458.

surements made of T_s and T_e, we estimate the ratio T_s/T_e to be today ≈365.242199.[*] It is convenient, therefore, to make a year last 365 days, as did the ancient Egyptians. The problem that arises, both for keeping track of history, and for making predictions about the future, is that after some time, this seemingly small discrepancy, equal to 0.242199, becomes a large and inconvenient error. Since, 0.242199 is almost 6 h, or one-fourth of a day, one simple solution is to add one extra day for every 4 years. Indeed, in the year 237 BC, the ruler of Egypt, Ptolemy III exactly proposed this modification to the Egyptian calendar. This proposal met some resistance in Egypt, but a few years later, Julius Caesar adopted the practice, and for this reason, this calendar became known as the *Julian* calendar.[†]

A day with one extra day is called a *leap* year. The Julian calendar assumes that a year is exactly 365.25 days long, which is, still, slightly greater than 365.242199. So now, we have an error in the opposite direction, albeit smaller. One solution to this new problem is the *Gregorian* calendar, named after Pope Gregory XIII, who was responsible for its adoption. In the Gregorian calendar, leap years are defined as those divisible by four, except not those divisible by 100, except not those divisible by 400. With this rule, a year becomes $365 + 1/4 - 1/100 + 1/400 = 365.2425$ days long, a much better approximation.

The methods just described for solving the calendar problem may be aptly described by the computer term *hack*. A hack is an effective, but clumsy, solution to a problem. A more elegant structural approach to designing rules for the introduction of leap years in calendars is based on the idea of rhythmic cycles. One such method is used in the design of the Jewish calendar. Here, a regular year has 12 months and a leap year has 13 months. The cycle has a total of 19 years that include 7 leap years.[‡] The 7 leap years are distributed as evenly as possible in the 19-year cycle. The cycle is assumed to start with creation as year one. The remainder obtained by dividing the year number by 19 indicates the resulting position in the cycle. The leap years are 3, 6, 8, 11, 14, 17, and 19. For example, the year $5765 = 303 \times 19 + 8$ and so is a leap year. The year 5766, which begins at sundown on the Gregorian date of October 3, 2005, is $5766 = 303 \times 19 + 9$, and is therefore not a leap year. Applying the Euclidean algorithm to the integers 7 and 19, as shown in Figure 20.1, yields E(7,19) = [x . . x . x . . x . . x . x . . x . .] with a durational pattern [3-2-3-3-2-3-3]. As before, we subtract columns in (a), (b), (c), (d), and (e) until there is a remainder of only one column in (e) consisting of the pattern [x..]. In (f), the three columns are separated. In (g), they are each rotated, and finally in (h), they are concatenated. If the seventh pulse is counted as the first pulse, we obtain the pattern [. . x . . x . x . . x . . x . . x . x], which describes precisely the leap year pattern 3, 6, 8, 11, 14, 17, and 19 of the Jewish calendar. Therefore, the leap year pattern of the Jewish calendar is a Euclidean necklace. Figure 20.2 depicts the necklace in an orientation that shows off its vertical mirror symmetry. Interestingly enough, this neck-

[*] Assar, G. R. F. (2003), p. 172. This number is derived from the definition of a *tropical year*, which is one revolution of the Earth around the Sun realizing two consecutive equinoxes. On the other hand, the *sidereal year* consists of 365.25636 days, and is determined by an orbit of the Earth around the Sun such that the Earth returns to the conjunction with the stars that are (or appear to be) fixed in the sky.

[†] Reingold, E. M. and Dershowitz, N. (2001). For the relationships between Euclidean rhythms, leap years, and drawing digital straight lines on a computer screen of pixels, see Harris, M. A. and Reingold, E. M. (2004) and Klette, R. and Rosenfeld, A. (2004).

[‡] Ascher, M. (2002), p. 48.

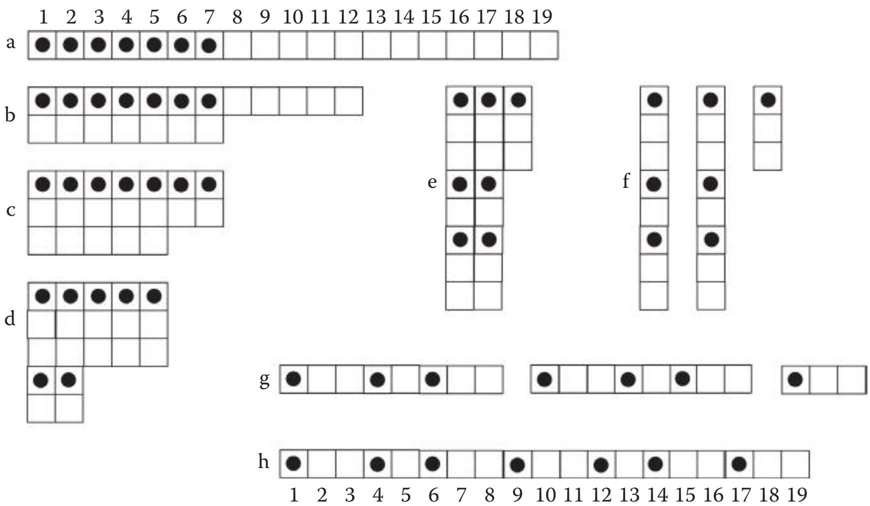

FIGURE 20.1 Generating the Jewish leap year calendar pattern using the Euclidean algorithm.

lace exhibits the same pattern of long–short intervals as the bembé rhythm and diatonic scale: [3-3-2-3-3-3-2] for the calendar versus [2-2-1-2-2-2-1] for the rhythm and scale. All the intervals in the diatonic scale are increased by one to obtain the Jewish leap year pattern. In musical terms, they have the same *rhythmic contour*.

Another structural design of a calendar that uses cycles is the Islamic calendar, which is based on the time between two successive new moons (*lunations*), and in which 1 year is defined as 12 lunations. This method gives ⊠10,632 days every 30 years, in which 11 leap years are employed. The common approximations used in the Islamic calendar put leap years at positions 2, 5, 7, 10, 13, 16, 18, 21, 24, 26, and 29 in the cycle yielding the duration pattern [3-2-3-3-3-2-3-3-3-2-3], shown in polygon notation in Figure 20.3. Applying Euclid's algorithm to the integers 11 and 30 yields E(11,30) = [x . . x . x . . x . . x . . x . x . .

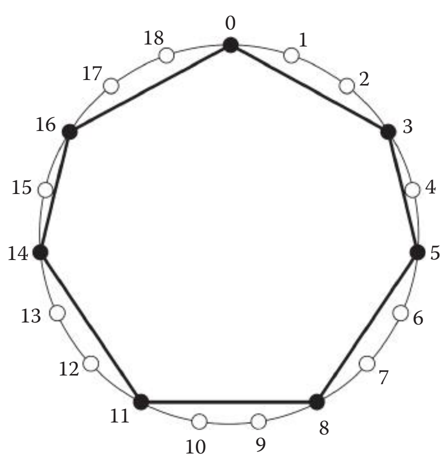

FIGURE 20.2 The Jewish calendar necklace with 7 leap years in a cycle of 19 years.

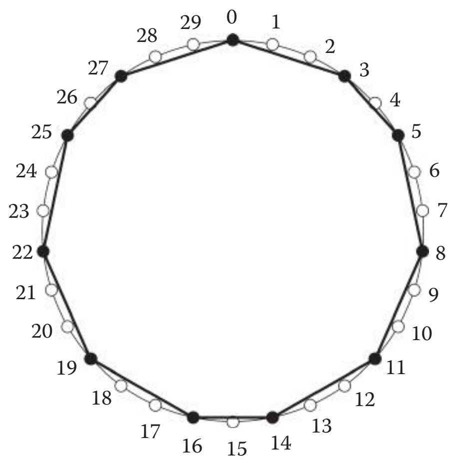

FIGURE 20.3 The Islamic calendar necklace with 11 leap years in a cycle of 30 years.

x . . x . . x . x . .]. If we start this rhythm on the 11th pulse, we obtain the pattern [. x . . x . x . . x . . x . . x . x . . x . . x . x . . x .], which describes precisely the leap year pattern 2, 5, 7, 10, 13, 16, 18, 21, 24, 26, and 29 of the Islamic calendar. Therefore, the leap year pattern of the Islamic calendar is also a Euclidean necklace.

The Jalali Persian calendar introduced in the year 1079 is also based on cycles. Omar Khayyám* contributed significantly to the design of this calendar. He proposed a cycle of 33 years with 8 leap years with the pattern [4-4-4-4-4-4-4-5]. Euclid's algorithm applied to the integers 8 and 33 yields the rhythm E(8,33) = [x . . . x . . . x . . . x . . . x . . . x . . . x . . . x . . . .], shown in polygon notation in Figure 20.4. Therefore, the Jalali leap year pattern introduced into the Persian calendar is also a Euclidean necklace.

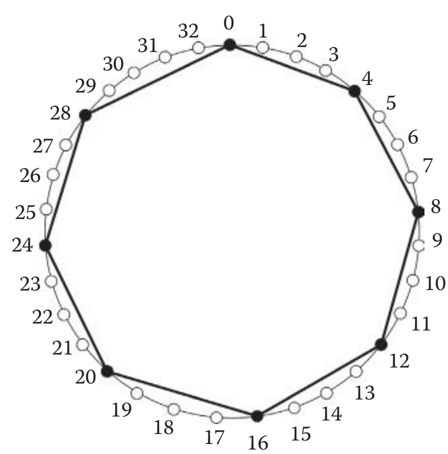

FIGURE 20.4 The Persian calendar necklace with 8 leap years in a cycle of 33 years.

* Richards, E. G. (1998), p. 235.

Around the year 3000 BC, the Sumerians, who settled in Mesopotamia, developed a lunar calendar that had 12 months of 29 or 30 days each. This calendar went through several revisions over the years. In the fifth century BC, this Babylonian calendar had seven intercalations in a cycle if 19 years in positions 1, 3, 6, 9, 11, 14, and 17, giving it a durational pattern [2-3-3-2-3-3-3], which is also a Euclidean rhythm.*

* Assar, *Ibid*., p. 174.

CHAPTER 21

Approximately Even Rhythms

In Chapter 19, it was demonstrated with numerous examples taken from music from around the world that for a multitude of numerical values of n and k, the number of pulses and onsets, respectively, there exist many rhythm timelines in cultures all over the world that have the property that they are Euclidean or maximally even. However, for every fixed pair of values of n and k, the Euclidean algorithm yields only one rhythm necklace. Consider for instance the Euclidean rhythm obtained when $n = 8$ and $k = 2$. Since eight is divisible by two, the rhythm obtained is [x . . . x . . .] = [4-4]. Repeating this pattern yields only a steady pulsation, not a very interesting rhythm. However, if we displace the second attack by one pulse, say to the left, we obtain the very interesting rhythm [x . . x . . . .] = [3-5]. This rhythm is quite common in Afro-Cuban music, where it is called the *conga*.* It has also been incorporated into *rock-n-roll* music, perhaps most notably as the mid-song electric guitar solo in the Beach Boy's 1964 best-selling ballad *Don't Worry Baby*, ranked by the *Rolling Stone Music* magazine as the 178th greatest song of all time.†

Adding one more attack to the conga, so that $n = 8$ and $k = 3$, yields again only one Euclidean rhythm E(3,8) = [x . . x . . x .] = [3-3-2], which when rotated yields the additional rhythms [3-2-3] and [2-3-3]. On the other hand, several other rhythms used in practice also consist of three onsets among eight pulses, but they are neither Euclidean rhythms nor rotations thereof. As the two examples above illustrate, to generate a larger, more inclusive class of "good" rhythms, the concept of maximally even has to be relaxed. One approach is to modify slightly, or mutate, a maximally even rhythm to generate other rhythms that are *almost* maximally even. In a brute-force approach, for a given pair of values of n and k, we could first generate all possible rhythms, then calculate according to some chosen distance function, the distance between all these rhythms and the maximally even rhythm, and finally select those rhythms that are close enough to the maximally even rhythm according to a preselected threshold. However, such an approach requires much computation. There are other more direct

* Rey, M. (2006), p. 192.
† http://www.rollingstone.com/music/lists/the-500-greatest-songs-of-all-time-20110407.

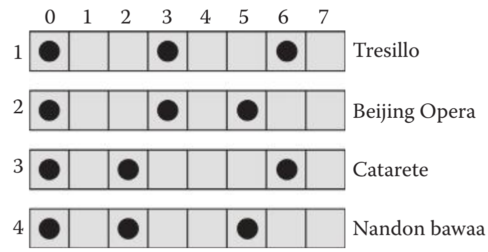

FIGURE 21.1 The four almost maximally even rhythms with three onsets and eight pulses.

ways of generating rhythms that are close to maximally even rhythms. As an example, consider again the ubiquitous tresillo rhythm at the top of Figure 21.1, and define a mutation operation as the displacement of one of its onsets (other than the first) by a duration of one pulse toward the left (anticipation). We can generate three new rhythms this way by moving either the second onset from pulse three to pulse two, the third onset from pulse six to pulse five, or both of these. The three rhythms produced in this way are used in music in many parts of the world. The second rhythm [3-2-3] is used in Beijing Opera, the third [2-4-2] is the *catarete* from Brazil, and the fourth [2-3-3] is a *bossa-nova* rhythm from Brazil, as well as the *nandon bawaa* from Ghana. Two of these are rotations of the tresillo, and therefore, they are also obviously maximally even, but the catarete is not.

This description of the algorithm for generating almost maximally even rhythms has another more geometric interpretation that uses a two-dimensional *onset-pulse grid*, as illustrated in Figure 21.2. First, construct a 3×8 rectangular grid with height three units and width eight units. The height marks the onset number, and the width the pulse number. Note that the width also indicates time, since there are eight pulses in the cycle. Next, draw a straight line connecting the corner point (0,1) on the lower left to the corner point (0,1) on the upper right.

Corresponding to each onset number, there is a horizontal dashed grid line intersected by a diagonal line (solid) that connects the two opposite corners of the rectangle. These intersection points have the time coordinates: two and two-thirds for the first, and five and one-third for the second. They divide the time span of eight pulses into three equal intervals. Onsets played at these positions would generate an isochronous perfectly even rhythm. In some forms of music, such as electronic music, a composer

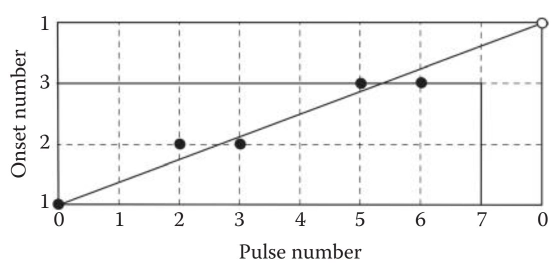

FIGURE 21.2 Generating the almost maximally even rhythms with eight pulses and three onsets.

may be perfectly happy to insert onsets such as these anywhere in the cycle, without respecting the integer locations of the pulses, in effect treating the time axis as a continuum of real numbers rather than integers. However, in the present context, and for our purposes, the onsets must be played only on the integer values of time, that is, the pulse numbers. The Euclidean rhythm, in this case, is obtained by snapping every intersection point to the pulse that is the nearest neighbor to the right. On the other hand, the family of *almost* maximally even rhythms is made up of the rhythms obtained by all combinations of snapping intersection points to both, the nearest right and the nearest left pulses.*

Figure 21.3 shows the onset-pulse grid diagram for generating the almost maximally even rhythms with 16 pulses and five onsets, and the resulting 16 almost maximally even rhythms are pictured in Figure 21.4. About half of these 16 rhythms are used in musical practice, and they are highlighted and labeled with one of their more common identifying names.†

It is worth noting that the rhythms generated in this fashion are not equally almost maximally even. All the rhythms do "live" in a thin strip near the diagonal line in Figure 21.3, and thus their unevenness is bounded from above, but if for each rhythm we sum the horizontal deviations of their attacks from this diagonal line, some are more even than others. This is not surprising since a generated rhythm may be obtained from the maximally even rhythm by a number of mutations that can vary between one and four.

Timelines with 12 pulses and five onsets also figure prominently in much music, and Figure 21.5 shows the onset-pulse grid diagram for generating them. There are again 16 almost maximally even rhythms in this case, and they are pictured in Figure 21.6. Again, about half of these rhythms are used in musical practice, and they are highlighted and labeled with their commonly assigned names.

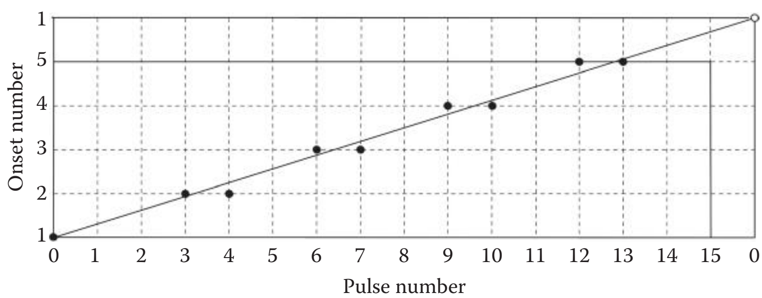

FIGURE 21.3 Generating the almost maximally even rhythms with 16 pulses and five onsets.

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* The field of computer science that deals with the automatic generation of symbol sequences provides many similar methods of generating binary sequences usually called *words* (rhythms in our context). See Lothaire, M. (2002) and Allouche, J.-P. and Shallit, O. (2002).

† Entry number 16 in Figure 21.4 is a bossa-nova variant frequently encountered in electronic dance music. See Butler, M. J. (2006), p. 83. It is used, for example, as a snare drum ostinato in Mario Più's *Communication*. See Butler, M. J. (2006), p. 147.

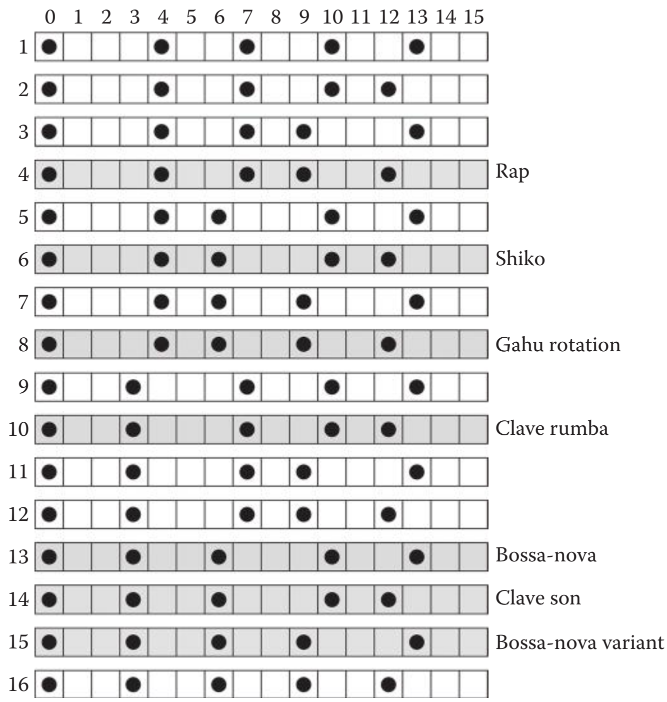


FIGURE 21.4 The 16 almost maximally even rhythms with 16 pulses and five onsets. (From Toussaint, G. T., Percussive Notes, 2011, November issue, pp. 52–59. With permission.)

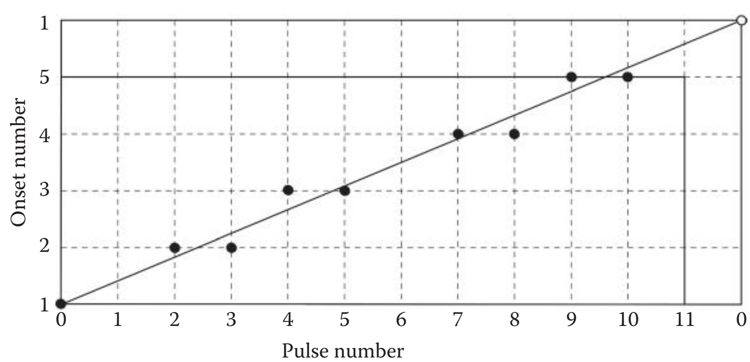


FIGURE 21.5 Generation of the almost maximally even rhythms with 12 pulses and five onsets.

In the preceding examples, all the almost maximally even rhythms generated from a seed rhythm (specified number of onsets) had the same number of onsets as the seed rhythm. This resulted from the fact that the number of pulses was large relative to the number of onsets, thus giving the onsets enough space to snap to their nearest pulses in either direction without "colliding" with other already assigned pulses. However, when the number of onsets is large relative to the number of pulses, the resulting overcrowding causes such collisions to happen, as illustrated with the timelines that have five onsets and 12 pulses that yield the onset-pulse grid shown in Figure 21.7.

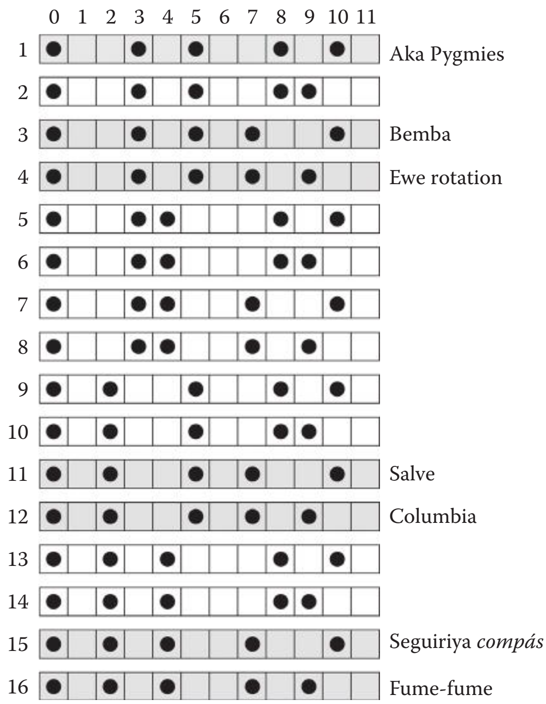


FIGURE 21.6 The 16 almost maximally even rhythms with 12 pulses and five onsets.

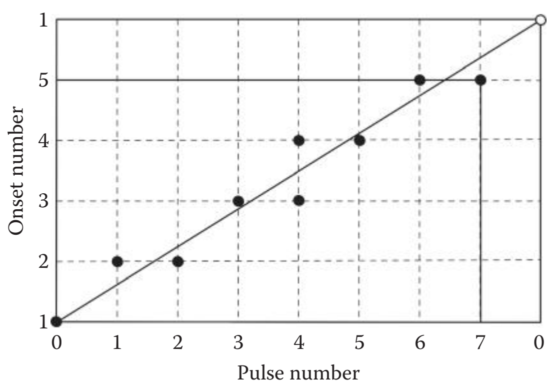


FIGURE 21.7 Generating the almost maximally even rhythms with eight pulses and five onsets.

The 16 almost maximally even rhythms obtained from this onset-pulse grid are shown in Figure 21.8. Again, about half of these are used in musical practice, and are identified by a name or genre in which they are used.* Note that this time, although the seed rhythm has five onsets, there are four rhythms numbered 3, 4, 11, and 12 that have only four onsets

* The names used here in no way suggest a complete list of either the countries in which the rhythms are played or what they are called in different cultures. For example, the Rumanian folk rhythm (third on the list) with durational pattern [2-2-3-1], when started on the third onset becomes [3-1-2-2], the time-keeping rhythm of the *sabar* of Senegal, played on the side of the drum with a stick. See Tang, P. (2007), p. 98.

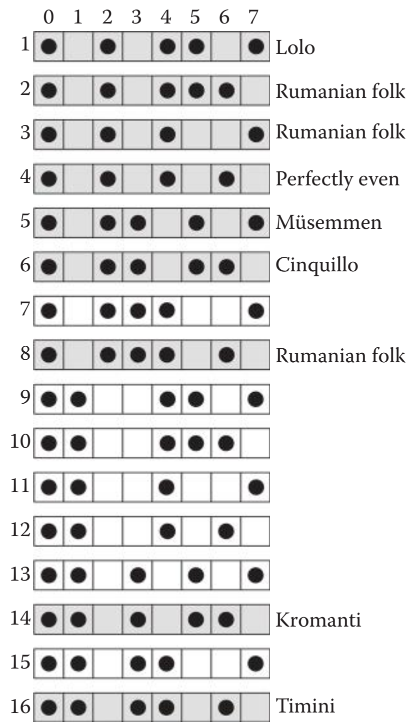


FIGURE 21.8 The 16 almost maximally even rhythms with eight pulses and five onsets.

due to collisions resulting from the snap operations. For example, in Figure 21.7, if the intersection point on onset line three is snapped to the right and the intersection point on onset line four is snapped to the left, then both intersection points are snapped to pulse number four.

In all the examples examined so far, the number of onsets is evenly divisible into the number of pulses. However, in the special cases in which this does not happen, the resulting maximally even rhythm is actually *perfectly* even, and the diagonal line intersects the horizontal grid lines exactly at the vertical pulse lines. When this happens, the almost maximally even rhythms obtained by moving the intersection points to their nearest right and left pulse positions have a tendency to acquire larger inter-onset intervals.* Consider, for example, the grid-line diagram for four onsets among 16 pulses in Figure 21.9. The resulting family consisting of eight almost maximally even rhythms generated with this grid are shown in Figure 21.10, where the four onsets of each rhythm are indicated with black-filled circles. All the rhythms have gaps of either five or six pulses. Nevertheless,

* There exist other methods for generating approximately even rhythms that handle the intersections of horizontal and vertical lines in a different manner. For example, if vertical and horizontal lines generate onsets and rests, respectively, when intersected by the diagonal line, and when all three lines meet at a point, an onset-rest pair is generated. Then the grid in Figure 21.11 yields the rhythm [x . x x x . x x x . x x x . x x]. Series, C. (1985), p. 21, describes a characterization of the class of rhythms obtained in this way. See also the *cutting sequences* in: Domínguez et al. (2009), p. 480, and the book by Lothaire, M. (2002), p. 109, for several related methods and references.

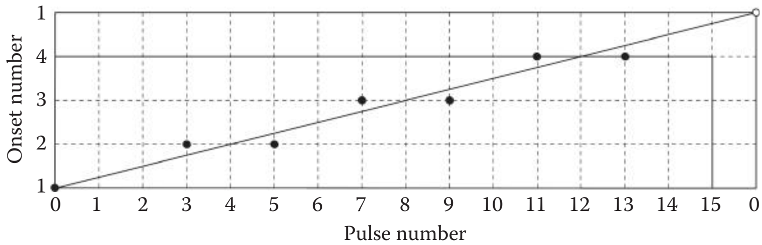


FIGURE 21.9 Generation of the almost maximally even rhythms with 16 pulses and four onsets.

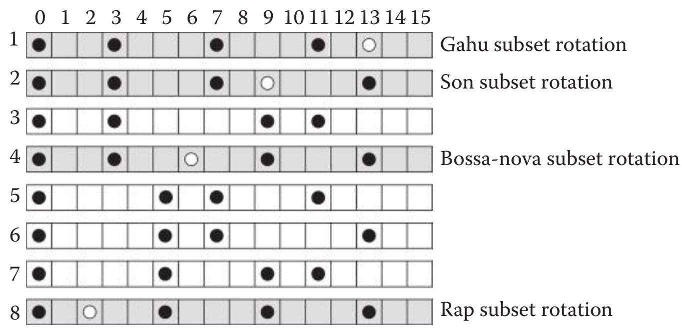


FIGURE 21.10 The eight almost maximally even rhythms with 16 pulses and four onsets. (From Toussaint, G. T., Percussive Notes, 2011, November Issue, pp. 52–59. With permission.)

four of these rhythms are interesting in that their onsets consist of subsets of rotations of well-known rhythms used in practice.

In Figure 21.10, the onsets marked with white-filled circles indicate those onsets that would have to be inserted for the rhythm to become the one labeled on the right. So, if an onset is inserted to rhythm No. 1 at pulse 13, the resulting five-onset rhythm would be the gahu if started at pulse 13. Similarly, if an onset is added to rhythm No. 2 at pulse nine, the resulting five-onset rhythm would be the clave son if started at pulse 13. Also, if an onset is added to rhythm No. 4 at pulse six, the resulting rhythm would be the clave bossa-nova if started at pulse three. Finally, if an onset is added to rhythm No. 8 at pulse two, the resulting rhythm would be the rap timeline if started at pulse nine.

For a second example, take the case of four onsets among 12 pulses, for which the grid-line diagram is shown in Figure 21.11, and the rhythms it generates in Figure 21.12.

Four of the rhythms in Figure 21.12 are derived from the rhythms labeled on the right. So, if an onset is inserted to rhythm No. 1 at pulse 10, the resulting five-onset rhythm would be the seguiriya meter if started at pulse 10. Similarly, if an onset is added to rhythm No. 2 at pulse nine, the resulting five-onset rhythm would be the buleria meter if started at pulse three. Also, if an onset is added to rhythm No. 4 at pulse four, the resulting rhythm would be the seguiriya meter. Finally, if an onset is added to rhythm No. 8 at pulse two, the resulting rhythm would again be the seguiriya meter.

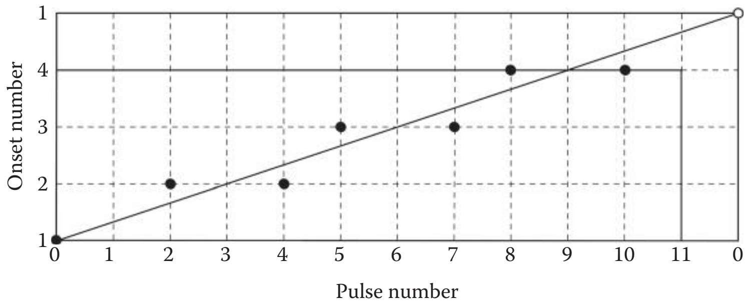


FIGURE 21.11 Generation of the almost maximally even rhythms with 12 pulses and four onsets.

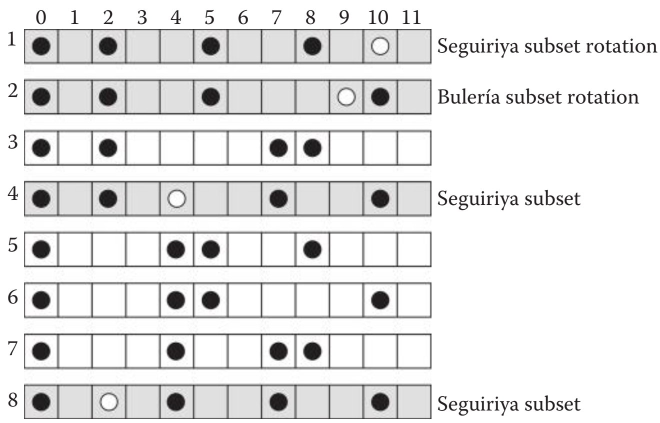


FIGURE 21.12 The eight almost maximally even rhythms with 12 pulses and four onsets.

As the examples examined in this chapter show, for small values of the number of pulses and onsets, all members of the families of almost maximally even rhythms are used in practice as timelines. Furthermore, for the important case of five onsets among 16 pulses, the almost maximally even set includes the clave son. However, for these and other similar values, as well as for larger values of onsets and pulses, although the almost maximally even sets include many rhythms that are used in practice, they also include rhythms that have not been adopted. Therefore, although the property of *almost maximal evenness* appears to be almost necessary for a rhythm to be good, it seems not to be sufficient to characterize the rhythms that have been adopted by cultures in the past. Certainly, this property by itself cannot characterize the clave son uniquely, nor therefore unequivocally explain how the clave son managed to become so successful throughout the world.

CHAPTER 22

Rhythms and Crystallography

Have you ever had the experience of viewing two different objects from one vantage point, concluding that they are identical, only to discover after obtaining more information, that they are in fact vastly different? The example in Figure 22.1 illustrates this point succinctly. The object on the left is a cube, and the one in the center a cylinder. They are poles apart from each other. Both objects look very different from the top, and yet from the front they look like the same square on the right.

In the 1920s and 1930s, crystallographers were studying the atomic structure of crystals with a powerful new tool that had just become available: *x-rays*. These scientists were interested in reconstructing the positions of the crystal atoms relative to each other, from only the distances between the pairs of atoms. However, x-rays are not as powerful as the computed tomography (CT) scanners of today that yield three-dimensional pictures of the objects under scrutiny. In those days, x-rays were bombarded through the crystals under investigation, providing only projections, or two-dimensional pictures that did not contain the exact coordinates of the atoms, but only evidence of their location. These pictures, called x-ray *diffraction patterns*, were then used to infer the distances between the atoms. The lofty final goal of this research project was nothing less than the construction of exact structural models of the molecules under investigation. Unfortunately, the scientists encountered a problem. To their chagrin, they discovered that, just as in the example of Figure 22.1, there exist pairs of molecules that have different atomic structures, but yield exactly the same collection of interatomic distances.

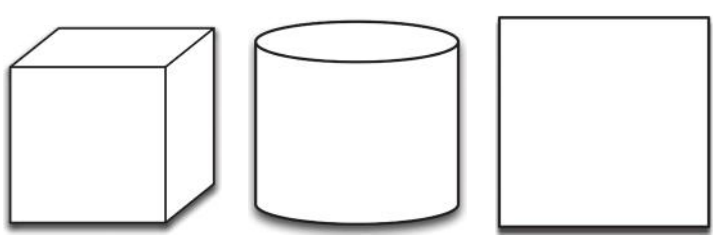


FIGURE 22.1 Two different objects that look the same from the front.

In 1944, the crystallographer A. Lindo Patterson published a landmark paper in the *Physical Review* titled "Ambiguities in the X-Ray Analysis of Crystal Structures."* In this paper, Patterson proposed to view crystallography in an original and unconventional manner. To understand clearly the anomaly evident in earlier experiments, instead of producing x-ray diffraction patterns from three-dimensional crystals in the laboratory, he analyzed one-dimensional idealized "crystals" with paper and pencil using the tools of distance geometry. Since one-dimensional "crystals" are periodic patterns much like the keyboard on a piano, which repeat octave after octave, Patterson took one period of the "crystal" and wrapped it around a circle for convenience. Thus, he obtained a circle with black-and-white equally spaced points. He called these sets *cyclotomic sets*. In mathematical language, they are called *circular lattices*. In that same paper, Patterson published an exhaustive combinatorial analysis of all possible different necklace patterns that could be obtained in a circle of n points by coloring k of them black and $n - k$ of them white, for $n = 8, 9, \ldots, 16$ and $k = 1, 2, \ldots, 8$. Recall that if one configuration can be brought into correspondence with another by rotations, reflections, or a combination of these operations, then the two configurations are considered to belong to the same bracelet. If reflections are left out of this definition, then we are left with necklaces. Clearly, if two cyclotomic sets are instances of the same bracelet, then their histograms of distances are also the same. For $n = 8$ and $k = 4$, there are eight different necklaces shown in Figure 22.2.

Note that although all eight patterns constitute eight different necklaces, that is, no two are congruent to each other, the third and fourth patterns have the same set of distance histograms as shown in Figure 22.3, where the two patterns are redrawn in such a way as to emphasize that they are complements of each other. Thus, Patterson provided for the first

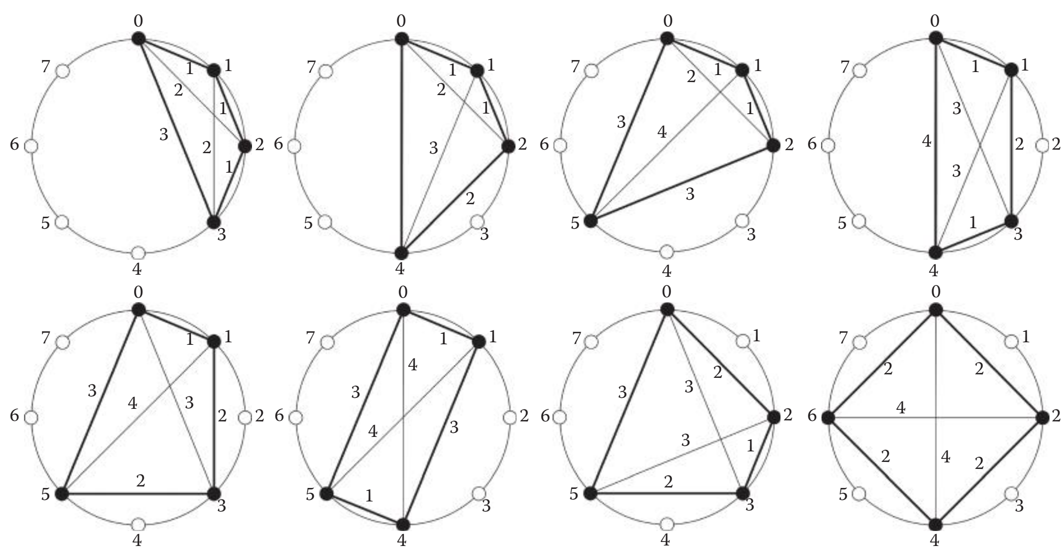


FIGURE 22.2 The eight rhythm necklaces obtained with $n = 8$ and $k = 4$.

* Patterson, A. L. (1944). See also Franklin, J. N. (1974), for other examples of homometric sets, and Erdős, P. and Turán, P. (1941) for a mathematical perspective.

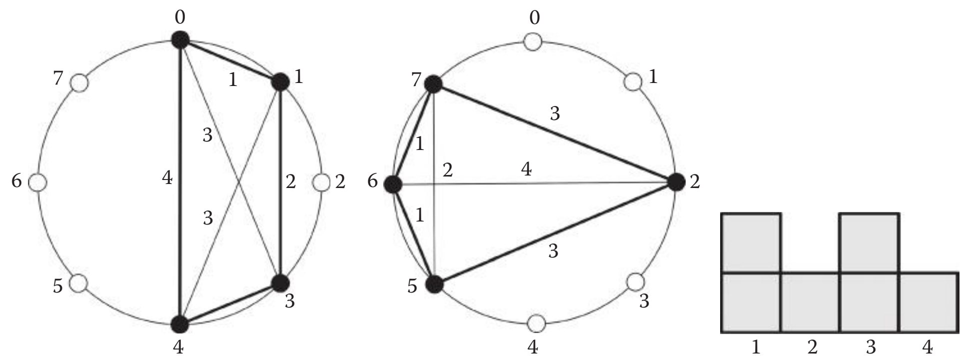


FIGURE 22.3 Two different rhythms that have the same interval-content histogram.

time, a precise geometric equivalent of the experimental anomaly discovered in the x-ray diffraction patterns of molecules. The black points represent atoms, and the numbers along the chords connecting the black points represent the distances between the atoms. That one of them cannot be rotated or reflected to obtain the other is evident from the simple observation that in the pattern on the left, the two intervals of length one are separated from each other by other intervals, whereas in the pattern on the right, they are adjacent. Distances one and three occur twice each, and distances two and four occur once each. As the reader may have observed, cyclotomic sets are identical to the clock diagrams of rhythms used throughout this book. As an interesting side comment, it is worth pointing out that these two rhythms are used together frequently in best-selling popular music. The rhythm on the left with inter-onset interval sequence [1-2-1-4] is a common bass pattern, whereas the complementary rhythm on the right is usually played on the high hat or the ride cymbal. The fifth, sixth, and seventh patterns in Figure 22.2 also represent rhythms that are heard frequently in sub-Saharan African music. It is doubtful, however, that Patterson had made this connection between crystal molecules and musical rhythms.

Patterson's discovery was a shocking revelation to crystallographers. If this example was a unique monster, an exception rather than the rule, then perhaps there was not much to worry about, since the molecules in the real world, of most interest to humanity, are more complex than this simple example. However, if there existed sufficiently many other such monsters, then the reconstruction of molecular structures from their interatomic distances would face a formidable obstacle. Patterson sent his example to the geometer Paul Erdős, one of the most prolific mathematicians of the twentieth century, and asked him if it might be possible to construct other similar examples.* Erdős, unfortunately, did not have good news for Patterson. He wrote back that there existed an infinite number of such monsters, and included the four-point example shown in Figure 22.4. These types of problems now form part of the discipline called distance geometry, which is concerned mainly with the reconstruction of configurations of points in space from their interpoint distances.†

* Hoffman, P. (1999).
† Skiena et al. (1990).

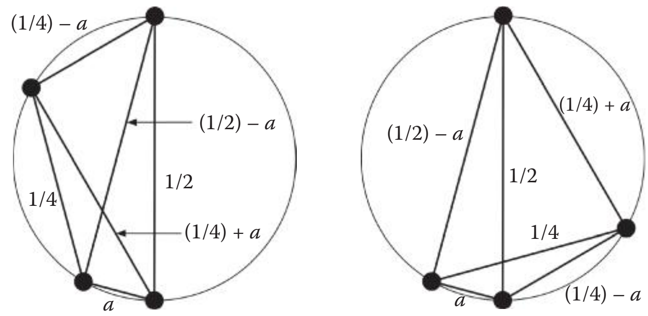


FIGURE 22.4 The infinite family of homometric pairs constructed by Paul Erdős.

This family of patterns is determined by a parameter a, which measures the arc-length (equivalent to time) along a circle of unit circumference, and may take any positive value less than one-half. As the reader may verify, it is impossible to either rotate or reflect one of the two patterns to obtain the other because in the pattern on the left, the intervals of length a and $(1/4) - a$ are not adjacent to each other, whereas in the pattern on the right, they are. Nevertheless, both patterns contain the same set of geodesic distances, namely: a, $1/2$, $(1/2) - a$, $1/4$, $(1/4) - a$, and $(1/4) + a$.

The infinite family of homometric cyclotomic sets that Erdős sent to Patterson is defined on the continuous circle, since the parameter a may take on any positive real value between zero and one-half. However, musical rhythms most often live in a discrete universe of an integer number of pulses. By choosing the value of a to be an integer, and setting the number of pulses in the cycle appropriately, we may create other examples of discrete homometric rhythms to add to Paterson's four-onset, eight-pulse example. For instance, the resulting homometric pair for 16 pulses with a equal to 1 is shown in Figure 22.5.

Patterson generalized the continuous example of Erdős shown in Figure 22.4 to the case of five points arranged on a unit-circumference circle, as pictured in Figure 22.6. Both rhythms in this family of homometric pairs have in common the equilateral triangle of sides $1/m$, where $m = k - 2$, and k is the number of points, in this case five. Furthermore, a and b are chosen so that $a + b = 1/(2m)$.

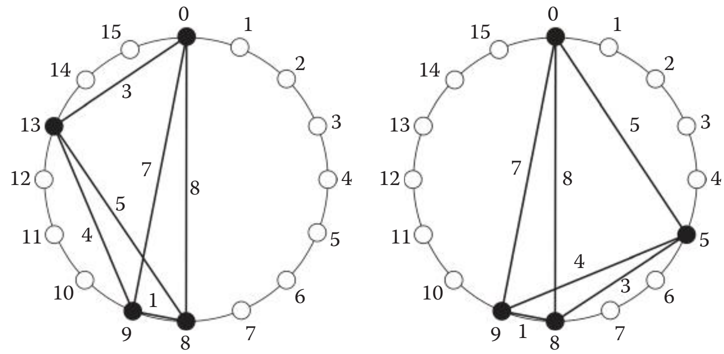


FIGURE 22.5 The Erdős construction for rhythms with 16 pulses and $a = 1$.

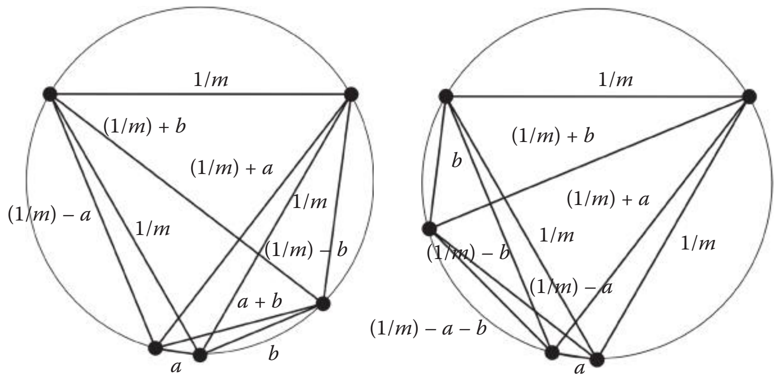


FIGURE 22.6 Patterson's continuous example for five points.

By selecting the number of pulses n, as well as the integer values for a and b, appropriately, it is possible to convert this continuous model to discrete version as well, as shown in Figure 22.7. Here, $n = 24$ so that $1/m$ becomes 1/3, and $a = 1$ and $b = 3$, so that the equation $a + b = 24/(2m)$ is satisfied.

By searching for cyclotomic sets with a higher number of black points, Patterson also found examples where more than two noncongruent sets had the same set of distances. One such triplet that contains six black points among 16 pulses (see Figure 22.8) is most relevant to the rhythmic analysis considered here. Note that the three rhythms contain all the possible inter-onset interval values ranging from one to eight, and have the interval vector (histogram) given by [2,1,2,2,2,3,2,1]. The intervals of length one and eight occur once each, the interval of length six occurs 3 times, and all the other intervals occur twice.*

It is truly amazing that among the rare instances of homometric triplets consisting of six onsets in a cycle of 16 pulses, there should occur the rhythm on the left in Figure 22.8,

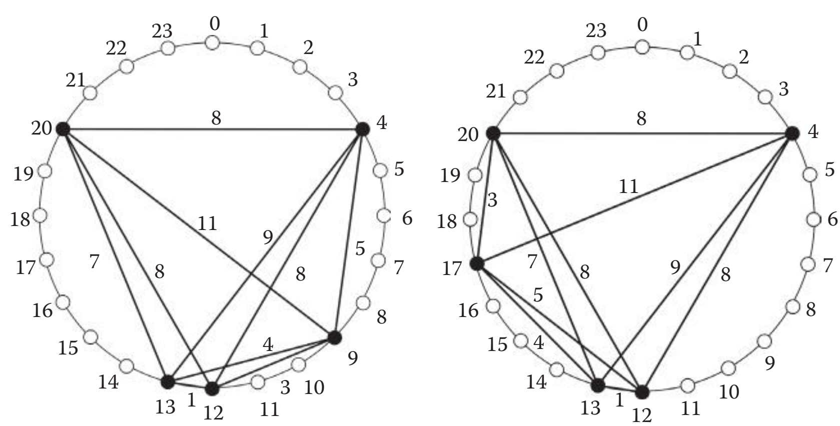


FIGURE 22.7 A discrete version of Patterson's example of Figure 22.6.

* In the pitch domain, David Lewin (1982) also discovered triples of homometric sets in the analogous 16-tone system.

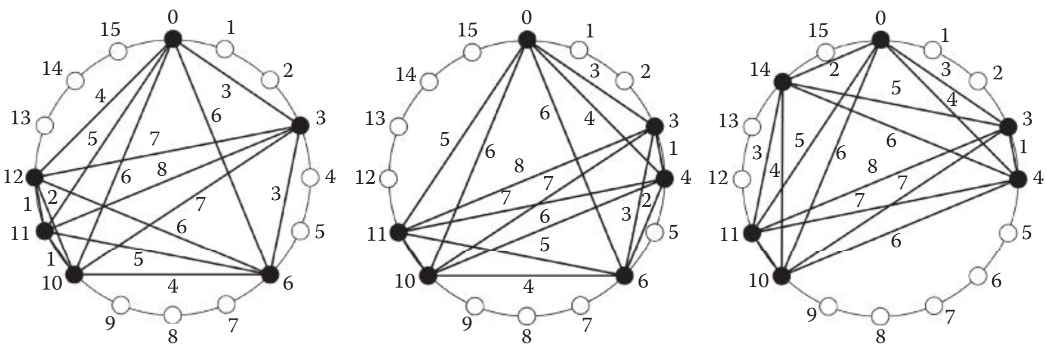


FIGURE 22.8 A homometric triplet discovered by Patterson.

which sounds and is almost identical to the five-onset clave son, the only difference being that this rhythm has one additional decorative onset at pulse number 11. Furthermore, the rhythm in the middle of Figure 22.8 sounds and is almost identical to the soukous timeline. The only difference here is that this rhythm has an additional decorative onset at pulse number four. To top it all off, all these four rhythms sound quite similar to each other. There are people in some parts of Africa who upon hearing these four rhythms would say that they were all the same rhythm. Musicologists may make the milder claim that they are *variants* of each other. It might be tempting to entertain the idea that if two rhythms have the same interval-content histograms, then they would necessarily sound similar. Indeed, some researchers in the field of music information retrieval compute a variety of features of these histograms to characterize different rhythms. Unfortunately, this is not the case, as the third example in Figure 22.8 illustrates. This rhythm sounds completely different from the other two, and yet has the same interval-content histogram. It turns out that finding features of rhythms that characterize their uniqueness and are good for measuring their similarity in a way that correlates well with human judgments is an extremely difficult problem.

In closing this chapter, it should be noted that homometric sets have been explored in music for some time in the pitch domain, where they are called Z-related sets.*

* Mandereau, J., Ghisi, D., Amiot, E., Andreatta, M., and Agon, C. (2011), Soderberg, S. (1995).

CHAPTER 23

Complementary Rhythms

At first glance, Figure 23.1 shows what appears to be a black candleholder, or vase, on a white background. However, if the viewer focuses attention on the contour boundary between the white and black regions, two white faces staring at each other over a black background may be perceived instead. This visual perception phenomenon is known as *figure-ground reversal*. As we gaze at visual stimuli such as these, the *figure* (or foreground) and the *ground* (or background) spontaneously switch their roles. This perceptual phenomenon has been exploited in the work of several artists, including Salvador Dali, who used it quite successfully in several paintings such as the *Slave Market with Disappearing Bust of Voltaire*.*

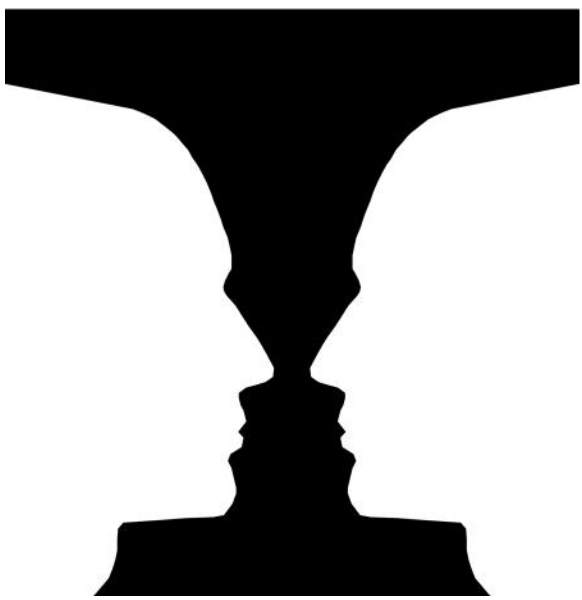


FIGURE 23.1 Visual figure-ground reversal.

* Fisher, G. H. (1967), p. 555.

A similar phenomenon occurs in the domain of aural perception of rhythms. Consider the two rhythms shown in Figure 23.2. Let the rhythm on the left be the lead rhythm (figure) played on a high-pitched conga, and the one on the right be the accompaniment (background) played on a low-pitched conga. If both rhythms are played at the same volume, then the listener's attention may shift back and forth spontaneously, or at will, to perceive either the rhythm on the left or the one on the right.*

Note that in Figure 23.2, the rhythm on the right consists of onsets at the positions of the silent pulses contained in the rhythm on the left, and vice versa. Such rhythms are called *complementary*.† Together, they fill the entire set of pulse locations in the cycle, and are thus also referred to as *interlocking*‡ rhythms. Interlocking rhythms constitute one of the main principles of rhythm integration in African drumming.§ In this particular case, both rhythms belong to the same necklace: one rhythm is a rotation of the other. Thus, this aural example is not completely analogous to the visual example in Figure 23.1, where the figure (candle holder) and the ground (two faces) are completely different patterns. However, analogous examples with two different complementary rhythms are also possible, as Figure 23.3 illustrates. The rhythm on the left is the illustrious *bembé* rhythm (also diatonic scale in the pitch domain), and the one on the right is a rotation of the equally prominent *fume-fume* rhythm (also pentatonic scale in the pitch domain). The "audible" beats of one rhythm are the "empty" beats of the other.¶ It is interesting that both the seven-onset *bembé* timeline and its complement are both highly prominent in sub-Saharan African traditional

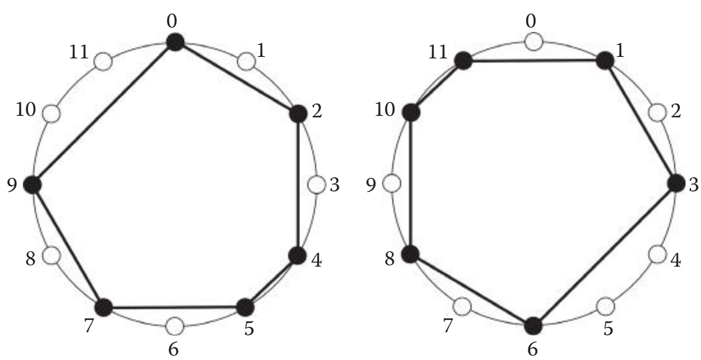


FIGURE 23.2 An example of aural rhythmic figure-ground reversal.

* See Anku, W. (2007) for a discussion of psychoacoustic considerations of rhythm in a cultural context.

† Morris, R. D. (1990).

‡ Kubik, G. (2010a), p. 42. The term *interlocking* rhythms sometimes also applies to situations where some attacks of the two rhythms coincide. Of course, with complementary rhythms, this generalization is automatically excluded.

§ Anku, W. (1997), p. 212, analyzes interlocking rhythms as well as overlapping, and adjacency and alternation rhythms. For further details, see Anku, W. (1995) and Anku, W. (2002a,b), as well as Cuthbert, M. C. (2006) for an evaluation and expansion of Willie Anku's theories.

¶ Sachs, C. (1943), p. 190, uses the terms "audible" and "empty" beats in discussions of complementary rhythms. He notes that Indian *tabla* drummers often play the "audible" beats with the right hand and the "empty" beats with the left. Kubik, G. (1999), p. 54, notes that: "One of the two phenotypes is usually dominant within a culture, while the other is implied or may sometimes be silently tapped with a finger." In other cultures, both rhythms are struck with equal force, one with each hand.

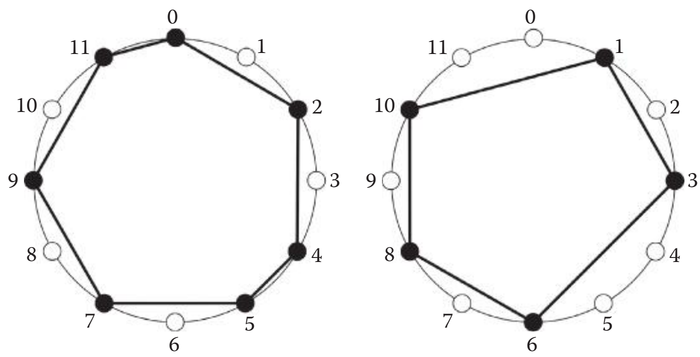


FIGURE 23.3 The *bembé* rhythm (left) and its complement (right).

music. Playing both together on different drums at the same volume also produces in the listener spontaneous figure-ground reversals between the two timelines.

Complementary rhythms that have the same number of onsets enjoy the very special and rather surprising property that their inter-onset interval histograms are always identical to each other. In other words, they are *homometric*. That the two complementary six-onset rhythms in Figure 23.2 are homometric is obvious since one rhythm is a rotation of the other, and homometricity is invariant to rotation and reflection transformations. What is surprising is that even if one rhythm is not a rotation or mirror image of the other, as long as both are complementary and have the same number of onsets, they are homometric. One such example is the three-over-four rhythm pictured in Figure 23.4 (left). This rhythm consists of the union of two regular rhythms: [4-4-4] at pulses (0,4,8) and [3-3-3-3] at pulses (0,3,6,9).* Clearly, the rhythm on the left is neither a rotation of the rhythm in the center nor a combination of a rotation and mirror image reflection, since the two adjacent inter-onset durations of size two lie next to each other in the first rhythm, and opposite to

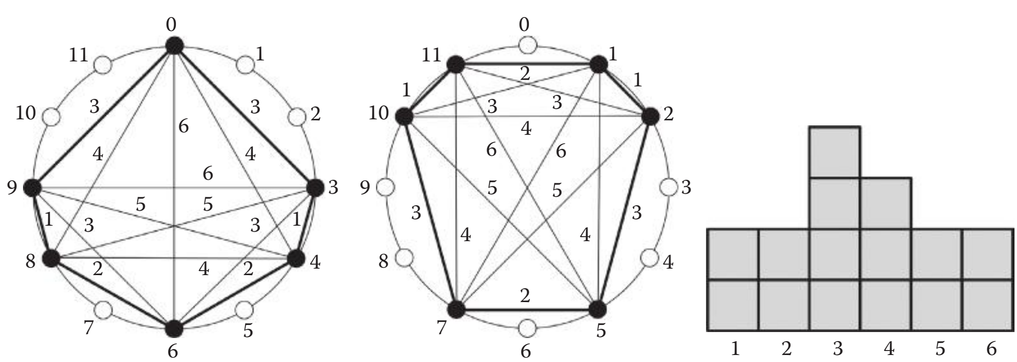


FIGURE 23.4 The three-over-four rhythm (left), its complement (center), and their common interval histogram (right). (From Liu, Y. and Toussaint, G. T. 2012. *International Journal of Machine Learning and Computing*, 2(3):261–265. With permission.)

* This is a common clapping pattern in West African music. See Jones, A. M. (1954a), p. 35.

each other in the complementary rhythm. Nevertheless, as the reader may readily verify, both rhythms have the same interval histogram (right), and are therefore homometric. This is sometimes called the *hexachordal* theorem in the music literature because it was originally proved there in the context of pitch and chords in a 12-tone system.

There is a simple and straightforward argument to show why the hexachordal theorem must be true. First, consider a rhythm of 12 pulses with an onset on every pulse and refer to Figure 23.5. The first diagram shows that distance one determines a 12-sided polygon. The second diagram contains two hexagons consisting of edges of distance two, for a total of 12 distances. The third figure contains three squares with side-lengths three, which makes 12 distances. The fourth figure contains four triangles that determine 12 edges of length four each. The fifth figure is one 12-sided star polygon with sides of length five, as a result of the fact that the numbers five and 12 are relatively prime. Recall that two numbers are relatively prime if they have no factors in common other than one. In this case, 5 and 12 are relatively prime. Finally, the sixth figure contains six bigons (two-sided polygons) or 12 distances of length six, one from each onset to its diametrically opposite onset.* It follows from these observations that when there is an onset at every pulse of a 12-pulse cycle, every distinct distance value occurs 12 times, and this generalizes to any number of pulses.

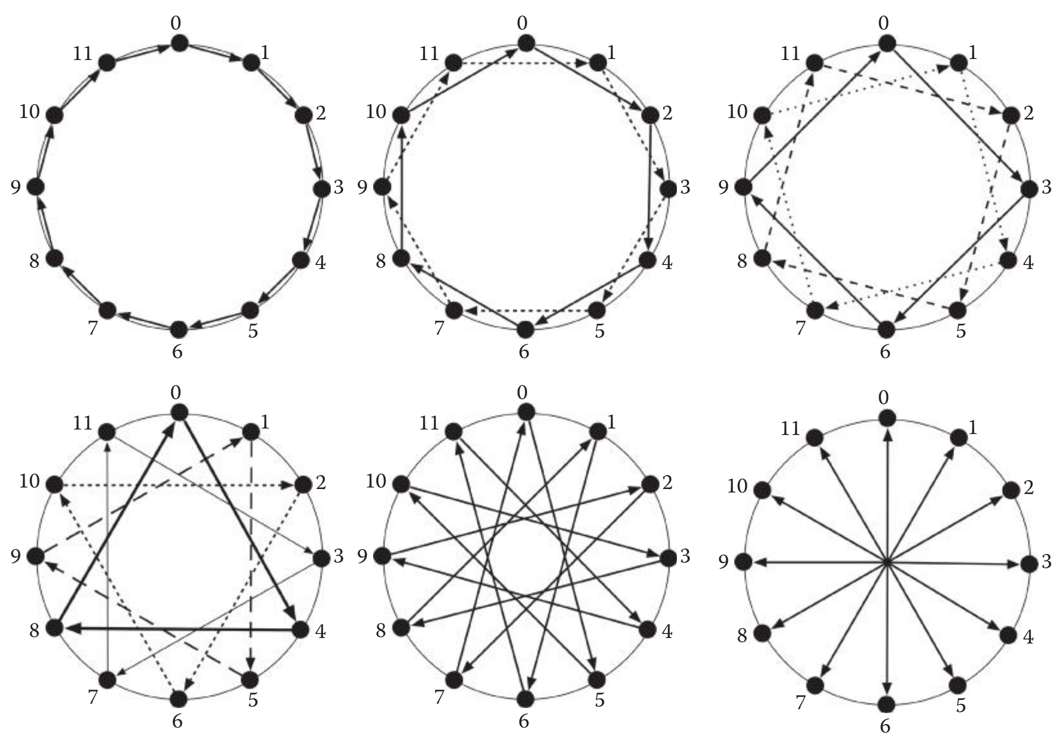


FIGURE 23.5 In a 12-pulse cycle, every distance occurs 12 times.

* Note that the distances are directed distances. Thus, between pulses zero and six, there are two distances, one from zero to six and one from six to zero.

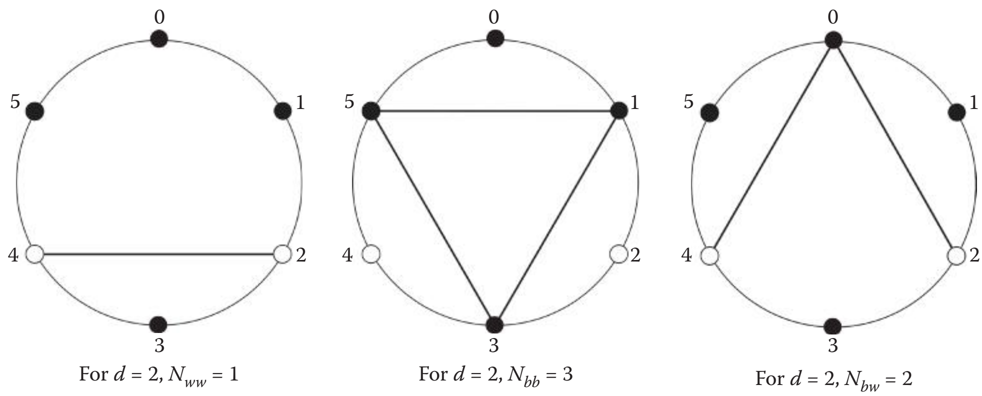


FIGURE 23.6 The number of white–white, black–black, and black–white pairs at distance 2.

The second step in the argument involves the analysis of how these distances change when not all the pulses are onsets. Accordingly, let there be n pulses in total, where n is an even number, and let there be p onsets (called black points) and $q = n - p$ silent pulses (called white points). For any fixed distance $d = 1, 2, \ldots, n/2$, let N_{ww}, N_{bb}, and $N_{bw} = N_{wb}$, be the number of edges of distance d connecting two white points, two black points, and a black–white pair, respectively. For example, in Figure 23.6, for the case of $n = 6$, $k = 4$, and $d = 2$, the number of white–white, black–black, and black–white pairs is 1, 3, and 2, respectively.

In particular, the second step in the argument establishes the following two simple relationships between these five quantities:

$$p = N_{bb} + (1/2)N_{wb}$$

$$q = N_{ww} + (1/2)N_{wb}$$

When all n points are black, the quantities N_{ww} and N_{wb} are equal to zero, and $N_{bb} = n$. Therefore, $q = 0$ and $p = n$, which confirms that $p + q = n$. It must be shown that these relationships continue to hold when some black points are changed to white. Initially, all points are black and one of them will be changed to white. This case is illustrated in Figure 23.7 for the more general situation in which some black points have already been changed to white. For concreteness, and without loss of generality, the black onset at pulse seven is changed to white. For any specified distance value d, this case assumes that there are two black onsets that realize this distance; call them d-neighbors. In changing this onset seven from black to white, the number of black–black d-neighbors N_{bb} goes down by 2, the number of white–black d-neighbors N_{wb} goes up by 2, and the number of white–white d-neighbors N_{ww} remains unchanged. It follows from the two preceding equations that p goes down by 1 and q goes up by 1, confirming that $p + q = n$.

The case when both d-neighbors are white is illustrated in Figure 23.8. This change causes N_{wb} to decrease by 2, N_{ww} to increase by 2, and leaves N_{bb} unchanged. It follows again

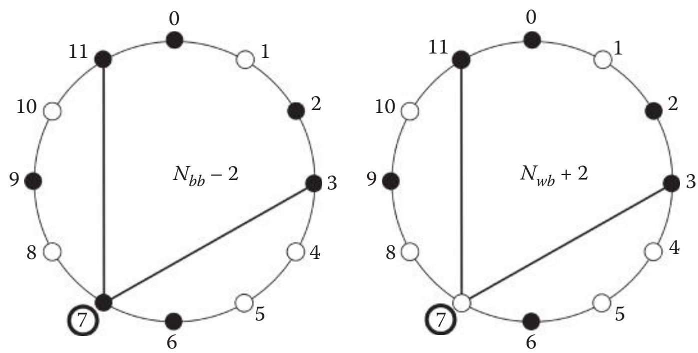


FIGURE 23.7 Both d-neighbors of a black point are black.

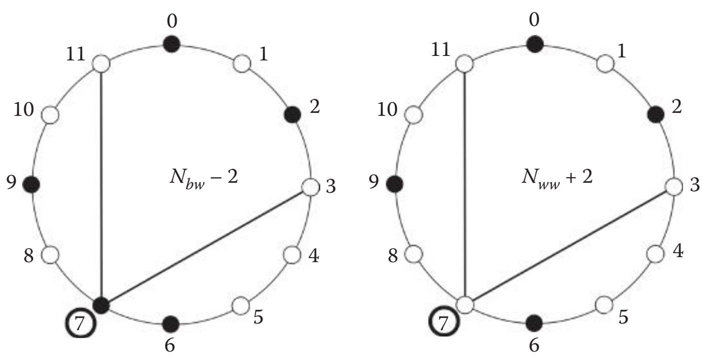


FIGURE 23.8 Both d-neighbors of a black point are white.

from the two preceding equations that p goes down by 1 and q goes up by 1, again validating that $p + q = n$.

Finally, the case where one neighbor is black and the other white is pictured in Figure 23.9. Now, N_{bb} goes down by 1, N_{ww} goes up by 1, and N_{bw} remains unchanged. Substituting these changes into the two equations makes p go down by 1 and q go up by 1, once more supporting the claim that $p + q = n$.

This case analysis establishes that no matter how many, or which, black points are changed from black to white, the preceding equations that relate p and q to N_{ww}, N_{bw}, and N_{bb} remain true.

The final step in the argument for proving the hexachordal theorem involves setting p equal to q in the above equations, which yields the equation:

$$N_{bb} + (1/2)N_{wb} = N_{ww} + (1/2)N_{wb}$$

Since the two terms on each side of the equality that contain N_{wb} cancel each other out, we conclude that $N_{bb} = N_{ww}$, which proves the result that for any distance value d, the number of times it occurs in the rhythm (N_{bb}) is the same as in its complementary rhythm (N_{ww}).

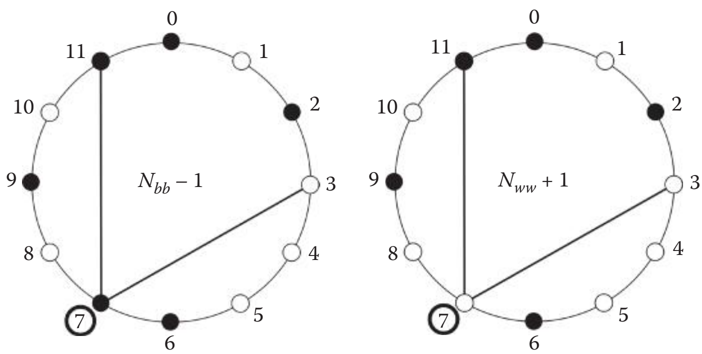


FIGURE 23.9 One d-neighbor of a black point is black and the other white.

In 1995, I discovered a large scientific literature concerned with the hexachordal theorem and related results on homometric sets, in the crystallography journals, that the music theory community was unaware of. I presented these findings at the *Fourth International Workshop on Computational Music Theory*, at the Universidad Politecnica de Madrid, Escuela Universitaria de Informatica, on July 24–28, 2006.[*] The crystallography literature contained a variety of different proofs of the hexachordal theorem, which I presented at a *Special Session on Mathematical Techniques in Music Analysis*, at the 113th Annual Meeting of the American Mathematical Society, in New Orleans on January 6, 2007.[†]

The term "homometric" was introduced in 1939 by Lindo Patterson.[‡] The earliest proof of the hexachordal theorem in the music literature is due to Milton Babbitt[§] and David Lewin.[¶] It used some powerful machinery from topology. Later, Lewin obtained new proofs using group theory. Later still, Eric Regener found an elementary simple proof of this theorem.[**] This theorem was known to crystallographers at least 20 years earlier.[††] It was claimed to have been proved by Lindo Patterson around 1940 but he did not publish a proof. In the crystallography literature, the hexachordal theorem is called Patterson's theorem.[‡‡] The first published proof in the crystallography literature is due to Buerger; it is based on image algebra, and is nonintuitive. A much simpler and elegant elementary induction proof was later found by Iglesias, which has been simplified in the description presented above.[§§] More recently, another simple and elementary induction proof was discovered by Steven Blau.[¶¶] Several other proofs of this theorem have been published over the years in the music theory, crystallography, and mathematical literatures. Some are shorter than the

[*] Toussaint, G. T. (2006a).
[†] Toussaint, G. T. (2007).
[‡] Mardix, S. (1990).
[§] Dembski, S. and Straus, J. N., Eds. (1986).
[¶] Lewin, D. (1959, 1960, 1976, 1977, 1982).
[**] Regener, E. (1974).
[††] Patterson, A. L. (1944).
[‡‡] Buerger, M. J. (1976).
[§§] Iglesias, J. E. (1981).
[¶¶] Blau, S. K. (1999).

proof described above, a few are more general, and others require advanced mathematical knowledge.*

The simplified version of Iglesias' proof presented above establishes a more general result, from which the hexachordal theorem, among other results discussed by Iglesias, follows as a corollary. However, if we are interested in proving just the hexachordal theorem, then the simplest proof I have heard comes from the Harvard mathematician Noam Elkies via Dmitri Tymoczko. It is fitting to close this chapter with this gem.

Using the same notation as above, for any fixed distance $d = 1, 2, \ldots, n/2$, let B, and W denote the black and white sets of points, respectively, each of cardinality $n/2$. For each value of d, a point in B is d units away from two other points, one advancing in a clockwise direction around the circle, and the other in a counterclockwise direction. Therefore, B has n distances of size d. Now, let N_{ww}, N_{bb}, and $N_{bw} = N_{wb}$ be the number of distances d between two white points, two black points, and a black–white pair, respectively. It follows that $N_{bb} + N_{bw} = n$. By the same argument, it follows that $N_{ww} + N_{wb} = n$. Therefore, $N_{bb} + N_{bw} = N_{ww} + N_{wb}$, and since $N_{bw} = N_{wb}$, the two terms cancel out leaving $N_{bb} = N_{ww}$.

* For additional proofs of the hexachordal theorem, see Rahn, J. (1980), Chapter 5, Senechal, M. (2008). Amiot, E. (2007) gives a concise proof of the hexachordal theorem using a mathematical tool called the fast Fourier transform (FFT). See also Hosemann, R. and Bagchi, S. N. (1954) for further information on homometric structures. For a nice discussion of homometry in musical distributions, see Mandereau et al. (2011).

Radio Astronomy and Flat Rhythms

Chapter 22 explored the unlikely connection between musical rhythms and crystallography. This chapter considers another surprising connection, this time between musical rhythm and radio astronomy. When I say "surprising," I mean surprising at first thought. Once it is realized that two areas are concerned with distances between pairs of elements, there is bound to be some connection. Radio astronomers are interested in receiving signals from outer space to discover new planets in other solar systems, perhaps stumble on alien intelligent life, as well as answer a variety of questions related to the structure of matter in the universe. For this purpose, they construct colossal and expensive radio telescopes or dish antennas: larger dishes provide better signals. However, very large dishes become prohibitively expensive. One approach to deal with this problem is to use several small dishes instead of one large one, and to arrange them some distance apart, thus ensuring that the waves of the signals arriving at each dish are out of phase with each other, resulting in interference. This interference causes the amplitude of the resultant wave obtained by superimposing the two waves to exhibit greater variation. For example, Figure 24.1 shows a two-element radio telescope at a unit distance apart. By analyzing this interference between the signals received in the two receivers, astronomers can improve signal detection.

By using more than two elements, much greater interference information is obtained, and the increased signal detection capability makes more efficient use of the individual elements. The two-element radio telescope provides only one separation distance between them that can be used to analyze the interference pattern between the two signals. However,

FIGURE 24.1 A two-element radio telescope separated by a unit distance.

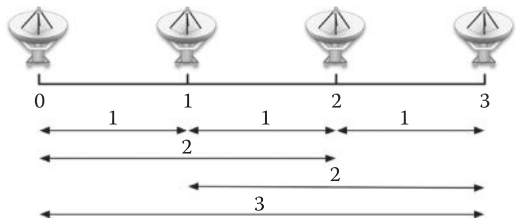


FIGURE 24.2 A linear four-element radio telescope with uniform spacing that realizes three distinct distances.

using several elements allows the possibility of realizing distances between all the pairs of elements. For example, the four-element radio telescope shown in Figure 24.2 realizes six pairwise distances [1, 1, 1, 2, 2, 3]. However, this arrangement is far from optimal since among the six distances realized, this set contains only three distinct distances: distance one is repeated three times and distance two is repeated twice, thus introducing considerable redundancy. In this application, redundancy is wasteful. What is required is a placement of the elements in such a way that as many distinct distances as possible are realized.

It is not too difficult to spread out the elements of a radio telescope so that all pairs of distances between them are different, especially if the number of elements used is not large, and one has unlimited space available. The example with four elements shown in Figure 24.3 realizes the six different distances [2, 3, 4, 5, 7, 9]. If each unit in this example represents 1 km in the real world, this telescope would need a space 9 km long. In practice, it is desirable to use as little space as possible, and still realize the maximum number of different distances. Figure 24.4 shows the optimal placement of four elements. The distances realized are [1, 2, 3, 4, 5, 6], and the required total length is only six units.

The problem of distributing radio telescope elements so that all the distances between pairs of elements are different is structurally identical to the problem of designing *Golomb rulers*. Ordinary rulers that are filled with many marks on both sides are used for measuring distances and drawing straight lines between pairs of points. A Golomb ruler on the other hand is a ruler with very *few* marks.* Furthermore, since one can only measure distances between pairs of marks, the marks in a Golomb ruler are arranged so that as many different distances as possible can be measured. Although the design of such rulers

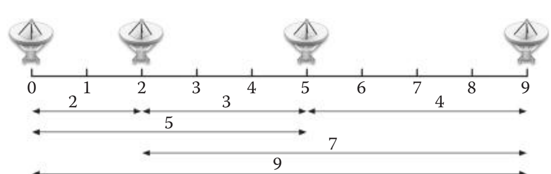


FIGURE 24.3 A linear four-element radio telescope array of length nine that realizes six distinct distances.

* Alperin, R. C. and Drobot, V. (2011).

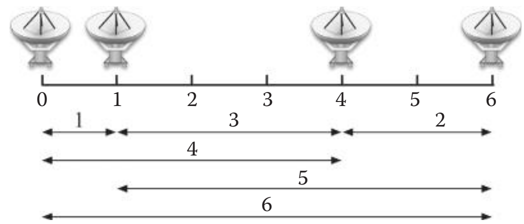


FIGURE 24.4 A linear four-element radio telescope array that realizes six distinct distances, and has length six.

was first investigated by Simon Sidon* in 1932, they are named after the mathematician Solomon Golomb, who was one of several independent rediscoverers of the problem, and made them popular. A Golomb ruler with k marks is called *optimal* if no other shorter Golomb ruler with k marks exists. Furthermore, if the Golomb ruler measures *all* the distances ranging from one to the length of the ruler, it is called *perfect*. Figure 24.5 shows a perfect Golomb ruler with four marks.

It is not easy to characterize optimal Golomb rulers, and thus obtain an efficient algorithm for generating all of them. So far, optimal Golomb rulers have been found with up to 26 marks only. The search for an optimal ruler with 27 marks continues to this day. However, sufficient conditions have been found for generating special cases of Golomb rulers. For example, Paul Erdős and Pál Turán showed that for every odd prime number p, a Golomb ruler with p marks may be constructed with the following algorithm: $2pk + (k^2 \bmod p)$, where k varies between 0 and $p - 1$, and *mod* stands for the modulo function, or what remains when k^2 is divided by p. If we let $p = 5$, this formula yields marks with integer values 0, 11, 24, 34, and 41. These five marks determine the 10 distinct distances given by 7, 10, 11, 13, 17, 23, 24, 30, 31, and 41. Although much shorter Golomb rulers with five marks exist, the power of this algorithm lies in its generality. A five-mark Golomb ruler with length only 11 is shown in Figure 24.6.

The problem of whether there exist rulers with a different set of marks that measure the same set of distances has also been investigated. This is the problem of the existence of homometric sets on the line, also called the *turnpike* problem in computer science. Figure 24.7 shows two six-mark homometric rulers discovered by R. Hosemann and S. N. Bagchi in 1954. These are not Golomb rulers because some distances occur more than once: distances two and six each occur twice. However, these two rulers have another redeeming

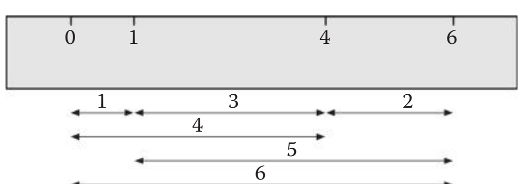


FIGURE 24.5 A *perfect* Golomb ruler with four marks.

* Erdős, P. and Turán, P. (1941).

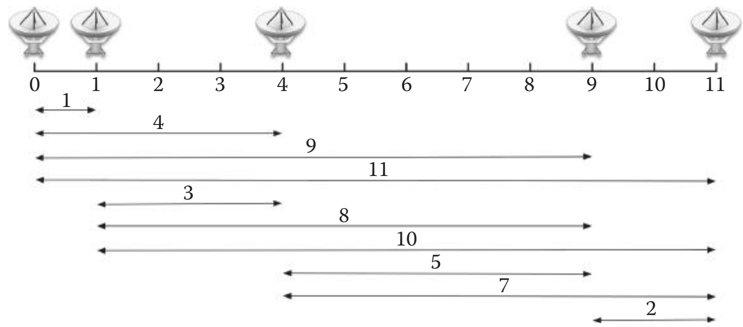


FIGURE 24.6 A Golomb ruler of length 11 with five marks.

property, and that is that they are able to measure *all* the integer distances between 0 and 11. Such rulers have been called *spanning rulers* by Roger Alperin and Vladimir Drobot, who studied some of their properties.* They also defined a *minimal spanning ruler* as one that stops being a spanning ruler if one of its marks is erased.

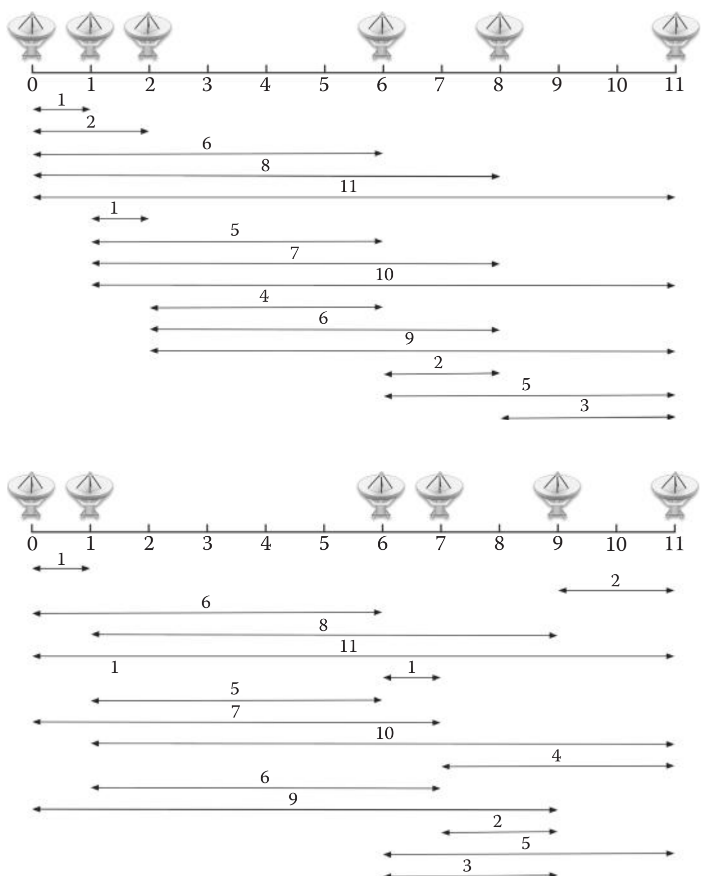


FIGURE 24.7 Two six-mark homometric *spanning* rulers that are not Golomb rulers.

* Alperin and Drobot, *op. cit.*

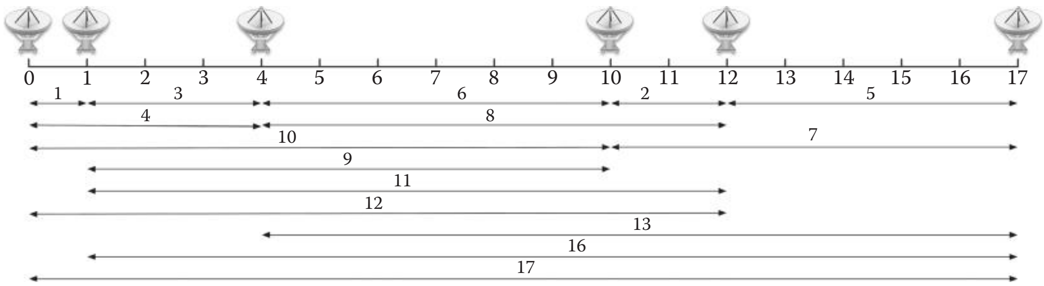


FIGURE 24.8 First Golomb ruler of Bloom's counterexample.

In 1974, J. Franklin provided a variety of constructions and examples of pairs of homometric rulers. However, like the example in Figure 24.7, none of his examples were Golomb rulers. Franklin's rulers were consistent with a published theorem dating back to 1939 due to S. Piccard,* which stated that if the sets of distances determined by the marks of two rulers had no repetitions (i.e., the rulers were Golomb), then a necessary and sufficient condition for the two distance sets to be equal is that the two rulers should be congruent, meaning that one ruler may be translated or rotated to bring it into perfect correspondence with the other. However, in 1975, Gary Bloom discovered a counterexample to Piccard's theorem.† He constructed two Golomb rulers shown in Figures 24.8 and 24.9 that measure the same set of distances between all pairs of their marks, and yet are not congruent.

By connecting the ends of a Golomb ruler together to form a circle, and considering the marks on the ruler as possible positions for the locations of the onsets of musical notes, the ruler determines a rhythm. If we replace the straight-line distances between the marks on a ruler by geodesic distances on the circle that represent time durations, then the Golomb ruler problem becomes that of constructing rhythms in which all the inter-onset durations are distinct. This means that the inter-onset duration histograms of such rhythms are flat in the sense that all histogram bins have height either one or zero. Accordingly, we will use Jon Wild's terminology, and call such rhythms *flat* rhythms.‡ It is very easy to create flat

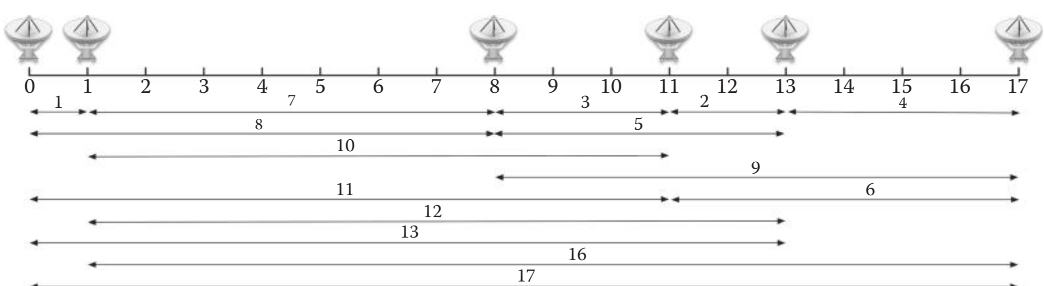
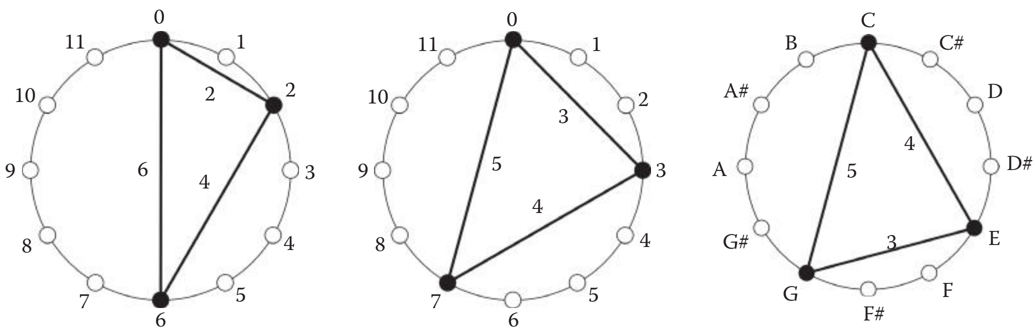


FIGURE 24.9 Second Golomb ruler of Bloom's counterexample.

* Piccard, S. (1939).

† Bloom, G. S. (1977).

‡ Wild, J. (2007) introduced the notion of *flat interval distributions* in the context of pitch class sets. Here, Wild's terminology is adapted to rhythms.



2–4–6 Rhythm   3–4–5 Rhythm   C major chord

FIGURE 24.10 Rhythms and chords with distinct intervals.

rhythms if the number of onsets is 3, because then there are only three inter-onset intervals. One, merely, has to ensure that the three distances are distinct. From the geometrical point of view, this means that the rhythm triangle should not be *isosceles*. Figure 24.10 shows two flat rhythms, left and center, with inter-onset interval sequences [2-4-6] and [3-4-5], respectively. On the right is the C major chord visualized as a triangle connecting the C, E, and G notes. It also has three distinct intervals [4-3-5].

The difficulty in creating rhythms with distinct inter-onset intervals arises when there are more than three onsets, although such rhythms do exist. Two examples of flat rhythms with four onsets and 16 pulses are shown in Figure 24.11. Both rhythms have exactly the same set of distances (all except two and six), and are therefore also homometric rhythms.

Just as there are perfect Golomb rulers that measure all the integer distances up to their maximum value, so also there are rhythms that contain *all* durations exactly once, thus producing flat histograms without gaps. Inspired by the notion of perfect Golomb rulers, we will call such rhythms *perfect* flat rhythms. Two examples of perfect flat (and homometric) rhythms with four onsets and 12 pulses are given in Figure 24.12. Note that they are also *minimal spanning* rhythms since deleting any onset from either of them prevents them from being able to measure more than three durations. In the pitch domain, these two

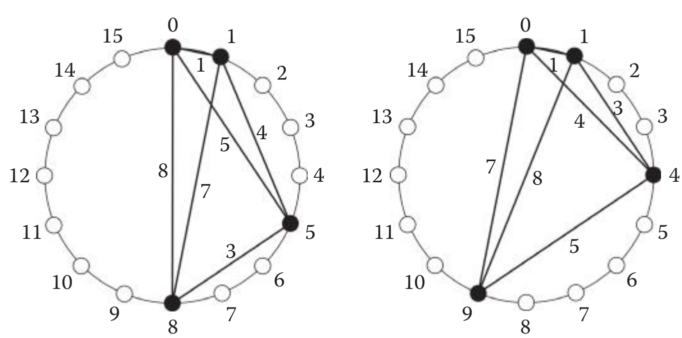


FIGURE 24.11 Two four-onset, 16-pulse homometric rhythms with flat interval-content histograms.

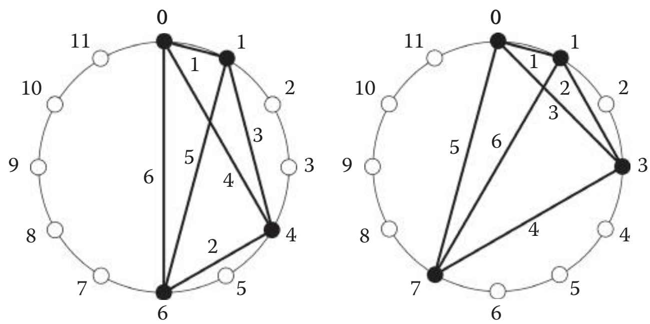


FIGURE 24.12 Two four-onset, 12-pulse minimal-spanning, homometric rhythms with *perfect* flat interval-content histograms.

polygons are the well-known *all-interval tetrachords*. These two chords have been explored extensively, by both music theorists and composers, for several decades.*

One may wonder how many more pairs of noncongruent perfect homometric rhythms exist in a 12-pulse time span. Recall that rhythms that are congruent may be transformed into each other by rotations and/or reflections, and thus are instances of bracelets. It turns out that the bracelets pictured in Figure 24.12 are the only two possibilities. An elementary case analysis will prove the point.† Since the time span has 12 pulses, there are a total of six distinct durations to be realized by all pairs of onsets. Therefore, the number of pairs of onsets must equal 6, implying that the number of allowable onsets must satisfy the equation $k(k-1)/2 = 6$. The only solution of this equation is $k = 4$. It follows that only quadrilaterals need to be considered in our search. Since the distance six must occur once in this quadrilateral, it must be either a diagonal or an edge of the quadrilateral. Consider first the case where it is a diagonal. Without loss of generality, let this diagonal connect pulses zero and six. Then, there are eight possibilities for constructing the quadrilateral such that one onset lies in each half-circle determined by the diagonal, and the sum of the two edges on each side equals six, as shown in Figure 24.13. The first and second quadrilaterals, (also the third and fourth) are reflections of each other about the vertical axis through pulses zero and six. In addition, the first and fourth (also the second and third) are reflections of each other about the horizontal axis through pulses nine and three. Similar relations hold for the bottom four quadrilaterals.

The four quadrilaterals at the bottom of Figure 24.13 are very different from the four at the top, in that they do not contain the interval of length three. Instead, they have two instances of the interval of length five. Therefore, they may be discarded from consideration. The four at the top do contain all six intervals, but they are all instances of the same bracelet. Thus, we have found one of the candidates in this set.

The second situation arises when the duration of length six is an edge of the quadrilateral. Six arrangements of the other three edges are possible as shown in Figure 24.14.

* Block, S. and Douthett, J. (1994), p. 35. See Goyette, J. (2011) and Childs, A. P. (2006) for discussions on the structural and transformational properties of the all-interval tetrachords, as well as additional references.

† The proof given here is similar to that of McCartin, B. J. (1998), p. 360.

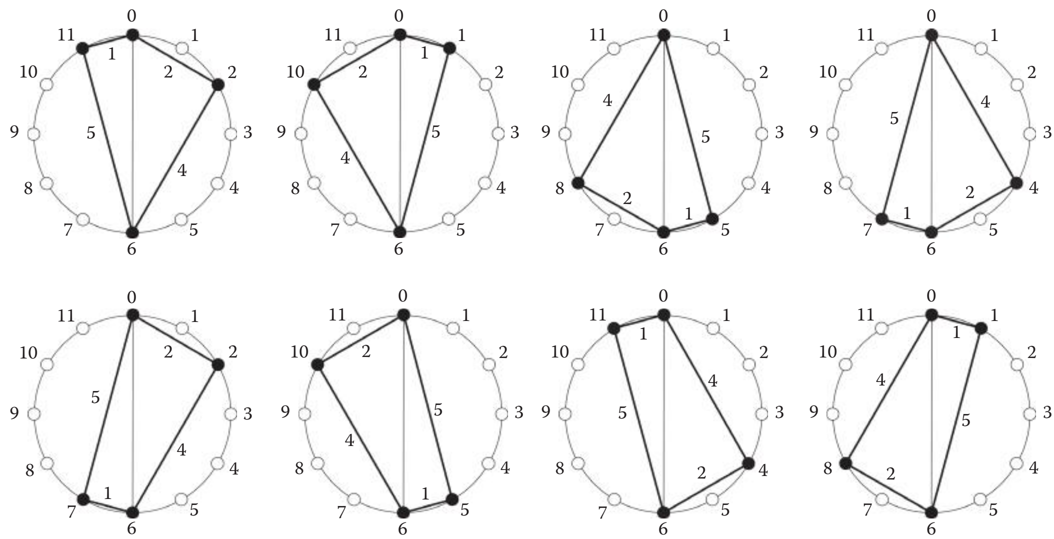


FIGURE 24.13 The eight cases in which the interval of length six is a diagonal.

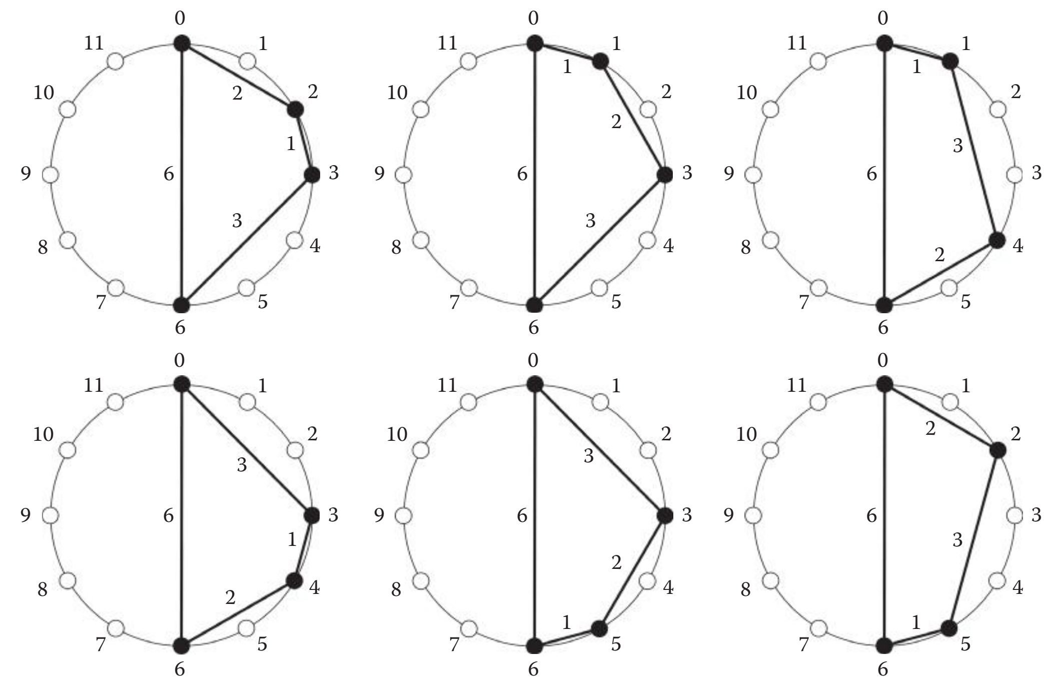


FIGURE 24.14 The six cases in which the interval of length six is an edge.

The middle edges may have lengths one, two, or three, and for each of these three choices, each of the two remaining edge lengths may be inserted at the top or the bottom. Notice that in the four arrangements on the left, the diagonal incident to pulse three has duration three, which in all four cases is already present as an edge. Hence, these four candidates may be discarded. The two candidates on the right do contain all six intervals, but they are reflections of each other about the axis through pulses nine and three, and are therefore instances of the same necklace. In conclusion, there are only two rhythm bracelets that have perfect flat interval histograms.

CHAPTER 25

Deep Rhythms

In the land of the pharaohs in ancient Egypt, the builders of the majestic pyramids and temples that adorn the surroundings of the Nile river made use of an amazing low-tech, biodegradable, zero-radiation, inexpensive, and low-maintenance computer to find solutions to a variety of geometric problems that they encountered in their daily lives: the *knotted rope*. As the name suggests, this computer consists of a rope of suitable length tied together at the ends, and interspersed with a preselected fixed number of equally spaced knots. There is evidence that one popular model of this computer comprised 12 knots as shown in Figure 25.1, in two configurations: loose (left) and taut (center). The users of this computer knew from experience in the field that if three people holding the rope at the knots numbered one, five, and eight walked away from each other until all three strands of rope between them were taut, the final shape of the rope would be a triangle.* Furthermore, and this is the crucial point, they were confident that the triangle would be a *right-triangle*: the angle made between the two short sides of lengths three and four is 90° angle. One of the most useful applications of this computer was, therefore, the construction of 90° angles in architecture. To illustrate this application, assume, for example, that the workers had to build a new wall that made a 90° angle with another wall already built, and refer to

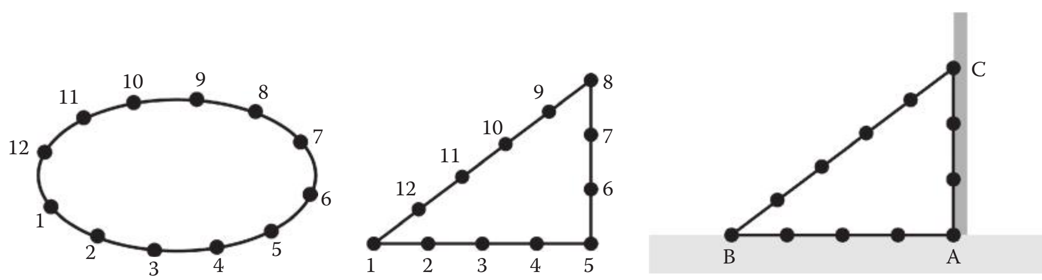


FIGURE 25.1 The knotted string computer for constructing right angles.

* Ross, C. (2007), p. 157. For an alternate view, see Rowe, D. E. (2006), p. 52, who argues that the use of the knotted rope for laying out right angles in ancient Egypt is speculation, and that there is little reliable evidence that the builders were aware of the Pythagorean theorem. Imhausen, A. (2006) discusses additional ancient Egyptian mathematics.

Figure 25.1 (right), where the old wall is shown in the horizontal position. Assume further that this new wall is required to meet the old wall at the point marked A. First, one worker holds the rope at the fifth knot and stands at position A. The second worker then takes knot number one, and walks away from the first worker along the old wall until the rope between them is taut, ending at position B. Finally, the third worker takes knot number eight, and walks away from the other two workers adjusting his or her position until both strands are taut. The engineers were convinced that if the new wall was built in a straight line from A to C, the angle BAC would be a 90° angle.

You may ask yourself how the workers came to be so assured that in a triangle with side lengths consisting of 3, 4, and 5 units (called a 3-4-5 triangle), the angle between the two short sides was a right angle. In those days, the workers followed algorithms outlined in manuals issued by the pharaohs. Very likely, in one such manual, the procedure outlined above for constructing walls at 90° angles with the knotted rope was followed by an official statement of the form: *If this algorithm is followed, then the correct solution is guaranteed to be found*. Next to this statement would usually be stamped the seal of approval of the pharaoh. This process might be called a proof by governmental decree. It would be several thousand years later before the correctness of this algorithm was formally established using a more democratic method: a mathematical proof.

In the sixth century BC, the great mathematician Pythagoras of Samos in ancient Greece proved a wonderful theorem about right-angled triangles, of which the 3-4-5 triangle used in the knotted rope computer is a special case.* This theorem, now called Pythagoras' theorem, is taught to every child in school. The theorem states that in a right-angled triangle, the area of the square with side equal to the longest side of the triangle (called the hypotenuse) is equal to the sum of the areas of the squares with sides equal to the other two sides of the triangle. The proof of Pythagoras' theorem that appears in Euclid's *Elements* is rather lengthy and detailed. Since then, hundreds of different proofs have been discovered, some of them quite elegant, short, and simple. One example that falls in the latter category is illustrated in Figure 25.2. The right-angled triangle in question is the dark gray triangle with sides a, b, and c, which occurs four times in the left diagram. It is required to be proved that the area of the square with side a plus the area of the square with side b (both drawn in light gray on the right diagram) equals the area of the large white square with side c (on the left diagram). The two complete squares, on the left and right diagrams, both have sides of length $a + b$, and are therefore equal. Also, each complete square contains four copies of the dark gray right-angled triangle, arranged in different ways. Therefore, removing these four triangles from both complete squares leaves equal areas. The remaining area on the left is the white square of side c, and the remaining area on the right consists of the two light-gray squares of sides a and b, thus proving the theorem.

This proof of Pythagoras' theorem holds for any positive values of a, b, and c, as long as the given triangle has a right angle. For the special case $a = 3$, $b = 4$, and $c = 5$, a simple and

* An ancient Babylonian clay tablet, dating back to 1800 BC, known as Plimpton 322 at Columbia University, contains other such Pythagorean triples of numbers, leading some authors to suggest that the Pythagorean theorem may have been known well before Pythagoras. See Robson, E. (2001, 2002) and Polster, B. (2004), p. 4. However, knowledge of examples that satisfy a theorem does not by itself imply knowledge of the theorem; for that, a proof is required.

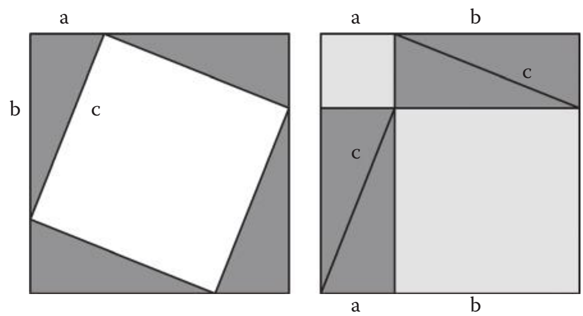


FIGURE 25.2 An elegant, simple, and short proof of Pythagoras' theorem.

original proof illustrated in Figure 25.3 may be found in the ancient Chinese classic book *Chou Pei Suan Ching*, probably written during the Han dynasty period dated between 500 BC and AD 200. Given a triangle with a 90° angle flanked by sides of lengths three and four, it must be proved that the area of the square with side length *eh* is $3^2 + 4^2 = 9 + 16 = 25$. First, four copies of the triangle (shaded in light gray) are embedded in a 7×7 square array consisting of 49 unit squares, as shown in the figure. This leaves out four identical triangles at the corners of the array, as well as one small dark gray square in the center. The area of the entire array is 49. Therefore, the area of the gray and white triangles is 48. But the area of the white triangles equals the area of the gray triangles. Therefore, the area of the gray triangles is 24. The area of the square with side *eh* is equal to the area of the four gray triangles plus the small unit square in the middle, and is thus 25, as was required to prove.

When students are asked how they would prove that the knotted-rope algorithm used by the ancient Egyptians to construct a right angle is correct, some are quick to answer: *apply the theorem of Pythagoras*. They are then surprised to discover that Pythagoras' theorem does not do the job. The theorem of Pythagoras proves that if we are given a right-angled triangle, then the square of the hypotenuse equals the sum of the squares of the other two sides. The theorem required to prove the correctness of the knotted-rope algorithm would have to state that if we are given a triangle in which the square of the hypotenuse equals the sum of the squares of the other two sides, then the angle opposite the hypotenuse must be a right angle. In other words, we require the *converse* of Pythagoras' theorem. It is the

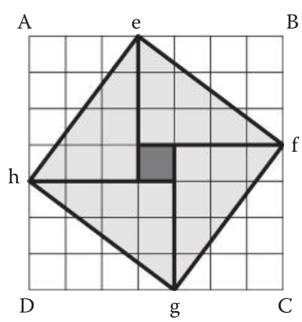


FIGURE 25.3 Proof of the Pythagoras theorem in the Chinese classic *Chou Pei Suan Ching*.

difference between the logical statements *if* and *only if*. The good news is that a proof of this converse theorem also appears in the *Elements* of Euclid. In fact, it appears just after the proof of Pythagoras' theorem.

Pythagoras was also a great music theorist, and laid some of the foundations of scales, chords, and tuning in music. The quotation *There is geometry in the humming of strings* is attributed to him. Pythagoras was referring to the pitch aspects of music in relation to the plucking of strings of different lengths. However, he could just as well have added that there is geometry in the drumming of drums. Pythagoras experimented with strings of different lengths that had the same tension and discovered that they sounded well together when the ratios of their lengths were related by small integers such as 1:2 and 2:3. In the ubiquitous diatonic scale used today, the major and minor three-note chords (triads) are considered to be the most stable and resonant because they are made up of the three most stable and resonant intervals: three, four, and five, the same integers that make up the smallest integers that satisfy the Pythagorean theorem. The C major and minor chords are shown as triangles in Figure 25.4. Interestingly, even though the major and minor chords have exactly the same intervals, the fact that the order of the intervals is reversed in going from one chord to the other makes the chords sound quite different from each other.

There are many ways in which 12 may be divided into three intervals, and the partition used by the major and minor chords is very special. The most even division would yield a triangle with three edges of the same length of four. Then, there are many partitions that would use two intervals of the same length, such as [3-3-6]. In addition, there are partitions that have very small and very large intervals such as [1-2-9]. The major chord on the other hand is the partition closest to [4-4-4] that has three distinct distances.

More than 2500 years after Pythagoras, in the twentieth century, another great and prolific mathematician, Paul Erdős, asked a very simple geometry question that also concerns distinct distances, musical rhythms, and scales, and that still has mathematicians baffled. Erdős asked if one can arrange n points in the plane with the restrictions that no three of the points should lie on a line, and no four of them may lie on a circle, so that for every value of $i = 1, 2, \ldots, n - 1$, there is a distance determined by pairs of these points that occurs

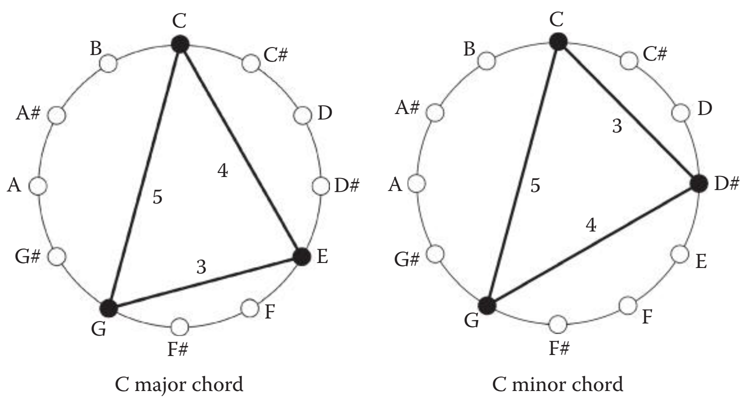


FIGURE 25.4 The C major and minor triangles with arc-lengths 3, 4, and 5.

exactly i times. In other words, each distance realized by pairs of points should occur a *unique* (distinct) number of times. We call such a set a *deep* set. The corner points of a rectangle of width a and height b does not yield unique distances because all three distances, a, b, and c, where c is the diagonal of the rectangle, occur the same number of times, namely twice. Furthermore, all four points lie on a circle, and thus violate one of the constraints. Therefore, the corners of a rectangle do not constitute a deep set.

You may ask yourself why Erdős disallowed the points to lie on a line or a circle. The answer is that the problem is too easy and uninteresting when these allowances are made. Consider, for example, the seven points on a line separated by unit distances, pictured in Figure 25.5. As the picture makes clear, distance six occurs once, distance five occurs twice, distance four three times, distance three four times, distance two five times, and distance one six times. In other words, no distance occurs the same number of times as any other distance. In other words, the multiplicity of each distance is unique.

A similar situation arises with points evenly spaced out on a circle. Consider the circle with 12 pulses and seven onsets at pulses zero through six, and refer to Figure 25.6. Again, each circular arc-length (geodesic distance) occurs a unique number of times. This remains true for any number of pulses and onsets as long as the onsets span at most one semicircle.

However, let us return for a moment to the original problem that Erdős posed for the two-dimensional plane. An example of a valid solution for $n = 4$ is illustrated in Figure 25.7. Two of the four points are located on the horizontal axis at coordinates 1 and –1, and the other two points are located on the vertical axis at coordinates –1 and $\sqrt{3}$. The histogram of the three distinct distances that occur between the points is shown on the right. The smallest distance $d_1 = \sqrt{2}$ occurs twice, the next largest distance $d_2 = 2$ occurs three times forming an equilateral triangle, and the largest distance $d_3 = 1 + \sqrt{3}$ occurs once between the two points on the vertical axis.

So far, the only solutions that have been found for this problem are for $n = 2, 3, 4, 5, 6, 7$, and 8. The solution for $n = 8$ illustrated in Figure 25.8 is quite elegant. The first diagram shows the arrangement of the eight points, along with all their pairwise distances. Since the eight points are located at the vertices of a triangular lattice, it helps to see the distances

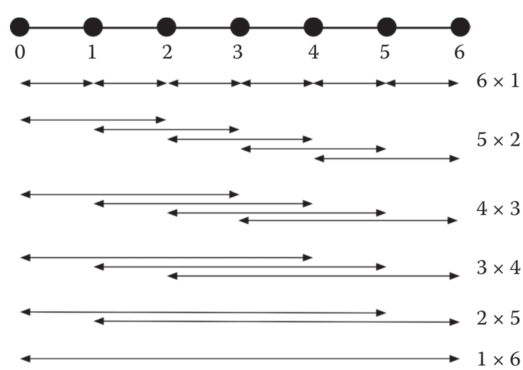


FIGURE 25.5 For evenly spaced out points on a line a unit distance apart, every distance occurs a unique number of times.

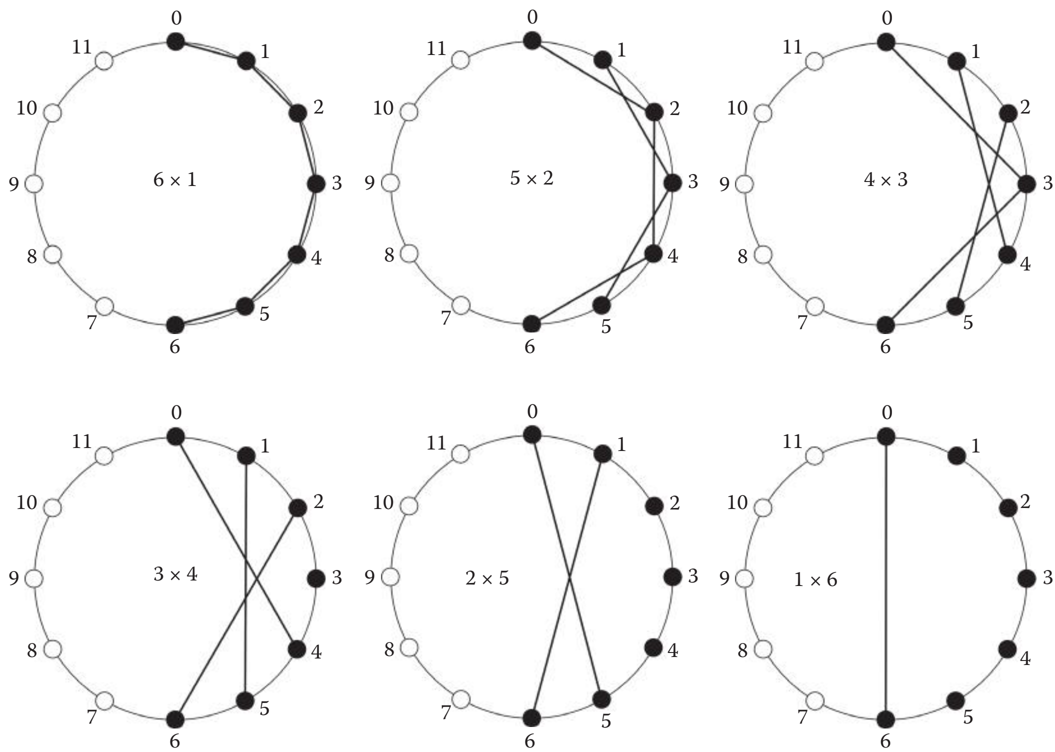


FIGURE 25.6 For evenly spaced out points in a semicircle, every distance occurs a unique number of times.

embedded in the triangulation. The remaining seven diagrams show each of the seven distinct distances by themselves in order of increasing multiplicity. The distance equal to one side of a triangle occurs once. The distance equal to two sides of a triangle occurs five times. The distance equal to two heights of a triangle occurs four times. For the remaining distances, it helps to view every pair of adjacent triangles as a lozenge, and combine adjacent lozenges into parallelograms. Then, the distance corresponding to the long diagonal

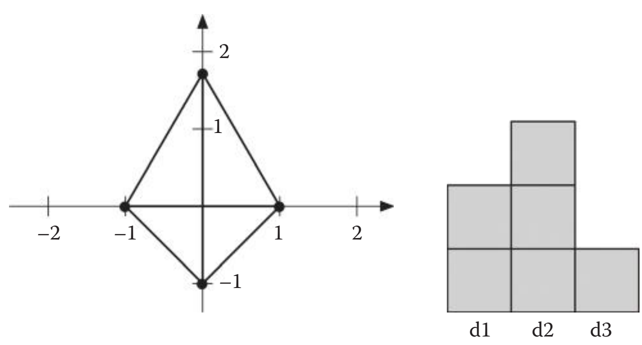


FIGURE 25.7 A set of four points with distinct distance multiplicities (left) and their histogram (right).

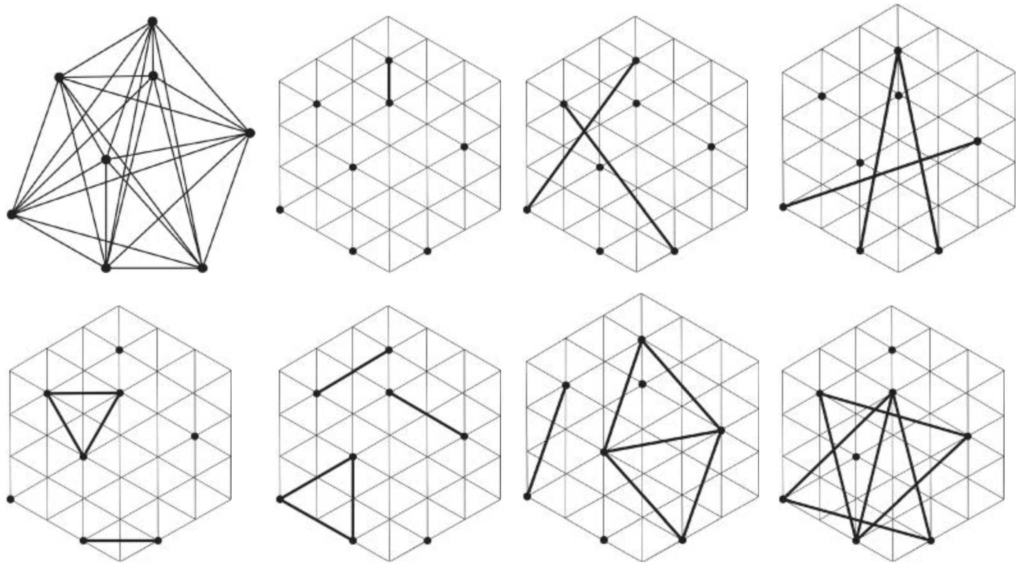


FIGURE 25.8 A *deep* set of eight points in the plane.

of a 1 × 2 parallelogram occurs six times. The distance equal to the long diagonal of a 1 × 3 parallelogram occurs seven times. The distance equal to the long diagonal of a 1 × 4 parallelogram occurs three times. Finally, the distance equal to the long diagonal of a 2 × 3 parallelogram occurs twice.

As already pointed out, one reason why the problem Erdős posed is so difficult is that the points must lie in the plane: a two-dimensional universe. However, if the problem is considered in a one-dimensional universe, the circle, then the problem is more tractable, even for the case when the points are not restricted to lie in a semicircle. Furthermore, it is precisely in this form that the problem is most relevant to musical rhythms and scales.

The simplest deep rhythms are those with three onsets—triangles. Geometrically, they are represented by *isosceles* triangles, that is, those that have two sides equal and one side different from the other two. For example, for three onsets and 12 pulses, there are four possible such rhythms with the apex of the isosceles triangle positioned at pulse zero, as illustrated in Figure 25.9. From left to right, these rhythms have durations [1-2-1], [2-4-2],

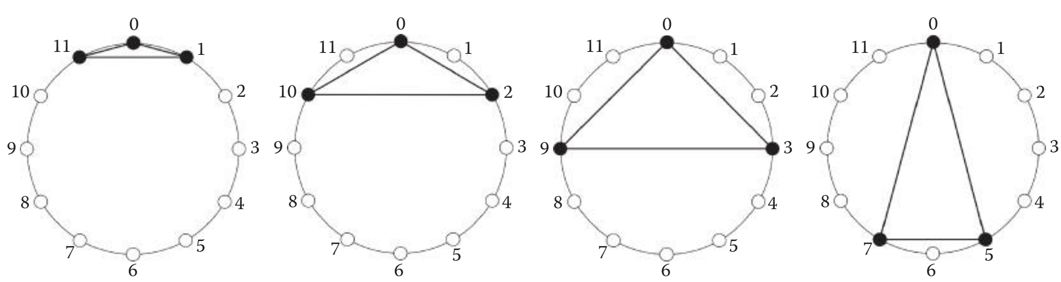


FIGURE 25.9 The four deep rhythms with three onsets and 12 pulses anchored at pulse zero.

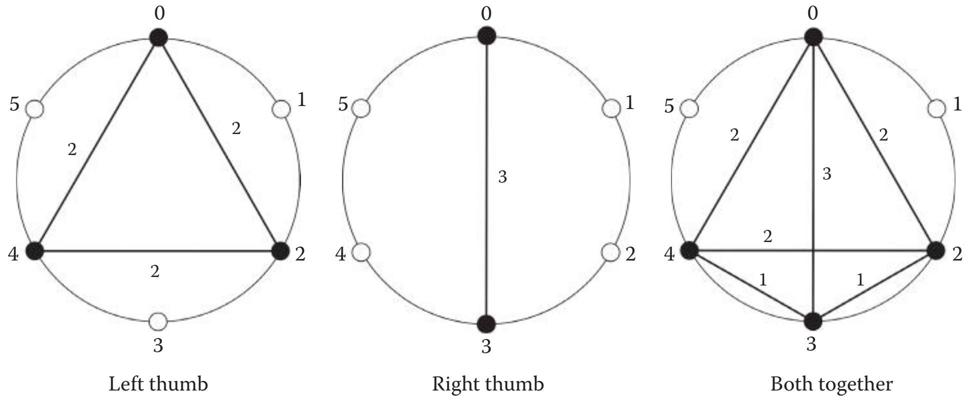


FIGURE 25.10 The *nyunga-nyunga mbira* rhythm of the Shona people of East Africa.

[3-6-3], and [5-2-5]. Since this vertex may be anchored on any of the 12 pulses, the total number of such rhythms is 48. For more than three onsets, deep rhythms are less straightforward to generate.

Deep rhythms with four onsets contain six distances. One method with which to realize a multiset of six distances is to have three distinct distances: one that occurs once, one twice, and a third three times. An easy way to obtain three equal distances is by means of an equilateral triangle. The smallest example with an equilateral triangle that contains more than three pulses is the three-in-six rhythm shown in Figure 25.10 (right). It is the rhythm called *nyunga-nyunga* played on the *mbira* (also called the *sanza*, *kalimba*, and *thumb-piano*) by the Shona people of East Africa. The thumb-piano illustrated in Figure 25.11 is an instrument consisting of a resonating chamber made from one-half of a calabash (or gourd) shell covered with a slab of wood that contains a hole, on which is mounted a series of long metal blades that are plucked with the thumbs. The rhythm in Figure 25.10 is the composition of two regular rhythms, [2-2-2] played with the left thumb (left), and [3-3]

FIGURE 25.11 The *thumb-piano*. (Courtesy of Yang Liu.)

played with the right thumb (center). When the two are combined (right), a deep rhythm is obtained, with distance two occurring three times, distance one twice, and distance three once. Note that by convention, if a distance spans the diameter of the circle, it is considered to occur once rather than twice. This rhythm is also a Korean shaman timeline played on a *tsching* (a Korean gong), a rhythm of the Cuban "canto de clave,"* and the rhythm of the *zarabanda*.† When the rhythm is started on the fourth onset to obtain the durational pattern [2-2-1-1], it is the fundamental rhythmic pattern of the Colombian *bambuco*.‡

Another unforgettable deep rhythm with four onsets is George Gershwin's rhythmic riff in *I Got Rhythm*, given by [. . x . . x . . x . . x . . . .].§ Here, distance seven occurs once, distance six, twice, and distance three, three times. But perhaps the most illustrious rhythm that contains the *deepness* property is the pattern of seven onsets among 12 pulses that defines the bembé rhythm shown in Figure 25.12 (left), along with all the distances between every pair of onsets indicated. As the histogram on the right clearly shows, every distance ranging from one to six occurs a distinct number of times. As already pointed out, this pattern is identical to the diatonic scale pattern in the pitch domain. Indeed, it was in this domain that in 1966 Terry Winograd and in 1967 Carlton Gamer studied scales with this property and christened them *deep* scales.¶ It is for this reason that rhythms possessing this property are here called *deep rhythms*.

Another well-known example of a deep rhythm is the five-onset, 12-pulse *fume-fume* timeline (also the pentatonic scale) shown alongside its interval-content histogram in Figure 25.13.

There is a simple algorithm that will generate deep rhythms that are more interesting than those that have all their onsets in one semicircle. This algorithm will generate different families of deep rhythms that depend on a parameter *d*, and is illustrated in Figure 25.14

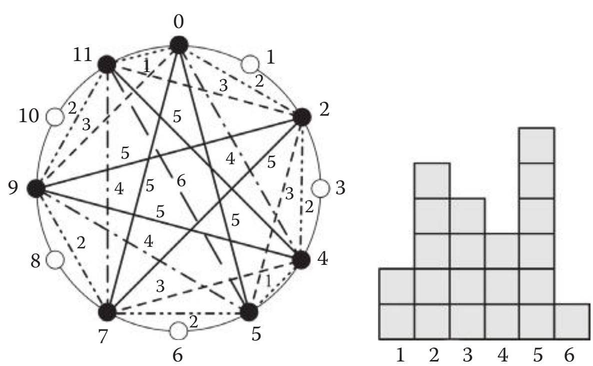


FIGURE 25.12 The *bembé* deep rhythm and its interval-content histogram.

* Orovio, H. (1992), p. 111.

† Van der Lee, P. (1995), p. 202. In this *zarabanda* rhythm, the second and fourth onsets are played with a lower pitch to obtain [x . **x** x **x** .].

‡ Varney, J. (2001), p. 140. In this *bambuco* rhythm, the first and third onsets are played with a lower pitch to obtain [**x** . x . **x** x]. See also the PhD thesis: Varney, J. (1999).

§ Crawford, R. (2004), p. 163. See also its relation to acid jazz discussed in Chapter 15.

¶ Johnson, T. A. (2003), p. 41.

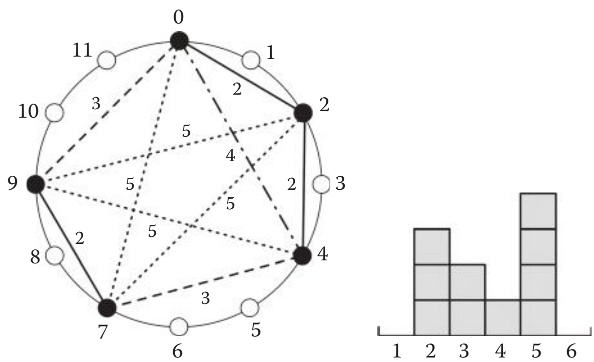


FIGURE 25.13 The *fume-fume* deep rhythm and its interval-content histogram.

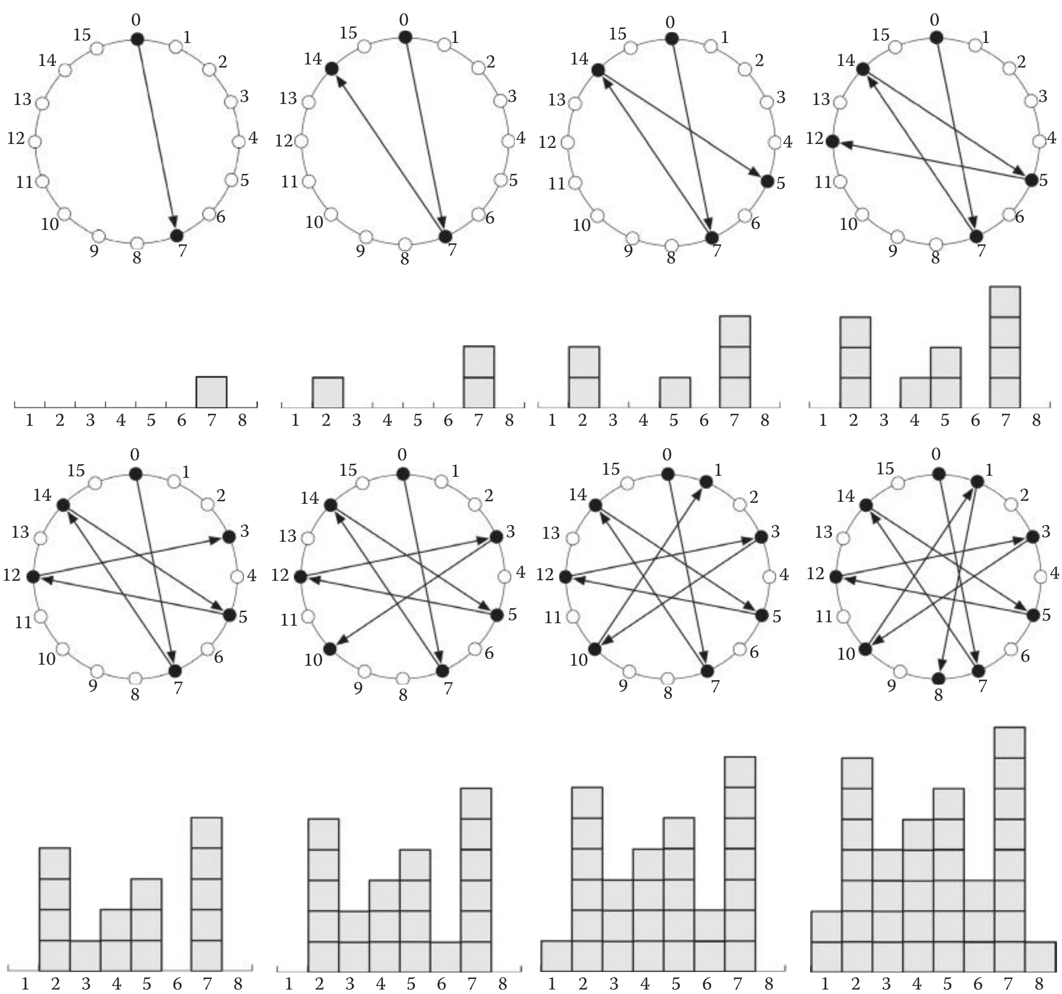


FIGURE 25.14 Illustrating the algorithm for generating *deep* rhythms with $d = 7$ and $n = 16$.

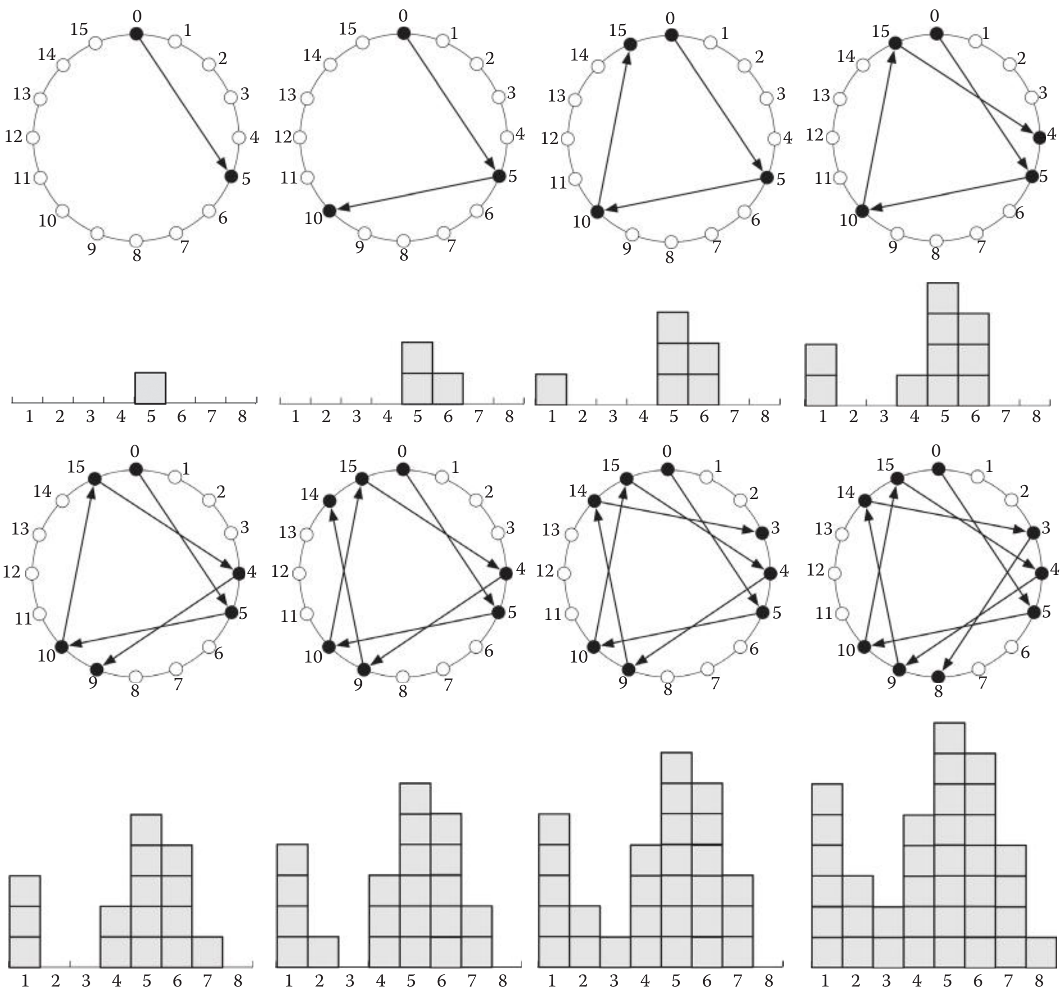


FIGURE 25.15 Illustrating the algorithm for generating deep rhythms with $d = 5$ and $n = 16$.

for a cycle with 16 pulses.* In this case, the algorithm will generate deep rhythms with any number of onsets between two and nine. The algorithm places the first onset at pulse zero. To place the remaining onsets, the pulses in the cycle are scanned in a clockwise order, and a visited pulse is selected as an onset every time a distance d has been advanced. The value of d can be any integer that is relatively prime to the number n of pulses in the cycle, that is, n and d have no common divisors other than one. Therefore, for $n = 16$, d may take on the values three, five, and seven. The example in Figure 25.14 uses $d = 7$. Therefore, the remaining eight onsets are selected in order at pulses 7, 14, 5, 12, 3, 10, 1, and 8. Below each rhythm is shown its histogram of the inter-onset distances, and it can be seen that after every insertion of a new onset, the new rhythm remains *deep*. Also worth noting is that when d is relatively prime to n and d is small enough, a new attack is never inserted in a

* The algorithm described here is used in the pitch domain to generate scales. There exist a variety of similar algorithms that generate different families of scales that may be applied to the time domain as well. See Clough, J., Engebretsen, N., and Kochavi, J. (1999) and Carey, N. and Clampitt, D. (1996, 1989) for further details and references.

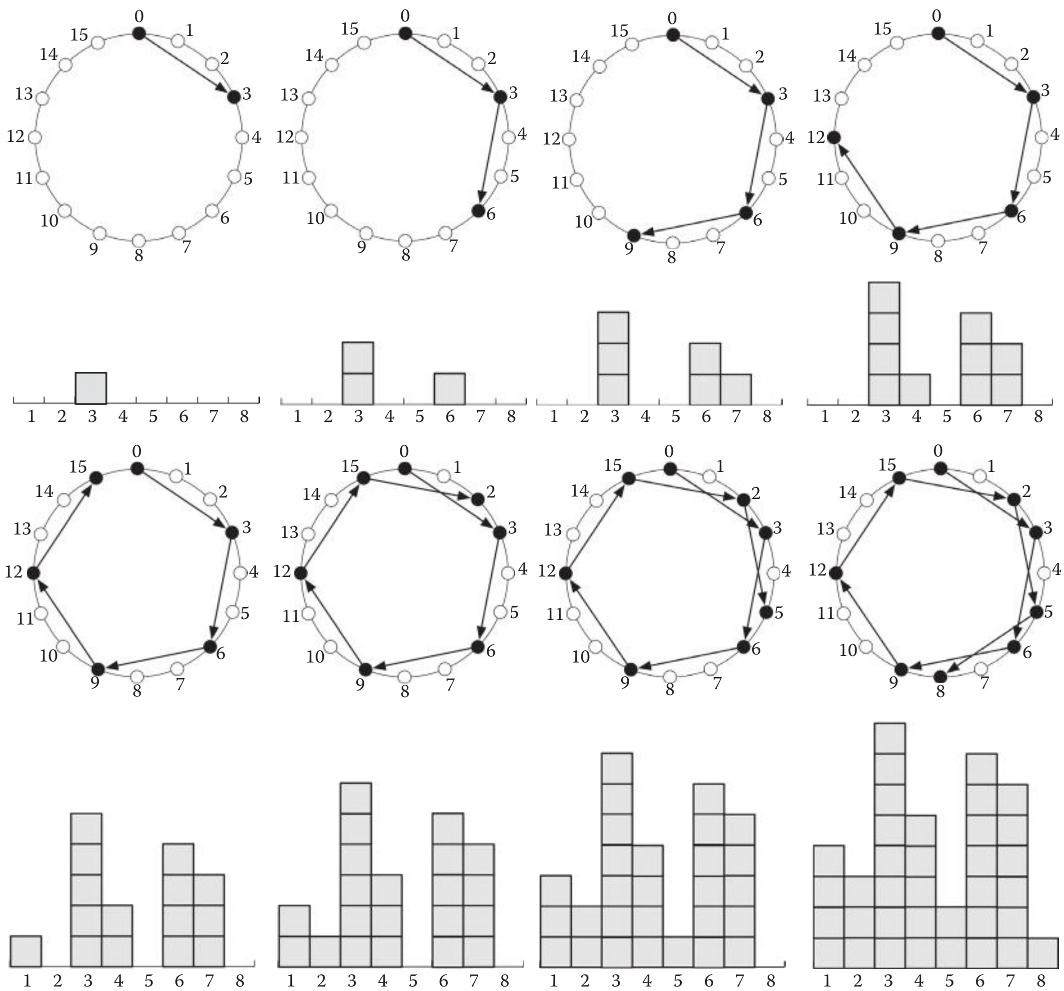


FIGURE 25.16 Illustrating the algorithm for generating deep rhythms with $d = 3$ and $n = 16$.

location diametrically opposite to an already-existing attack, thus preserving the property of rhythmic oddity. If the *Hop-and-Jump* algorithm of Chapter 15 is stopped when seven attacks are obtained, then the rhythm necklace generated is the same as the seven-attack rhythm necklace generated with the procedure described here.

For $d = 5$, the algorithm generates the deep rhythms and histograms shown in Figure 25.15. Note that the two families of rhythms are quite different from each other. For $d = 7$, the rhythms generated quickly become maximally even, whereas for $d = 5$, the onsets tend to group together into three separate clusters, making them *minimally* even. This shows that deepness and maximal evenness are properties of a very different nature.

Figure 25.16 shows the deep rhythms and histograms generated by the algorithm when $d = 3$. This family of rhythms is also quite different from the other two families. Whereas for $d = 5$, the onsets group together into three separate clusters of three rhythms each, making them minimally even, when $d = 3$, the onsets cluster into four smaller groups of two rhythms each.

CHAPTER 26

Shelling Rhythms

In 1956, Elvis Presley released his own faster and more syncopated version of a slow blues song recorded 4 years earlier by Willie Mae "Big Mama" Thornton. For both Thornton and Presley, this song, titled *Hound Dog*, rose to the top of the charts. In Elvis Presley's rendition, the rhythmic patterns produced by the snare drum, the bass, and the handclaps are typical traditional African rhythms, a common thread that meanders through the rockabilly music of the 1950s.* Less well known is the fact that these rhythms have two interesting mathematical properties: they are all *deep*, and three of them exhibit what is called the *shelling* property with respect to the *deepness* property. The four rhythms that permeate the song are shown in Figure 26.1 along with their inter-onset interval histograms directly underneath each of them. From these histograms, it is clear that the first three rhythms are deep: they have the property that each interval in the rhythm appears a unique number of times.† The bass pattern on the left is the well-known Cuban *tresillo* pattern. In "Big Mama" Thornton's version, the less syncopated duration pattern of the bass is [x . . . x . x .]. The clap pattern in Elvis' version (second from left), known as the *habanera* rhythm,‡ is obtained by adding an onset at pulse position five (here adding refers to replacing a silent pulse with a sounded one). In "Big Mama" Thornton's version, the unsyncopated clap pattern is a straight pulsation given by [. . x . . . x .]. Elvis' snare-drum pattern, on the right is the signature "backbeat"§ pattern of "rock-n-roll" music, and plays the role of the clap in "Big Mama" Thornton's version. If the snare-drum onset at pulse two

* It is ironic that a non-African-American such as Elvis Presley should modify the music of the African-American blues singers by incorporating more syncopated African rhythms into their blues songs. Elvis appropriated several other songs from Black Americans that went on to become hits, such as *Mystery Train*, by Little Junior Parker, and *That's All Right*, by Arthur "Big Boy" Crudup. See Fryer, P. (1998), p. 2.

† In the case of a two-onset rhythm, such as the snare drum backbeat in the rightmost diagram of Figure 26.1, with onsets on pulses two and six, the duration interval of four units may be counted once or twice depending on the context. In this case, given two onsets *a* and *b* that determine two durations [*ab*] and [*ba*], we take the smallest of the two as the single duration determined by the pair of onsets. Thus, if [*ab*] = [*ba*], as in the snare drum example, we count just one of the intervals.

‡ Rey, M. (2006), p. 192.

§ Wade, B. C. (2004), p. 61.

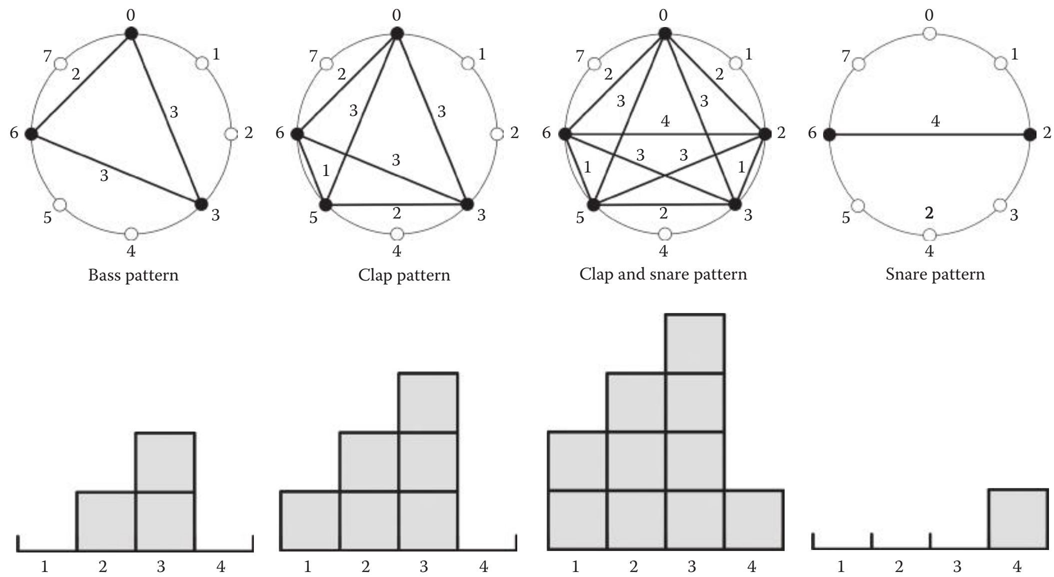


FIGURE 26.1 The rhythms and interval histograms used in Elvis Presley's *Hound Dog*.

is added to the clapping pattern, then the well-known *cinquillo* pattern (second from right) is obtained.[*]

A rhythm admits a *shelling*, with respect to some property *P* if there exists a sequence of insertions or deletions of onsets so that, after each insertion or deletion, the rhythm thus obtained continues to have property *P*. Again, here and throughout this chapter, the terms "insertions" (or deletions) refer to replacements of a silent (or sounded) pulse with a sounded (or silent) pulse. In the present case, starting with the bass pattern, onsets may be inserted in positions five and two, in this order, while maintaining the deepness property of the resulting rhythms. Similarly, starting with the cinquillo rhythm and deleting the onsets in the reverse order (two before five), preserves the deepness property. In fact, since, by convention, a pair of onsets defines just one duration interval corresponding to the shortest circular arc determined by the pair, another onset may be deleted from the bass pattern on the left, while still maintaining the deepness property, albeit at its most elemental level. Deleting the third onset at pulse six yields the Cuban timeline [x. . x. . . .] called the *conga*,[†] which at a faster pace becomes the *charleston* rhythm.[‡] Recall that this rhythm, probably the simplest, minimalist, yet effective, electric guitar solo ever used in a rock-n-roll megahit, is played in the middle section of the 1964 song *Don't Worry Baby*, by the Beach Boys. Their solo ends by inserting the third onset, thus converting it to the tresillo. Deleting the first onset of the tresillo yields [. . . x. . x.], also a Cuban rhythm called the conga.[§]

[*] According to Brewer, R. (1999), p. 307 (Example 10), the clapping pattern in Elvis Presley's *Hound Dog*, is the five-onset *cinquillo*. However, in the recording, the clap on pulse two cannot be heard as it is drowned out by the attack on the snare drum.

[†] Rey, *op. cit.*, p. 192.

[‡] Middleton, R. (1983), p. 252.

[§] Mauleón, R. (1997), p. 17.

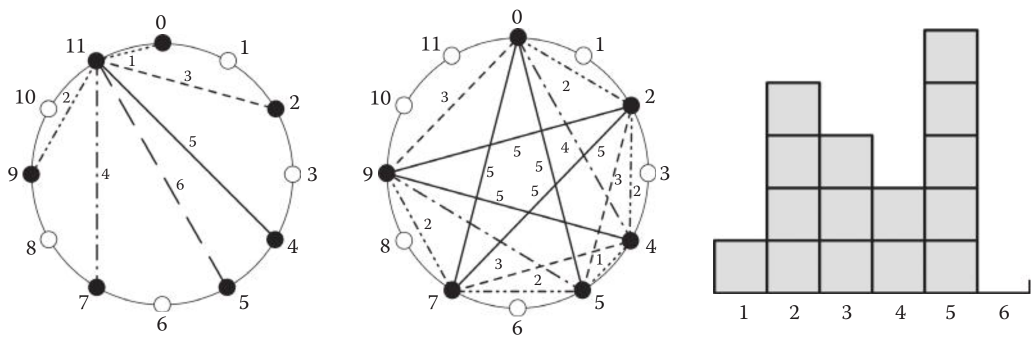


FIGURE 26.2 Deleting the onset at pulse 11 (left) from the *bembé* rhythm yields a Yoruba bell timeline (center) with a deep histogram (right).

The 12-pulse ternary timelines used in much sub-Saharan African music provide another example of a family of rhythms that admit shelling with respect to the deepness property. Consider the *bembé* rhythm and its interval-content histogram shown in Figure 26.2. There are seven different onsets each of which may be deleted to obtain a six-onset rhythm. Each of the seven onsets has a spectrum of six distances determined by the other six onsets. To ensure that the rhythm remains deep when an onset is deleted, the onset to be deleted must contain a spectrum with exactly one instance of every distance present in the histogram of the rhythm before the onset is deleted. In that way, the height of each column of the histogram will be reduced by one unit, thus preserving the deepness property. The spectrum for the onset at pulse 11 is pictured in Figure 26.2 (left). Note that only the onsets at pulses 5 and 11 have spectra that contain all the distances, and therefore only one of these two onsets may be deleted. If we remove the onset at pulse 11, we obtain the rhythm shown in the middle with its resulting histogram on the right. This rhythm is also a bell timeline used by the Yoruba people of Nigeria. Of the six onsets, only the onset at pulse five has all five distances in its spectrum. Therefore, only this onset may be deleted next if deepness is to be preserved, and doing so results in the fume-fume rhythm.

Another characteristic property of rhythms is the *all-interval property*. A rhythm with this property contains all the possible duration intervals, and we may ask whether a shelling exists that maintains this property. In addition, if a rhythm admits the deletion of some onsets that preserves the all-interval property, does it admit a sequence of deletions that will yield a *minimum* all-interval rhythm? A rhythm is a minimum all-interval rhythm if it has the minimum number of onsets such that deleting any of its onsets results in a rhythm without the all-interval property. Figure 26.3 contains two minimum all-interval rhythms in a 12-pulse time span.

To illustrate the process of all-interval shelling, consider the bembé rhythm (isomorphic to the diatonic scale in the pitch domain). Since it is a deep rhythm, it obviously also has the all-interval property. Can this rhythm be shelled to the two minimum all-interval bracelets given by durational patterns [1-3-2-6] and [1-2-4-5]? That it can for one and not for the other is illustrated in Figure 26.3, which shows the seven initial onsets of the bembé with black-filled pulses. Since we must retain the interval of duration one, we must keep

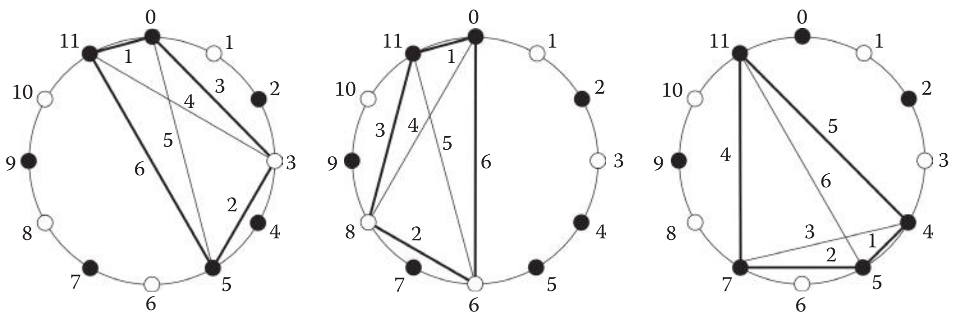


FIGURE 26.3 The two *minimum* all-interval rhythm bracelets.

either the interval between pulses 0 and 11 or pulses four and five. Because the rhythm is symmetric about the line through pulses two and eight, we need to only be concerned with one of these choices. Therefore, let us keep the onsets at pulses 0 and 11. Since we need an interval of duration six, the onset at pulse five must be kept. Thus, the only remaining choice for the fourth onset is at pulse three, which is invalid since it is not contained in the bembé.

The only other possible choice for this bracelet, which keeps pulses 0 and 11, is shown in the middle diagram, but here, onsets are needed at pulses six and eight, neither of which is contained in the bembé. Therefore, this minimum all-interval bracelet is not reachable from the bembé via deletion shelling operations.*

Consider then the second all-interval bracelet contained in the bembé shown in the right diagram of Figure 26.3. This rhythm is a shelling descendant of the bembé, and it may be reached via several shelling sequences. One sequence is as follows. First, the onsets at pulses zero or four may be deleted. Then, the onset at pulse two may be dropped, followed by the onset at pulse nine. Another sequence first deletes the onset at pulse two, and then either zero followed by nine or four followed by seven.

The two minimum all-interval subsets of the bembé rhythm shown in Figure 26.3 are *perfect* in the sense that each inter-onset interval is used exactly once. However, there may exist shelling sequences that lead to minimum all-interval descendants that are not perfect, such as the two shown in Figure 26.4, where the onsets that may be deleted are shaded gray. A symmetric pentagon (left) is obtained by deleting the onsets at pulses seven and nine. That this is a minimum may be ascertained by inspection. If either onset 11 or 5 is dropped, then interval six is lost. If either onset zero or four is deleted, interval four is lost, and if the onset at pulse two is removed, interval three is no longer present. An asymmetric pentagon (right) is obtained when the onsets at zero and seven are dropped. It may also be verified by inspection that this rhythm is all-interval minimal.

* Music theorists may find that this example follows trivially from the generative theory of diatonic scales. A generated scale may be constructed by repeatedly adding a constant integer around the pitch circle. For further details, see Gamer, C. (1967) and Johnson, T. A. (2003), p. 83. However, this book is not aimed primarily at music theorists, and it is hoped that the general idea expounded in this chapter, of shelling with respect to *any* property (either in the pitch or time domain), will suggest new avenues for research in music theory.

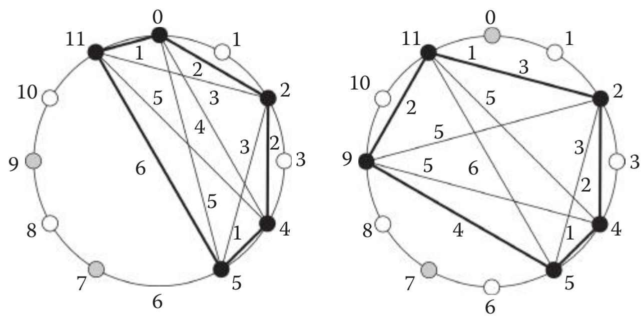


FIGURE 26.4 Two minimum all-interval shelling descendants of the *bembé*.

The process of shelling rhythms is a natural technique that is routinely applied by drummers during their solo improvisations, and it can also be used as a composition tool. Indeed, the minimalist composer Steve Reich applied this technique, which he called *rhythmic construction*, for the first time in his 1973 piece titled *Drumming*, and described it as "the process of gradually substituting beats for rests (or rests for beats)."* From the empirical examination of preferred rhythms used in several cultural traditions,† it follows that deepness is a mathematical property that appears to reflect this cultural preference. Therefore, the process of shelling rhythms while maintaining this property provides an algorithm that may find application to the automatic generation of "good" rhythms. Whether deepness by itself is a property that has a psychologically robust reality has yet to be determined. Reich, however, did not preserve the deepness property in the shelling employed in *Drumming*. The first and last eight measures of this 75 min composition that consists of 100 measures are shown using polygon notation in Figure 26.5. The first eight are labeled (1) through (8) and the last eight (93) through (100). The polygon notation makes it easy to discover the structural rhythmic properties utilized in both the *construction* and *reduction* phases at the beginning and ending of the piece. The first observation is that, apart from the obvious fact that the ending is not the same as the beginning played backwards, the rhythms with three, four, and five onsets (triangles, quadrilaterals, and pentagons) all lie in a half-cycle (semicircle) in the construction phase, but do not in the reduction phase. The rhythms with five and six onsets exhibit mirror symmetry (about the line through pulses 10 and 4) in the construction phase (measures five and six), but not in the reduction phase (measures 95 and 96). Nevertheless, measure 95 does have 180° rotational symmetry about the same line through pulses 4 and 10. Furthermore, the rhythms in measures 6 and 95 have an interesting relationship to each other. If the rhythm in measure six is decomposed into two half-cycles by cutting it along the line through pulses 4 and 10, then reversing the order of the lower chain of inter-onset intervals converts the rhythm of measure six to that of measure 95. Referring to Figure 26.6,

* S. Reich, S. (2002), p. 64.
† See the list of Euclidean rhythms in Toussaint, G. T. (2005c).

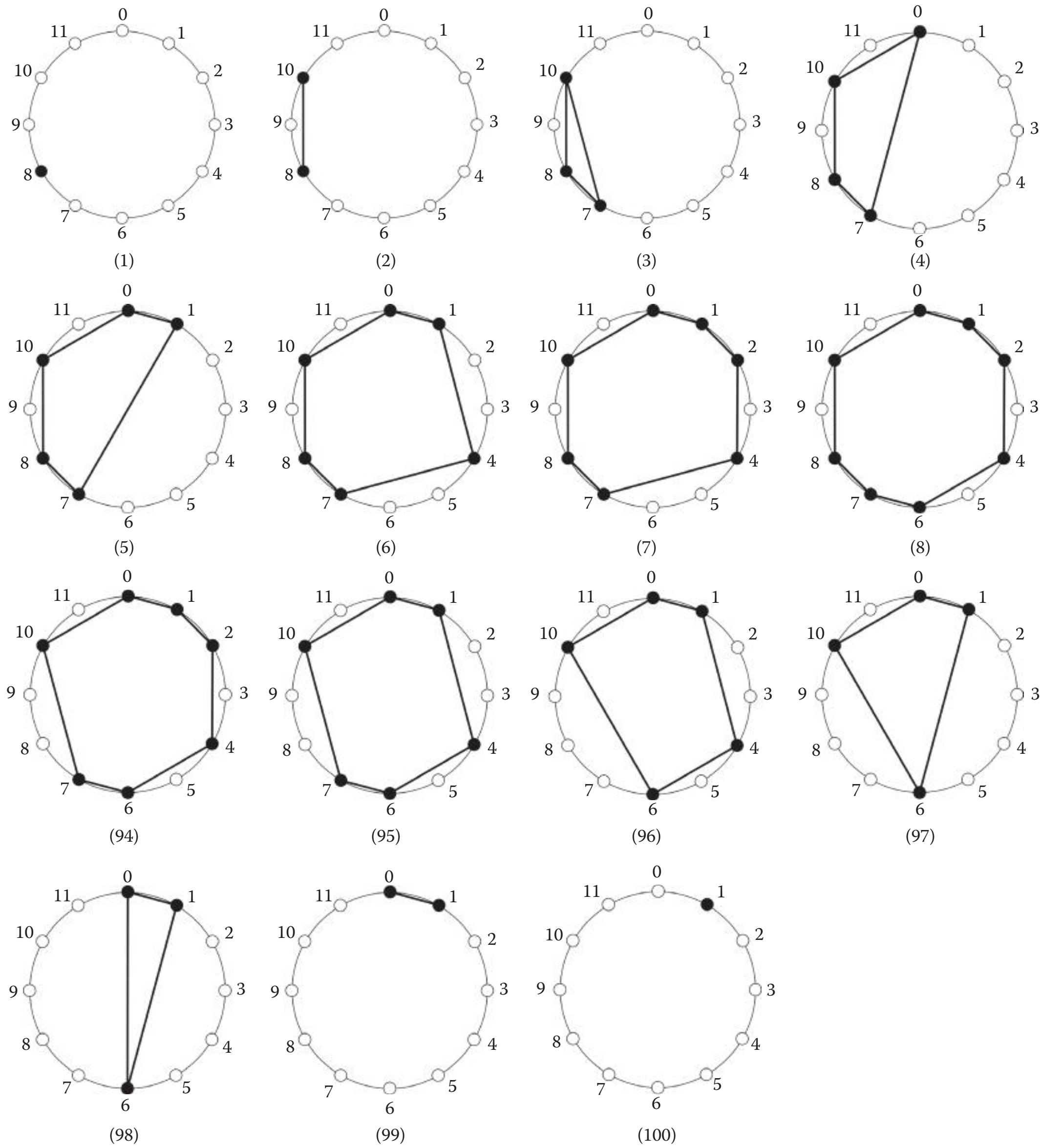


FIGURE 26.5 The *construction* and *reduction* method employed in Steve Reich's *Drumming*.

this is equivalent to performing a mirror symmetry operation of the shaded half-cycle about the line through pulses one and seven.*

Also noteworthy is the rhythm used in measure 97, which has a perfectly flat inter-onset interval histogram: it contains exactly one instance of every interval ranging from 1 to 6 (this is the infamous all-interval tetrachord in pitch theory). Finally, it is worth pointing out that the rhythm used in measure six is the union of two regular rhythms:

* In the computational geometry literature, such a restructuring operation performed on a polygon is called a *flip-turn*. See Toussaint, G. T. (2005e).

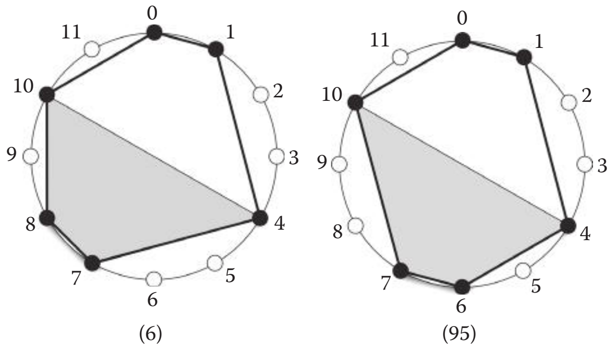


FIGURE 26.6 Transforming the rhythm of measure six to that of measure 95 by flipping over the shaded half-cycle using mirror symmetry about the line through pulses one and seven.

one with three attacks at pulses zero, four, and eight, and one with four attacks at pulses 1, 4, 7, and 10. This combination of rhythms is used frequently in the music of sub-Saharan Africa.

The geometric analysis of the shelling technique that Steve Reich used in the construction and reduction phases of the beginning and ending of his piece *Drumming* reveals that the main property that is preserved during this process may be characterized by the notion of symmetry. This symmetry is sometimes perfect mirror symmetry as in measures 5, 6, 8, and 93, sometimes rotational symmetry as in measure 95, and at other times near-mirror symmetry.* Thus, we may characterize Reich's construction and reduction techniques used as being dominated by *symmetry-preserving rhythmic shelling.*

A convenient and useful representation of shelling rhythms is via graphs, in which the nodes (vertices) of the graphs represent rhythms, and pairs of nodes are connected with edges provided that their corresponding rhythms share some specified property. As a simple example, consider all the 16 rhythms consisting of four pulses.† If an edge is connected between every pair of nodes corresponding to pairs of rhythms that have the property that they differ from each other by only one attack, then the graph obtained is the hypercube shown in Figure 26.7. In this figure, all the rhythms that have the property of deepness are shaded black for the three-attack deep rhythms, and gray for the elementary (trivially deep) two-attack rhythms. Furthermore, the edges connecting pairs of deep rhythms are highlighted in bold lines. Thus, the shaded vertices and bold edges determine the *deep-rhythm-subgraph* of the hypercube. Note that this subgraph is *connected*. This means that with shelling, we can traverse from any one deep rhythm to any other deep rhythm by a sequence of insertions or deletions of attacks, while preserving the deepness property of all rhythms traversed along the way.

* Traditionally, mirror symmetry, or symmetry in general, is viewed in a binary manner: an object has or does not have symmetry. However, measures exist for determining the degree of symmetry possessed by an object. For further details, see Zabrodsky et al. (1992) and Eades, P. (1988). The concept of near-symmetry is also discussed in the context of voice-leading in Tymoczko, D. (2011).

† The ideas described apply to rhythms with any number of pulses but the drawings for more than four pulses become rather cumbersome.

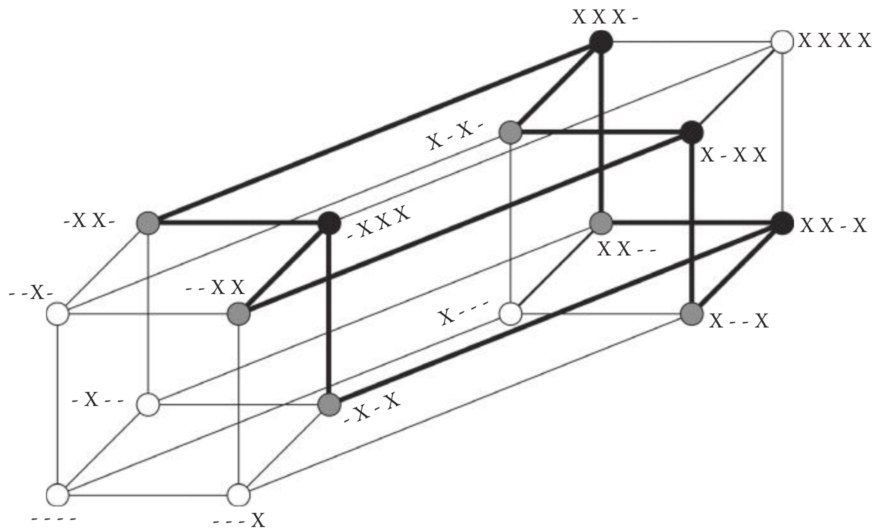


FIGURE 26.7 The hypercube representation of all four-pulse rhythms that differ by at most one attack. Deep rhythms are shaded and connected with bold edges.

Just as we constructed a graph for the deepness property shown in Figure 26.7, a similar subgraph of the hypercube may be constructed for any other property of a rhythm, by connecting the two vertices of the corresponding rhythms provided they share that property. The representation of rhythms using graphs, as described here, was inspired by similar ideas that were previously explored by music theorists working in the pitch domain.* In 1964, Allen Forte defined what he called a *set-complex* $K(T)$ for chords. For any chord T, another chord S is a member of the set-complex $K(T)$ provided that S is a subset or superset of either T or its complement. Furthermore, S must have a number of elements different from T or its complement, and the number k of elements of both S and T is bounded by $3 \leq k \leq 9$.† With such a loose definition, Forte discovered that these set-complexes were too large to be useful, and subsequently refined his definition to a subcomplex that he called $Kh(T)$. To belong to $Kh(T)$, a chord must be included in *both* T and its complement. In 1974, Eric Regener went further in the direction outlined above. First, he defined the general inclusion graph (lattice) of all pitch class sets. Every possible chord corresponds to a node in the graph. Two nodes, corresponding to chords T and S in this graph, are connected with an edge, if T contains S, and T has exactly one more element than S.‡

* I am indebted to Dmitri Tymoczko for bringing this connection to my attention.

† Forte, A. (1964).

‡ Regener, R. (1974), pp. 208–209.

CHAPTER 27

Phantom Rhythms

PICTURE YOURSELF STANDING IN FRONT OF a large kettle drum holding a mallet, and playing a three-onset, four-pulse rhythm with adjacent inter-onset durational pattern [2-2-4]. While playing this rhythm, observe the motion of your mallet as it travels through space. In particular, focus on the distance between your hand and the drum, as a function of time. Chances are that this distance function looks something like the curve shown in Figure 27.1, which illustrates two cycles of the rhythm, and shows the time points (pulses) at which your mallet strikes the instrument, namely zero, two, four, zero, two, four, and zero. The numbering here repeats every eight pulses because the period of this cyclic rhythm consists of eight pulses.

We have all heard the expression *what goes up must come down*. A natural corollary to this expression is *what goes up and comes down must reach a point of maximum height somewhere in between the going up and the coming down.** The points in time at which your arm achieves this maximum height are likely to be the *midpoints* of the inter-onset intervals, indicated in Figure 27.1 by vertical lines at the pulses numbered one, three, six, one, three, and six. These midpoints themselves may be interpreted as determining another (silent) rhythm lurking in the subconscious mind, a phantom of the rhythm actually sounded. If we represent the original rhythm [2-2-4] on a circle, as in Figure 27.2 (upper left), then the resulting rhythm of silent beats becomes the rhythm pictured

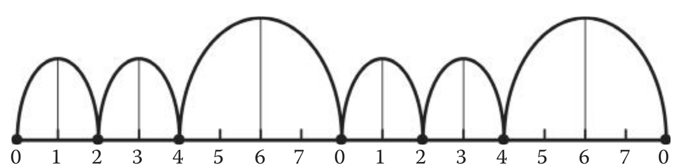


FIGURE 27.1 The vertical motion of a hand, arm, or mallet as a function of time.

* In the real Newtonian world, this happens with footballs and baseballs if they do not go into outer space. In the mathematical world, where almost anything can happen, we have to add of course the assumption of function continuity of some sort.

in Figure 27.2 (upper right), and is sometimes called the *shadow* of the rhythm [2-2-4].[*] Although the motor image here is not aligned with the sounding image,[†] the muscles of the arm change their function in a significant way at these midpoints in time, and thus it is reasonable to hypothesize that the motor system of the brain must register these distinctive moments, either consciously or subliminally.[‡] For this reason, some musicologists such as Gerhard Kubik believe that these shadow rhythms are physiologically and psychologically relevant to the proper study and understanding of rhythm cognition and performance, going as far as to claim that motion and motor action are essential to the explanation of rhythm.[§]

Shadow rhythms should not be confused with *subjective* rhythms. Subjective rhythms are those that are *perceived* by the listener, but not actually produced acoustically. That is, a subjective rhythm is an aural illusion that exists in the mind of the perceiver, but cannot be measured with scientific equipment in the outside world, since it lacks a concomitant acoustic signal. In other words, a subjective rhythm cannot be detected with electronic measuring devices in a recording of the sounded rhythm. A shadow rhythm on the other hand, although consisting of silent beats, is embedded in the acoustic signal of its progenitor, and can be detected in a sound recording of its progenitor by calculating the midpoints of the progenitor's adjacent inter-onset intervals. This is not to suggest that a subjective rhythm cannot be measured with scientific equipment that is able to read brain activity, such as fMRI or MEG. Indeed, such techniques have revealed that a silent beat, that is, a nonexistent acoustic input, can trigger a brain response if a beat was expected to occur at a specified point in time, but failed to materialize.[¶] Of course, the motor action rhythm produced by the points of maximum height of the arm also lacks an acoustic

[*] The term "shadow rhythm" as utilized here is adopted from the work of Jay Rahn (1996), p. 82, and should be distinguished from the term "shadow metre" employed by Eytan Agmon, who uses the term in the context of a conflict between two metrical divisions of equal duration. Agmon calls the primary and secondary divisions, the metre and shadow metre, respectively, and no midpoint relation between the two metres is implied. Agmon, E. (1997), p. 64. According to Agmon, the term "shadow metre" was coined by Frank Samarotto, *Ibid*., p. 74. See also London, J. (2004), p. 81. It is also at variance with the term "shadow" of a rhythm as used by Reinhard Flatischler (1992), p. 118, who uses the term to mean the *complement* of a rhythm, composed of the unused pulses of the original rhythm.

[†] Scherzinger, M. (2010) refers to those rhythms where the sounding image is aligned with the motor image, as *kinesthetic patterns*.

[‡] It would be interesting to try to determine using an MEG whether there exist any neural correlates of these points in time.

[§] For example, in his comparative analysis of sub-Saharan African rhythm with European rhythm, Hornbostel (1928), p. 52, particularly in the context of striking drums or balafons, with sticks or mallets, proposes that whereas the European analyzes rhythm via hearing, the African generates rhythm by means of motion. This is probably a caricature with little scientific basis (a European also has to raise the arm with a mallet before striking a drum). In any case, raising the mallet requires muscular contractions and strains the arm muscles before relaxing them, to allow the mallet, with the aid of gravity, to meet its target. Shadow rhythms also have implications for understanding the notion of syncopation. In Western cultures, the downbeat is generally associated with striking the drum downwards, thus defining the accents, whereas in other cultures, the tense lifting of the stick is considered to be the accent, thus swapping downbeats with upbeats, and turning syncopation upside-down. For further discussion, see Blacking, J. (1955), p. 20, Blacking, J. (1973), and Stone, R. M. (2007). In short, a rhythm that is considered syncopated in one culture may sound perfectly normal in another, and vice versa. However, as pointed out by Sachs (1943), p. 47, this distinction may lose much of its impact when light sticks or fingers are used at a high tempo. For further discussion on the topic of motor accents on up-strokes, see Kubik, G. (2010b), p. 87.

[¶] See Jones, M. R. and Boltz, M. (1989) and Grahn, J. A. (2009), p. 262. A delightful and in-depth coverage of the psychology of expectation in time is the book by David Huron (2007).

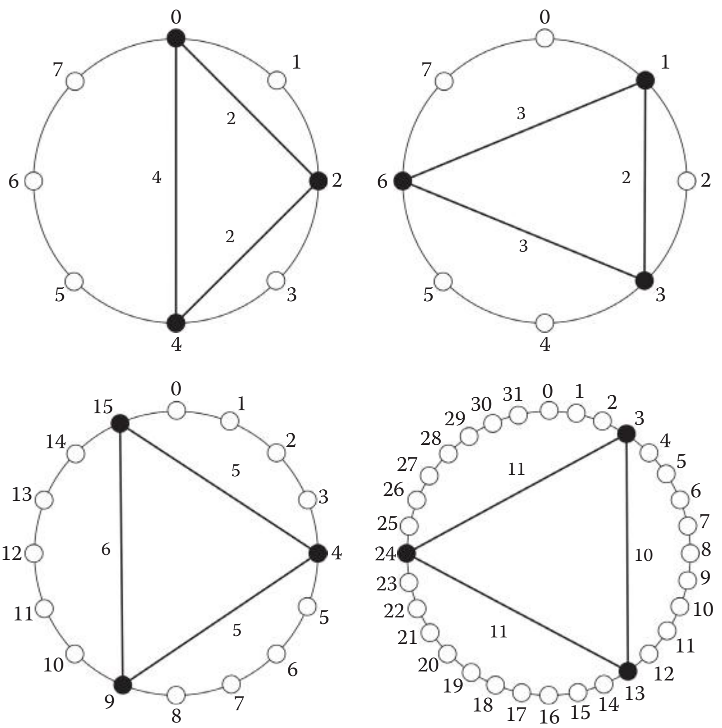


FIGURE 27.2 The rhythm [2-2-4] and its first three symmetric *shadows*.

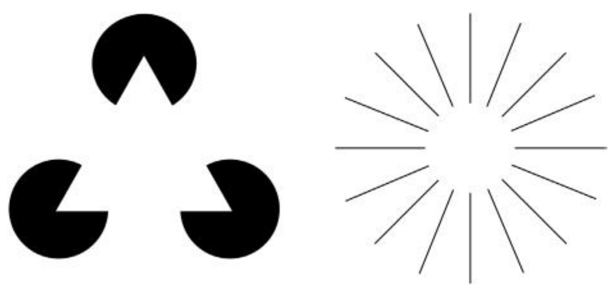


FIGURE 27.3 Two examples of *subjective* contours. (From Toussaint, G. T., *Percussive Notes*, 2011, November Issue pp. 52-59. With permission.)

signal. However, it has physical reality that may also be measured by a scientific instrument. Shadow rhythms may or may not be perceived depending on the particular context. Subjective rhythms are auditory analogs of *subjective contours* (also called illusory contours) encountered in the domain of visual pattern perception illustrated in Figure 27.3, where on the left we perceive the contour of a white triangle that is not present and on the right a circle that is absent.* It should be emphasized that the subjective contours referred

* The triangle in Figure 27.3 (left) is known as the Kanizsa triangle named after the Italian psychologist Gaetano Kanizsa, whose groundbreaking research into illusory contours at the University of Trieste during the 1950s stimulated much subsequent research in this area. Winckelgren, I. (1992), p. 1520.

to here are not rational conclusions resulting from cognitive deliberations about what must be out there on the paper, but actual perceptions in the mind. In other words, with respect to the subjective triangle on the left, what is relevant is not the deduction from some occlusion properties of objects in space, that there must be a white triangle superimposed on three black disks, but rather, that the materially nonexistent lines (contours defining the boundary of an imaginary triangle) joining the pairs of disks are perceived by the viewer. Similarly, with the figure on the right, the subjective contour does not refer to the fact that the viewer may conclude there is a white disk covering a set of rays that meet at a central common point, but rather that a nonexistent contour in the shape of a circle is perceived, a circle that has no visual signal on the paper that can be measured with scientific equipment such as a camera. As with the aural example of subjective rhythms, this is not to imply that subjective contours do not have neural correlates that can be measured with fMRI or MEG. It is worth noting that in both examples the "whites" of the triangle and the disk appear to be whiter than the "whites" outside the triangle and disk. This is not surprising, and indeed must be so if subjective contours are to be perceived.

Shadow rhythms and subjective rhythms should also be distinguished from what are called *inherent* rhythms, a term coined by the ethnomusicologist Gerhard Kubik, who originally defined inherent rhythms as those that are heard by the listener but not played by any single individual musician or instrument. Such rhythms were said to emerge from the interaction of different rhythms played on different instruments, or on the same instrument but with different tones.[*]

In the psychology literature, the phenomenon responsible for the existence of inherent rhythms is called *auditory stream segregation* or simply *streaming* for short.[†] It appears to have been discovered independently in 1950 by G. A. Miller and G. Heise (1950) in the laboratory at Harvard University, and 10 years later by G. Kubik, while doing field work in Kampala, Uganda, where he was learning to play the *amadinda* xylophone.[‡] Miller and Heise originally called the phenomenon the "melodic fission effect," Robert Erickson used the term "chanelling,"[§] but more recently, the term "inherent patterns" is being used. The perception of inherent patterns (or streaming) can be explained with the grouping principles of Gestalt psychology, and predominantly with the principle of *proximity*.[¶] Several rhythmic grouping principles based on gestalt principles were proposed in the celebrated

[*] Kubik, G. (1962), p. 33. Kubik illustrates how African instrumental music exploits the many illusions that are present in aural perception and in his 1962 paper describes four conditions for inherent rhythms to be generated by the acoustic signal: (1) The pitch intervals of the notes must be large, (2) the musical complex must be metrically unaccented, (3) the music must be played at a fast tempo, and (4) all the notes of the same pitch should form a recognizable whole or "gestalt."

[†] Bregman and Campbell (1971). An in-depth treatment of auditory streaming as well as a variety of auditory illusions and strategies for making rhythmic sense out of sequences of tones may be found in the book by Albert Bregman (1990).

[‡] Kubik, G. (2010b), p. 108, traces the history of *streaming*, its changing terminology, and its application in sub-Saharan African music composition. Wegner (1993) provides a detailed comparison of the psychological aspects of streaming discovered by Bregman and his colleagues with the musicological findings of Kubik. Musical streams are related to Yeston's music-theoretic concept of *strata*; an isochronous stream is a type of stratum. See Parncutt, R. (1994), p. 410, and Yeston, M. (1976) for further discussion.

[§] Erickson, R. (1975), p. 116, illustrates the concept of *chanelling* with a transcription of a voice and whistle piece performed by the Ba-Benzelé people of Central Africa.

[¶] Deutsch, D. (1999), p. 313.

theoretical work of Lerdahl and Jackendoff (1983).[*] Kubik is clearly aware of the gestalt processes that are involved in streaming, and yet oddly enough proposes that inherent patterns are analogous to the optical illusions illustrated in Figure 27.3, that give rise to subjective contours.[†] However, subjective contours do not have material existence, whereas the inherent patterns that listeners perceive do. Thus, inherent rhythms are not like subjective rhythms, and not analogous to the subjective contours of visual perception. Inherent rhythms emerge from clustering due to proximity relations. Perhaps, the easiest way to describe the difference between subjective contours and inherent patterns, at the risk of oversimplification, is as follows. With subjective contours, we perceive there is something there when objectively there is nothing there, whereas with inherent patterns, what we perceive is objectively there, and can be detected with a measuring device. This conclusion is at odds with Kubik's analysis. Concerning inherent patterns, he states: "they cannot be detected in a recording by electronic measuring devices. So far, no 'thinking robot' has been able to recognize these phantom images."[‡] In computational terms, Kubik implies that no algorithm exists that can extract from the complete input rhythm, the "gestalt" inherent patterns formed in the minds of the listeners. On the contrary, it will be shown that even a rather "stupid robot" can accomplish this task. McAdams and Bregman (1979) explored in depth some of the parameters that cause streaming.[§] In particular, they determined experimentally, the trade-off that exists between the sizes of the durational inter-onset intervals and the pitch intervals of the tones used, as well as the thresholds necessary for streaming to emerge. Thus, the problem of designing an algorithm to exhibit "gestalt" inherent patterns may be converted to a two-dimensional problem of clustering points in the plane in a perceptually meaningful way, a classical problem in the field of pattern recognition and computer vision. In one version of this problem (connect the dots), a set of points are arranged in the plane, and the robot must select which pairs of points should be joined with edges so as to create a perceptually meaningful line drawing. As an example, consider the set of points in Figure 27.4.

The human brain can perfectly easily perceive four distinct clusters of points, or separate "gestalt" inherent patterns that together suggest a face: namely two eyes, a nose, and a smiling mouth. Endowing a robot with artificial intelligence to accomplish such a feat was considered a difficult problem, and has received a great deal of attention from psychologists and computer scientists.[¶] Most methods are rather complicated, use heuristics, and contain parameters that must be tuned to give good results on the application at hand. However, it is possible to solve the problem with a simple geometric algorithm that does

[*] Although the work of Lerdahl and Jackendoff is primarily theoretical, it has received empirical verification by Deliège, I. (1987).

[†] *Ibid*., p. 115, Figure 30.

[‡] *Ibid*., p. 111.

[§] McAdams, S. and Bregman, A. S. (1979), pp. 26–43.

[¶] Zahn, C. T. (1971) proposed using a proximity graph called the minimum spanning tree for obtaining gestalt clusters from dot patterns. The minimum spanning tree connects all the points in one tree, and may be constructed by first connecting the closest pair of points, then the second closest pair, and continuing this way until all the points are part of the tree, as long as no cycles are produced, in which case such a pair is skipped. The disadvantage of this method is that to produce disconnected gestalt components, parameters must be defined and tuned to delete specific edges from the complete tree. See also Papari, G. and Petkov, N. (2005).

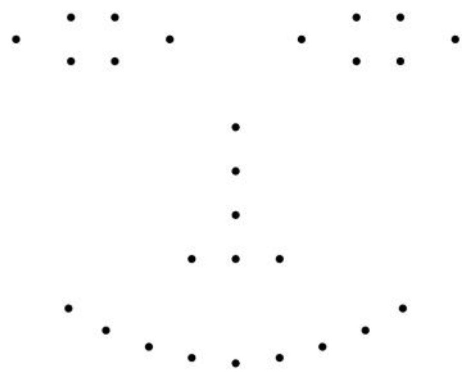


FIGURE 27.4 A set of points in the plane.

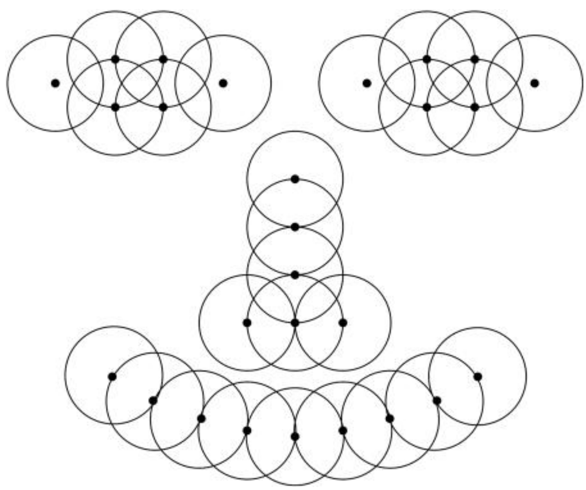


FIGURE 27.5 The dot pattern of Figure 27.4 with the nearest-neighbor circles.

not require any parameters to be tuned. Neither does the algorithm need heuristics from artificial intelligence to produce gestalt clusters that agree with human perception. Indeed, the algorithm may be described as part of the vision system of a "stupid" or "nonthinking" robot. The algorithm consists of two steps. The first step involves the construction of a circle around each dot such that the center of the circle coincides with the dot, and the radius of the circle is equal to the distance to its closest neighboring dot. If this is done for the dot pattern of Figure 27.4, then the diagram in Figure 27.5 is obtained.

The second step connects two dots with an edge provided that their corresponding circles overlap. Applying this step to the dot–circle arrangement of Figure 27.5 yields the line drawing of Figure 27.6. This line drawing of the points is called the *sphere-of-influence*

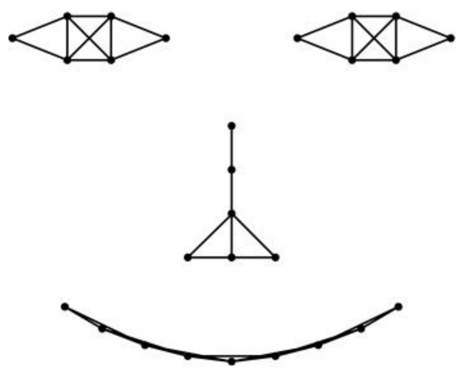


FIGURE 27.6 The sphere-of-influence graph (SIG) determined by the dot pattern of Figure 27.4.

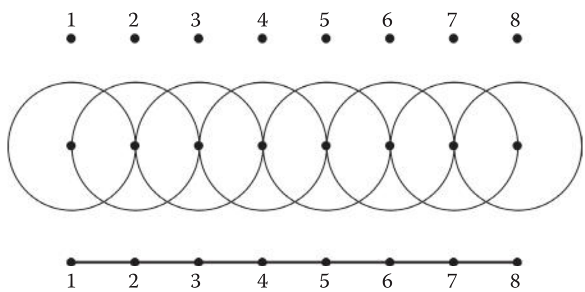


FIGURE 27.7 A sequence of points representing eight isochronous and isotonic tones (top), the circles centered at each point with radius equal to the distance to its nearest neighbor (middle), and the SIG of the eight points, consisting of a single connected component (bottom).

graph (SIG)* and is a computational model of the concept of the *primal sketch* in the theory of visual perception.†

Let us return to the problem of computing the gestalt inherent patterns from a rhythm. Consider a rhythm consisting of eight isochronous and isotonic pulses. If we represent time along the horizontal axis, and pitch along the vertical axis (as is done in Western music notation), then we can represent this rhythm by the dot pattern shown at the top of Figure 27.7. The center of the figure shows the dots with their nearest-neighbor circles, and the resulting SIG is pictured at the bottom. The SIG consists of a single connected component, and thus considers the rhythm as one gestalt without the emergence of any inherent patterns. Now, let the eight isochronous pulses be made up of two tones instead of one. In particular, let the pulses numbered one, two, four, and seven have an isotonic high tone,

* Toussaint, G. T. (1988).

† Marr, D. (1982). David Marr developed a theory of low-level, bottom-up, vision that he called *early processing*, which involved the construction of a *primal sketch*. Todd, N. P. (1994) proposed an auditory version of a primal sketch in the form of a multiscale model of rhythmic grouping. The SIG is a simple computational-geometric model of a primal sketch that is parameter-free. It is not intended to serve as an alternative theory of rhythmic grouping, but merely to illustrate that a simple algorithm is able to extract inherent rhythms.

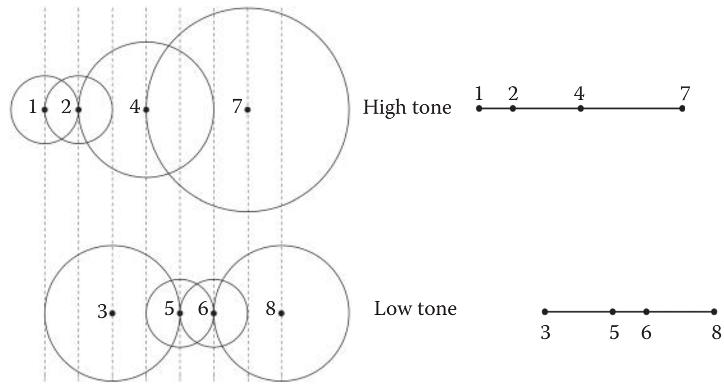


FIGURE 27.8 The same set of eight isochronous points of Figure 27.7, but this time with two tones. The left side of the figure shows the circles centered at each point with radius equal to the distance to its nearest neighbor, and the right side shows the SIG of the eight points.

and let those numbered three, five, six, and eight have an isotonic low tone. The new configuration of points, along with the nearest-neighbor circles, is shown on the left of Figure 27.8. The resulting SIG, shown on the right, now consists of two disconnected components indicating the emergence of two gestalt inherent patterns. This example illustrates the power of a variety of new computational tools, such as proximity graphs, that are becoming increasingly available to researchers in comparative musicology.* That the concept of streaming provides a useful tool for music analysis has already been established by John Roeder, for example, who used *pulse-streaming analysis* to study Schoenberg's middle-period music. Pulse streaming "represents polyphony as concurrent pulse streams created by regular recurring accents."† Hopefully, proximity graphs such as the SIG will some day find application to the automatic computation of pulse streams.

Before closing this chapter, let us return to the topic of shadow rhythms and a beautiful geometric question that they bring up, which is not only relevant to the theory of musical rhythm, but also an interesting mathematical puzzle in its own right. This problem becomes evident by comparing the polygonal representations of the rhythm [2-2-4] with its shadow at the top of Figure 27.2, which reveals that the shadow of the rhythm (top right) has inter-onset intervals with durations that have less variability (or are more regular) than its parent rhythm (top left). The variability of a rhythm's intervals may be conveniently measured by the ratio of its longest to its shortest duration interval. For the rhythm [2-2-4], this ratio is 4/2 = 2, whereas for its shadow it is 3/2 = 1.5. In other words, the shadow of rhythm [2-2-4] is more *regular* than [2-2-4]. A natural geometric question that arises is whether this property is true for all rhythms. Consider the second shadow of rhythm [2-2-4], that is, the shadow of the rhythm in the top-right of Figure 27.2. The midpoint of the interval between pulses three and six is the midpoint between pulses four and five. Therefore, to represent this shadow rhythm as points on the circle, a cycle of 16 pulses is required. The resulting

* Toussaint, G. T. (2005a), Tzanetakis et al. (2007), Cornelis et al. (2010).
† Roeder, J. (1994), p. 249.

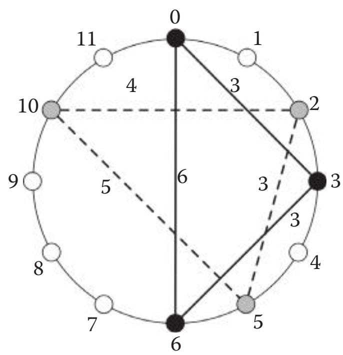


FIGURE 27.9 The *asymmetric shadow* of a rhythm.

shadow rhythm is shown at the bottom left and has a longest/shortest duration ratio of 6/5 = 1.2. Continuing this process, the shadow of this rhythm in turn requires a 32-pulse cycle, shown at the bottom right, which has a longest/shortest ratio of 11/10 = 1.1. In this example, the sequence of ratios of longest to shortest intervals, as we continue to apply the shadow operation, is the decreasing sequence 2, 1.5, 1.2, 1.1. It turns out that the shadow operation makes *all* nonregular rhythms more regular, and furthermore, if this operation is continued, eventually the rhythm becomes perfectly regular, that is, all adjacent inter-onset interval durations become equal.*

With the aid of slow motion film, the ethnomusicologist Gerhard Kubik made some interesting observations from the analysis of the upward and downward motions of the arms of xylophone players. In one study, he compared European players with Ugandan players, and found that the upward and downward motions between two consecutive attacks of the Europeans tended to have equal durations, but for the Ugandan players, the upswing lasted twice as long as the downswing. Therefore, it is natural to call the shadow rhythms resulting from the former process *symmetric* shadows, and from the latter process *asymmetric* shadows. Figure 27.9 shows an asymmetric shadow rhythm in which the upward swing lasts twice as long as the downward swing. The rhythm actually played is [x . . x . . x . . . . .]. Using this rule, and considering that the duration between the first and second onsets is three pulses long, the maximum height of the arm swing between these two onsets occurs at pulse two. Thus, the resulting two-thirds–one-third shadow rhythm is [. . x . . x . . . . x .]. Comparing this rhythm to its asymmetric shadow rhythm reveals that the ratio of the longest to the shortest duration interval has changed from 6/3 to 5/3, that is, the asymmetric shadow is still more regular than the parent rhythm. Even in the case of asymmetric shadow rhythms, if the shadow operation is repeatedly applied, the resulting rhythm will also eventually become perfectly regular.

For the special case in which the rhythms consist of just two onsets, the rhythms converge to the regular rhythms in a single step. In other words, the shadow of any two-onset irregular rhythm is a regular rhythm. This must be true because the bisector of any chord of a circle must intersect the center of the circle and determine a diameter. To see

* Hitt and Zhang (2001), p. 22.

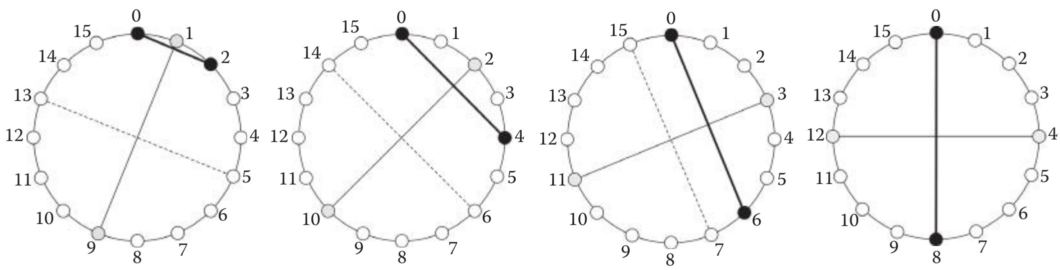


FIGURE 27.10 Some two-onset rhythms and their shadows.

this, examine the sample of rhythms pictured in Figure 27.10. In the first three cases, the rhythms have inter-onset intervals [2], [4], and [6], respectively, indicated with bold lines connecting the black-filled pulses. Their shadows are shown with thin solid lines connecting the gray-filled pulses, and the shadows of the shadows are shown in dotted lines connecting white-filled pulses. After the first step, the subsequent shadows continually alternate between two orthogonal diametrically opposite pulse pairs, forever. The right-most rhythm with inter-onset interval [8] is already regular.

The following figures exemplify some additional rhythms used in musical practice alongside their shadows. Figure 27.11 shows that one of the six distinguished timelines, the shiko on the left, has as its shadow a rotation of another distinguished timeline, the bossa-nova on the right. The bossa-nova starts on pulse eight. The fact that when a performer plays the shiko, his or her motions implicitly also perform the bossa-nova as a shadow rhythm suggests that the relationship of these two rhythms should be explored further from the psychological or neurological points of view to determine if shadow rhythms exhibit any neural correlates. Note, however, that the shadow operation is of course not symmetric. The shadow of the bossa-nova is not the shiko, but rather an even more regular rhythm that contains three intervals of duration three, and two intervals of duration 3.5.

One of the most popular metric patterns used in classical music, in Figure 27.12 (left) with durational pattern [4-4-2-2-4] or [2-2-1-1-2] has as its shadow a rotation of one of the most popular timelines used in rap music (right). The rap timeline starts on pulse number

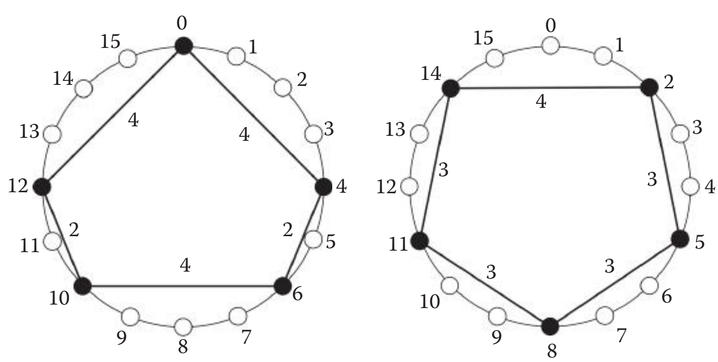


FIGURE 27.11 The *shiko* timeline (left) and its shadow (right).

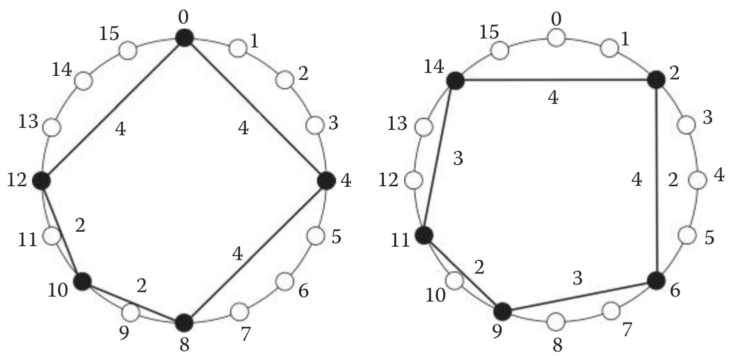


FIGURE 27.12 A popular classical music rhythmic ostinato (left) and its shadow (right).

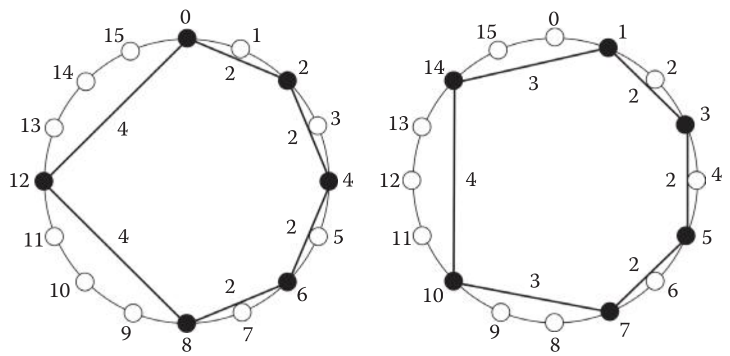


FIGURE 27.13 A popular timeline (left) and its shadow (right).

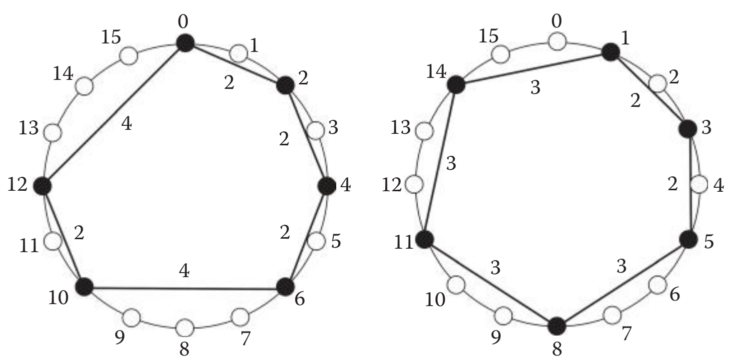


FIGURE 27.14 Another popular timeline (left) and its shadow (right).

two. A rotation of the pattern on the left, given by [4-2-2-4-4] or [2-1-1-2-2] is also the beloved metric pattern of the *Allegretto* in Beethoven's *Seventh Symphony*. In the words of Max Kenyon,* "this metre does more than appeal to the soul: it appeals to the body as well." Did Beethoven "hear" the shadow rap rhythm of this metric pattern when he chose it for

* Kenyon (1947) p. 170.

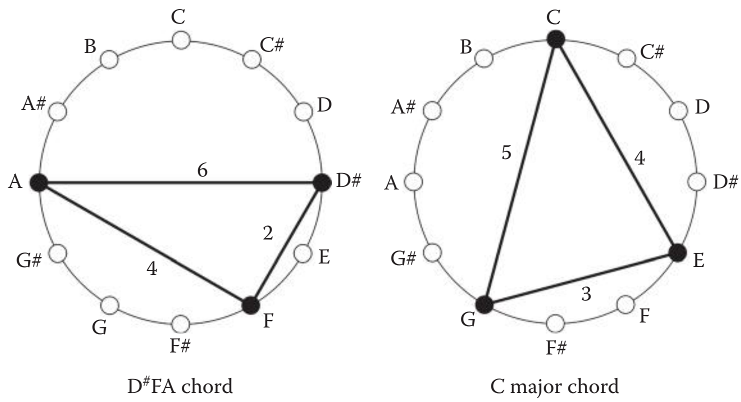


FIGURE 27.15 The C-major chord is the shadow of the D#FA-chord.

his composition? Do rappers rapping to the pattern [4-3-2-3-4] feel Beethoven's umbrella rhythm hovering over it?*

Two additional popular timelines used in sub-Saharan African music and their shadow rhythms are pictured in Figures 27.13 and 27.14.

In the pitch domain, chords also have their shadows of course. In Figure 27.15, the famous C-major chord consisting of the notes CEG (right) with internote intervals [4-3-5] is the shadow of the D#FA-chord (left) with intervals [4-6-2] with root F, because E is the midpoint of the dyad D#F, G is the midpoint of the dyad FA, and C is the midpoint of dyad AD#. Whether the shadow concept has interesting musical and psychological implications in the pitch domain remains an open question yet to be investigated.

* If the metric pattern used by Beethoven, [2-2-1-1-2], is rotated by a half-cycle, one obtains [1-1-2-2-2], the drumming timeline used in the *bugóbogóbo* music, the symbol of Tanzanian national identity. See Barz, G. (2004), pp. 20–21.

CHAPTER 28

Reflection Rhythms and Rhythmic Canons

WHILE IT IS DIFFICULT to define precisely what makes a "good" rhythm good, it is not hard to list properties that appear to contribute to a rhythm's goodness in the presence of other appropriate properties. Given that we know a certain rhythm to be good, we may examine its properties to gain insight into what makes it good. However, such properties are, by themselves, neither necessary nor sufficient to make a rhythm good. We have already examined several such properties in previous chapters. Another possible candidate for such a property is that the mirror-symmetric image of a rhythm about some axis of symmetry be equal to its complementary rhythm.* I call rhythms that have this geometric property *interlocking reflection* rhythms. These rhythms also exhibit the phenomenological property that if the rhythm and its complement are both played simultaneously, and their acoustic properties such as timbre or intensity differ, the listener has the impression that the rhythm is switching roles, sometimes acting as a figure with its complement as a background, or vice versa. This occurrence is analogous to the unstable perception caused by the cubic wireframe shown in Figure 28.1 (left) that has no depth cues, and which is sometimes perceived as a cube viewed either from the top (center) or from the bottom (right). Such figures are known as Necker cubes.†

This chapter presents, by means of examples, two simple methods to generate rhythms that exhibit the interlocking reflection property: it introduces the notion of rhythmic canons and relates reflection rhythms to rhythmic canons.‡ The first technique uses the well-known snare drum rudiments called *paradiddles*.§ These rudiments form the basis of modern drumming technique, and have been shown to increase speed, control, independence, accuracy, creativity, rhythmic innovation, and fluency. They have roots in African

* This idea in the pitch domain is called *inversional combinatoriality*, and was used as a compositional tool by Arnold Schoenberg. See Milstein, S. (1992), p. 6. Thus, it is interesting to investigate its role in the rhythmic domain.

† Penrose, R. (1992), Long, G. M. and Olszweski, A. D. (1999).

‡ See Toussaint, G. T. (2010) for a more thorough description of the methods described in this chapter.

§ Wanamaker, J. and Carson, R. (1984).

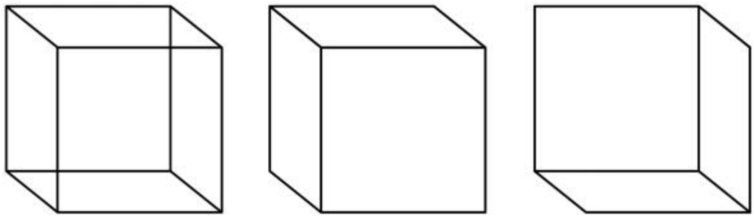


FIGURE 28.1 Unstable perception of the wireframe cube (also Necker cube).

balafon technique, and have inspired rock drummers as well as modern jazz pioneers.[*] The reader is warned that the mathematics, theory, and algorithms used in paradiddles are simple. This chapter is more about drumming technique, about left–right independence, and hand–foot independence, than about deep theory. Thus, the reader is advised to use the hands and feet while studying this chapter. Once the simple technique is mastered and has become automatic, a few simple rules thrown in will cause the emergence of elegant rhythmic complexity.

Paradiddle Method: To illustrate the first method, consider a time span of 16 pulses, and refer to Figure 28.2. First, insert an onset at every pulse in the cycle of the rhythm, as in (a), where the label "Right" indicates the rhythm is played with the right hand, and the label "period" indicates the length of the rhythmic pattern that repeats itself. If we denote by the symbols R and L, the striking of the drum with the right and left hand, respectively, then this rhythm repeats the pattern R 16 times. Next, transform this rhythm into a rhythm for

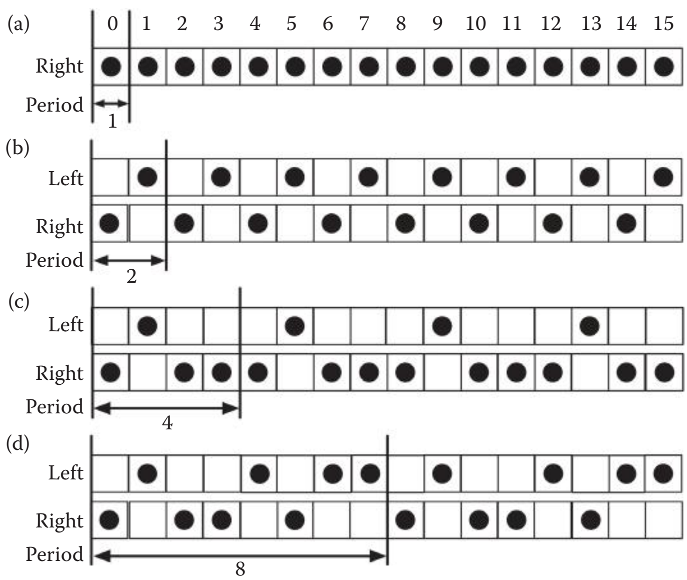


FIGURE 28.2 Constructing a *single paradiddle* reflection rhythm.

[*] Brown, A. (1990).

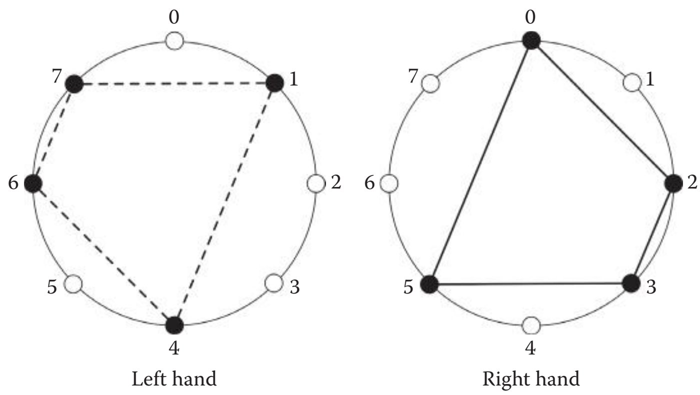


FIGURE 28.3 The left- and right-hand patterns of the *single paradiddle* drum rhythm.

two hands by alternating the onsets between the left and right hands, preferably each hand playing on a different drum, as in (b). The idea here is that the right and left hands should produce different sounds either in pitch or timbre so that the listener may perceive both the right-hand and left-hand rhythms simultaneously as two separate streams of pulsations. Note that this operation doubles the length of the period of the rhythms played by each hand. The rhythm thus repeats the pattern RL eight times. The next step in the process involves repeating the symbol R after one instance of the RLR pattern to obtain the string RLRR, as in (c). Note that this operation doubles the period again to a length of four. The final step involves alternating between the RLRR pattern and its mirror image LRLL, as in (d), to obtain the pattern RLRRLRLL. This operation again doubles the period of the rhythm giving it a time span of eight pulses. The rhythm heard on the right-hand drum has the durational pattern [2-1-2-3] shown in polygon notation in Figure 28.3 (right), with its mirror image complementary rhythm played on the left-hand drum (left). Note that the rhythm [2-1-2-3] is a deep rhythm, has the rhythmic oddity property, and when played backwards becomes [3-2-1-2], which, as we have seen, is the hand-clapping pattern in Elvis Presley's *Hound Dog*. The complete rhythm played with both hands is one of the three most important rudimentary snare drum exercises called the *single paradiddle*, and it is not conceived as two separate rhythms, but rather as a single resultant pattern.* Here, on the other hand, the paradiddle technique is used to generate the individual rhythms played with just the right or left hand. Interestingly, Steve Reich used this rhythm in his composition titled *Phase Patterns*, for four electronic organs, in which the right hand played [2-1-2-3], and the left hand played the complementary rhythm.†

For a second example of this method of generating good rhythms, consider the construction of a ternary rhythm with six onsets and 12 pulses. This time we start with a rhythm consisting of 24 onsets and 24 pulses as shown in Figure 28.4 (a). As in the previous

* See Koetting, J. and Knight, R. (1986), p. 60, for a discussion on the use of paradiddle structures in traditional African drumming. It is worth pointing out that in creating the paradiddles, it is not just a matter of taking any LR rhythm and following it by its complementary RL rhythm. It has to be only the exact pattern RLRRLRLLRLRRLRLL, and so on.

† Cohn, R. (1992), Example 1, p. 150.

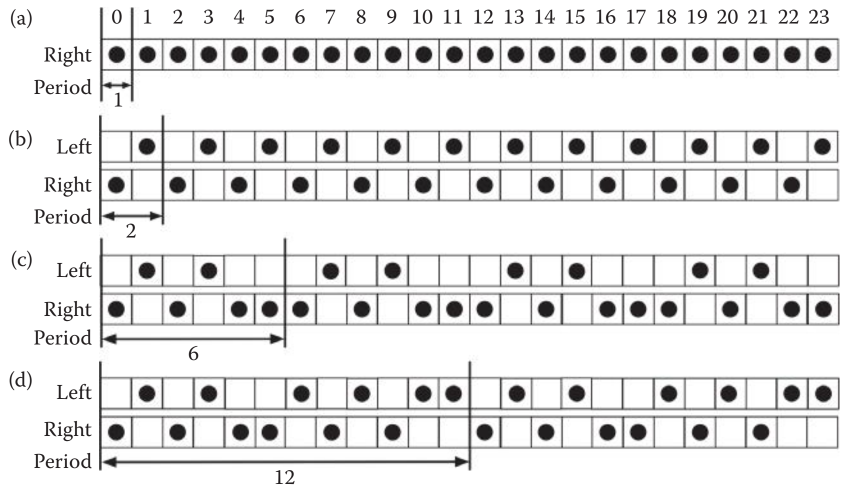


FIGURE 28.4 Constructing a *double paradiddle* reflection rhythm. (a) All pulses played with right hand, (b) all pulses played with alternating hands, (c) repeating the right hand at pulse 5, (d) concatenating the mirror image of pattern (c).

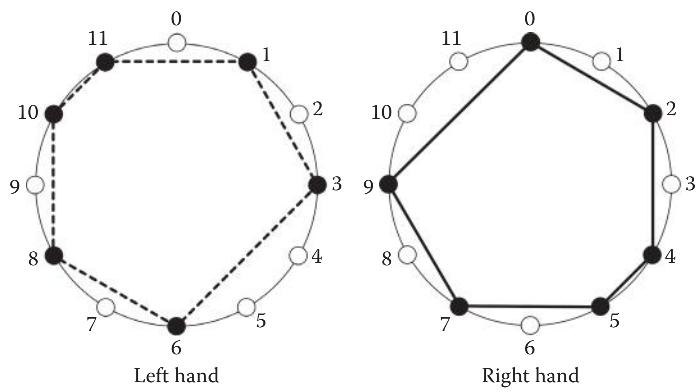


FIGURE 28.5 The left- and right-hand patterns of the *double paradiddle* drum rhythm.

example, this rhythm is first decomposed into its alternating right- and left-hand onsets as in (b). The next step shown in (c) is the only step in this process that changes. Instead of repeating the symbol R after the RLR pattern, now it is repeated after the RLRLR pattern to create an RLRLRR pattern of period 6. The final step (d) is the same as before: the RLRLRR pattern is alternated with its mirror image LRLRLL to create a rhythm with period 12. The rhythm played with the right hand, shown in polygon notation in Figure 28.5, has durational pattern [2-2-1-2-2-3], which has the rhythmic oddity property, is a deep rhythm, and is a popular Yoruba bell timeline from West Africa. It is also a Kpelle rhythm of Liberia.* The entire rhythm played with both hands is called the *double paradiddle* in rudimentary snare drum technique.

* Stone, R. M. (2005), p. 82.

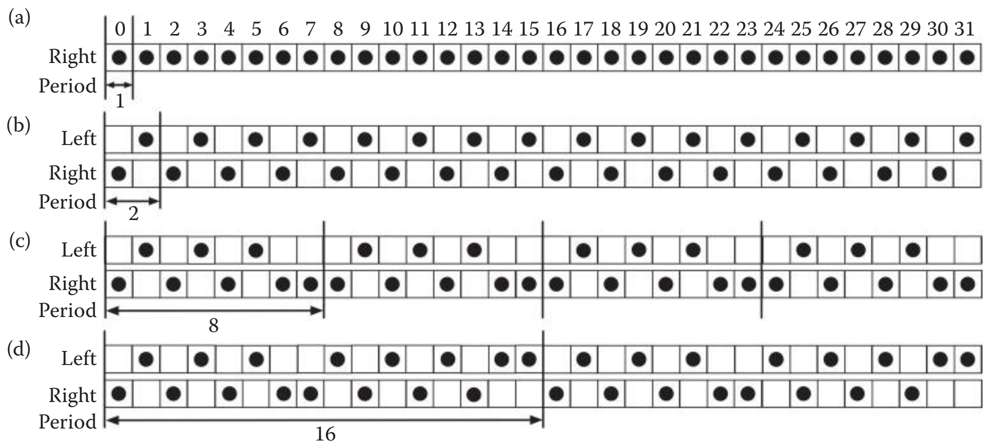


FIGURE 28.6 Constructing a *triple paradiddle* reflection rhythm. (a) All pulses played with right hand, (b) all pulses played with alternating hands, (c) repeating the right hand at pulse 7, (d) concatenating the mirror image of pattern (c).

For the third example that illustrates this method, let us generate a rhythm with eight onsets and 16 pulses. The process for the first two steps is the same as before, but this time let us start with 32 pulses, as shown in Figure 28.6, steps (a) and (b). The change comes again in step (c) where instead of repeating the symbol R after the RLRLR pattern, now it is repeated after the RLRLRLR pattern to create an RLRLRLRR pattern with period eight. The final step (d) is the same as before: the RLRLRLRR pattern is alternated with its mirror image LRLRLRLL to create a rhythm with period 16. The rhythm played with the right hand, shown in polygon notation in Figure 28.7, has inter-onset interval vector [2-2-2-1-2-2-2-3], which is a deep rhythm as may be seen from the histogram on the right. This rhythm also has the rhythmic oddity property. The entire rhythm played with both hands is called the *triple paradiddle* in rudimentary snare drum technique.

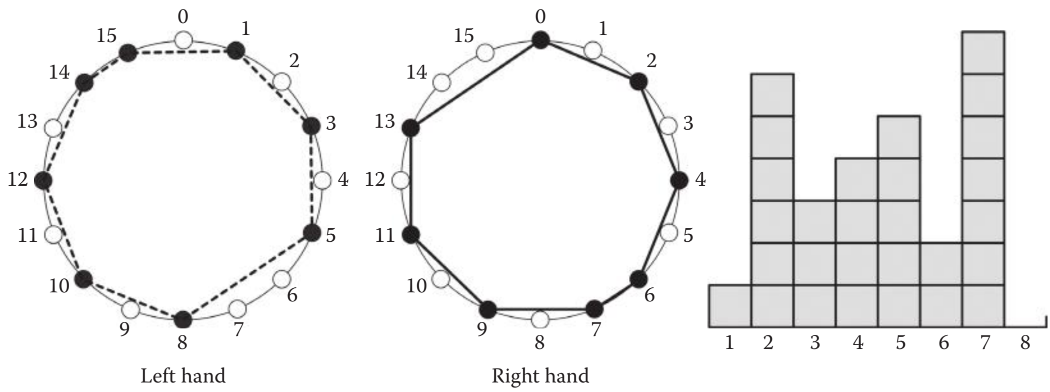


FIGURE 28.7 The left- and right-hand patterns of the *triple paradiddle* drum rhythm, and their interval content histogram.

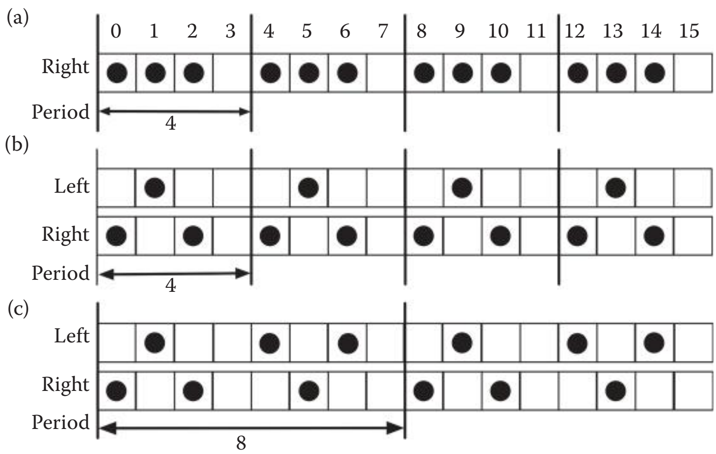


FIGURE 28.8 Constructing the [2-3-3] reflection rhythm using the *alternation* method. (a) The seed pattern played with one hand, (b) the seed pattern played with alternating hands, and (c) repeated reflection of left and right handed patterns.

Alternating-Hands Method: Whereas the *paradiddle method* described above first puts an onset on every pulse, and then creates a pattern by breaking the alternation between the right and left hands through the repetition of an onset with the same hand, the alternating-hands method maintains the alternation of right and left hands throughout the execution of the rhythm, and instead generates a new rhythm by transforming another (usually simpler) seed rhythm.

For the first example that illustrates the alternating-hands method, refer to Figure 28.8, and consider the simple four-pulse seed pattern [x x x .] shown in (a) with durational pattern [1-1-2]. This rhythm is a universally used simple pattern dating back to at least the *ars antiqua*, associated with prosody, and known as the *short–short–long* pattern. We have seen earlier that it is also a pattern used in the *Baiaó* rhythm of Brazil, a drum rhythm in South Indian classical music, and the *polos* rhythm of Bali. The second step decomposes this cyclic pattern into alternating right- and left-hand strokes as shown in (b). Finally, this rhythm is alternated with its mirror image creating on the right-hand drum an eight-pulse rhythm with inter-onset interval vector [2-3-3], as shown in (c). At the same time, the left hand plays a reflected (mirror image) version of this rhythm. The rhythms in (b) and (c) are typical rhythms played with the *krakebs*, or metal double castanets (illustrated in Figure 28.9) in the *Gnawa* trance music of North Africa, found mainly in Morocco and Algeria. One pair is used in each hand to produce a loud and distinctive hypnotic metallic sound that resembles hand-clapping.

The reader is invited to apply this alternating-hands method to three more seed patterns given by [1-2-1-2-2] (a version of the cinquillo) in Figure 28.10, the pattern [1-1-1-1-2] in Figure 28.11, and the pattern [2-1-1-2-2] (the popular classical music rhythmic ostinato) in Figure 28.12. As can be seen by following the procedure illustrated in these figures, the three patterns yield, respectively, the bossa-nova, fume-fume, and clave son timelines.

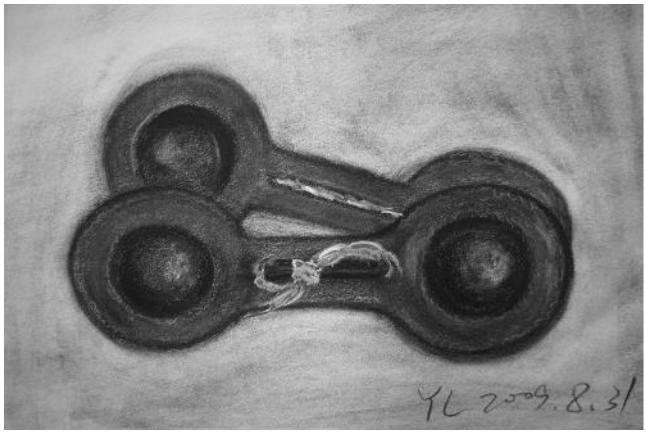


FIGURE 28.9 Moroccan krakebs (metal double castanets). (Courtesy of Yang Liu.)

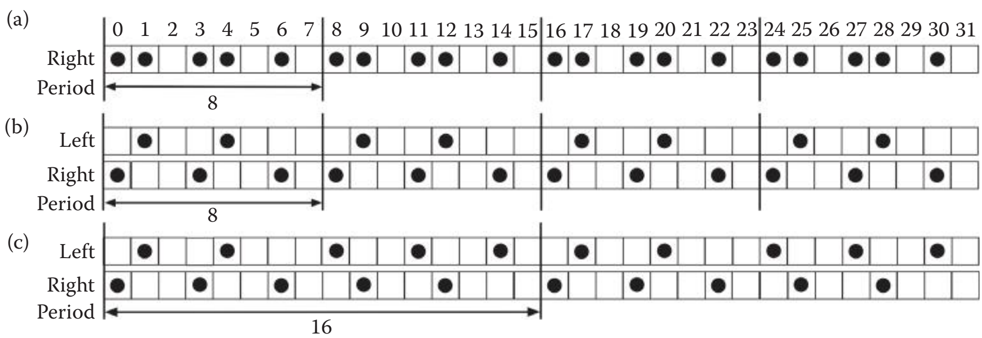


FIGURE 28.10 Constructing the bossa-nova necklace timeline with the alternation method. (a) The seed pattern played with one hand, (b) the seed pattern played with alternating hands, and (c) repeated reflection of left and right handed patterns.

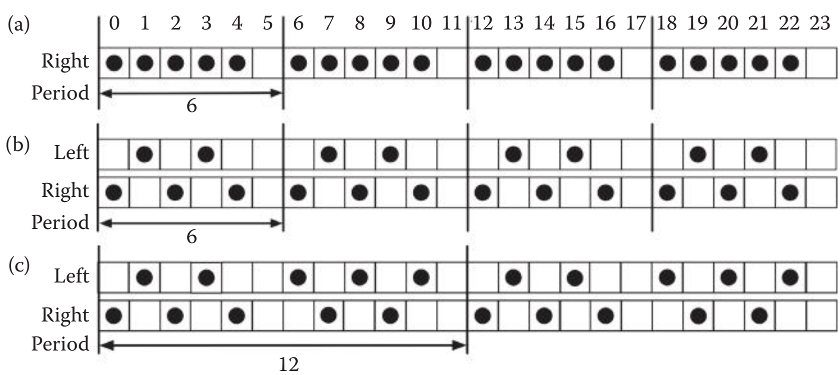


FIGURE 28.11 Constructing the fume-fume timeline with the alternation method. (a) The seed pattern played with one hand, (b) the seed pattern played with alternating hands, and (c) alternating the seed pattern with its left-right mirror image.

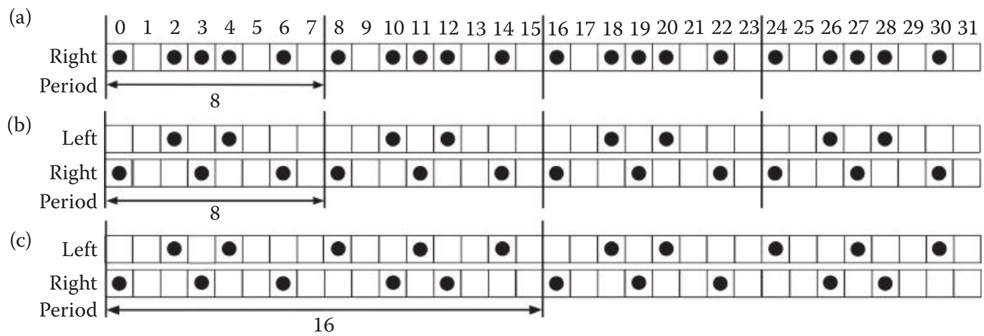


FIGURE 28.12 Constructing the clave son with the alternation method. (a) The seed pattern played with one hand, (b) the seed pattern played with alternating hands, and (c) repeated reflection of left and right handed patterns.

The simple alternating-hands method for rhythm generation that started with the seed pattern [x x x .] produced the pair of right-hand and left-hand rhythms given by the sequences [x . x . . x . .] and [. x . . x . x .], respectively. If we rotate these rhythms in a counterclockwise direction by two pulses, we obtain the rhythms shown in Figure 28.13. The right-hand rhythm is the ubiquitous [3-3-2] *tresillo* pattern and the left hand produces its reflection about the line through pulses 1 and 5. Here, the right-hand rhythm is shown with solid lines, and the left hand with dashed lines. These two rhythms may be viewed as having other noteworthy geometric relationships to each other. For one, the left-hand rhythm is a rotation of the right-hand rhythm by 180°, that is, half a cycle. For another, the left-hand rhythm is the *antipodal* version of the right-hand rhythm. In other words, the left-hand rhythm may be obtained from the right-hand rhythm by replacing every onset by its antipodal onset, that is, the onset diametrically opposed to it.

The [3-3-2] tresillo pattern, which constitutes the first half of the clave son, by itself is already imbued with a great deal of rhythmic power, as evidenced by its widespread popularity all over the world. Playing it alongside its mirror reflection about the line through pulses 1 and 5, which has a distinguishably different timbre, as in the case of the krakebs, may explain why it plays a role in Gnawa trance music. Furthermore, it is interesting to

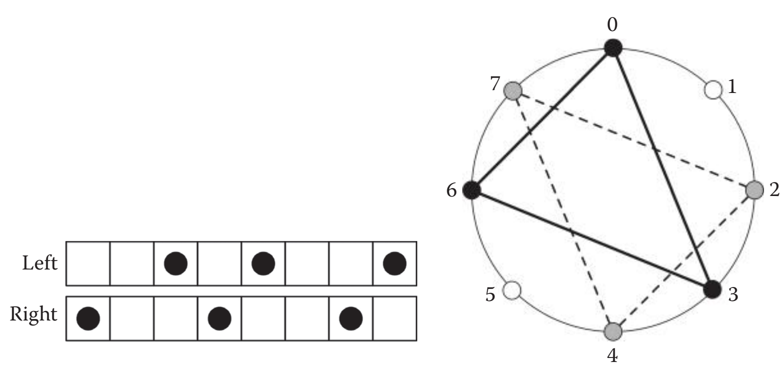


FIGURE 28.13 The tresillo rhythm and its reflection in box (left) and polygon (right) notations.

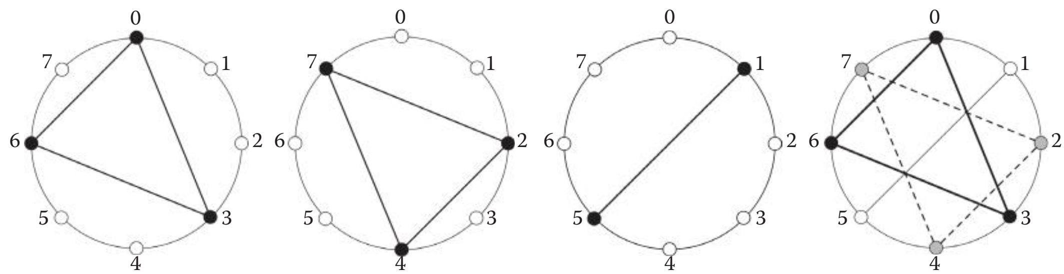


FIGURE 28.14 Three interlocking rhythms used in Balinese *Kecak* chanting.

note that if we add a third rhythm that consists of only the two onsets at pulses 1 and 5, as in Figure 28.14, then we obtain three interlocking rhythms, two of which are mirror images of each other reflected across a line (mirror) that is itself a third rhythm played, as shown in the right-most diagram of Figure 28.14. These three rhythms are used in the rhythmic *Kecak* group chanting music of Bali, known for its very powerful trance-inducing effects.[*] Here, each of the three rhythms is chanted by a different group of singers. However, it is not clear that rhythms with these geometric properties are in any way necessary to induce trance behavior, as is evidenced from the "self-stabbing" theme of the *Rangda/Barong* ritual. The rhythms used in this piece consist of seven regular perfectly metrically aligned patterns.[†]

Rhythmic Canons: A *rhythmic canon* is composed of two or more rotations of a rhythm played at the same time, with the constraint that each rotation (also called a *voice*) uses a tone or timbre that is distinguished from the others, and no two onsets (attacks) of different voices sound in unison. Rhythmic canons provide a popular and effective composition technique that can be traced back to Olivier Messiaen,[‡] and has recently received a great deal of attention from the music-theoretical and mathematical points of view.[§] Two of the three interlocking rhythms used in Balinese *Kecak* chanting, shown in the rightmost diagram of Figure 28.14 (the tresillo and its mirror reflection about the axis through pulses one and five, also shown in Figure 28.13), constitute a rhythmic canon in two voices: the first voice (solid lines) starts the rhythm [3-3-2] at pulse zero, and the second voice (dashed lines) starts the same rhythm [3-3-2] half a measure later at pulse four. In this case, the reflection of [3-3-2] about the line through pulses one and five generates a canon because it is equivalent to a rotation by four pulses (half a measure).

In the above example, although no two onsets from different voices sound together, the canon does not use all the onsets available, in particular the onsets at pulses one and five. A rhythmic canon in which every pulse receives exactly one onset is called a *rhythmic tiling canon*. The paradiddle method described at the start of this chapter for producing reflection rhythms yields rhythms performed with the right hand, which are not only

[*] McLachlan, N. (2000), p. 63.
[†] Becker, J. (1994), p. 48.
[‡] Messiaen, O. (1956).
[§] See Agon, C. and Andreatta, M. (2011), Hall, R. W. and Klingsberg, P. (2006), and Tangian, A. (2003) as well as the references therein for some history and a sampling of mathematical results concerning rhythmic canons.

reflections but also rotations of the rhythms performed with the left hand. Therefore, if different voices are assigned to the left-hand and right-hand attacks, the paradiddle method may be used to generate rhythmic tiling canons such as [2-1-2-3] and its complement in Figure 28.3, [2-2-1-2-2-3] and its complement in Figure 28.5, and [2-2-2-1-2-2-2-3] and its complement in Figure 28.7.

If, while reading this chapter, you found that these rhythms were too simple to play, then its time to introduce some additional rules to create more interesting rhythms. Remember though that the left–right patterns of the paradiddle always remain the same. For example, the single paradiddle is RLRRLRLLRLRRLRLL. Now assume you are playing four congas that have four different timbres, and that they are located in the space around you in the form of a square: two in the front and two in the back. Now you can add the rule that while playing the paradiddle pattern, the attacks should alternate between consecutive attacks on front congas and two on back congas, as in the pattern ffbbffbbffbbffbb. The resulting drum attacks you play should then be (Rf)(Lf)(Rb)(Rb)(Lf)(Rf)(Lb)(Lb)(Rf)(Lf)(Rb)(Rb)(Lf(Rf)(Lb)(Lb). If this is still too simple, then apply an asymmetric front back rule such as ffbffbffbffbffbf. With a drum set, you can do something similar with the feet that is different from the hand patterns. As you can see, the sky is the limit.

CHAPTER 29

Toggle Rhythms

In the *alternating-hands* method for generating rhythmic canons composed of "good" rhythms, described in the preceding chapter, the right and left hands continually take turns striking the instrument, analogous to the way our feet do on the ground while we walk, except that, depending on the method employed, the durations between consecutive right- and left-handed strokes may vary. For example, in the rhythm of Figure 28.8, the duration between the first onset (right-hand) and the second onset (left-hand) is one pulse, but the duration between the third onset (right-hand) and the fourth onset (left-hand) is two pulses. In the method described in this chapter, the rhythm emerges from the process of *accenting* the proper onsets with each hand while maintaining all the durations between left- and right-handed strokes equal to one pulse. In other words, some strokes may be louder than others, or they may differ in timbre, or tonality. Indeed, the soft sounds may even be so muted that they are inaudible, or the hand may stop just before coming into contact with the instrument. The important point is that the motion of the hands consists of a continuous pendular alternation of the right and left hands, such that all durations between adjacent pulses are equal. In other words, the downward motions of the hands trace *all* the pulses of the rhythm.*

Toggle rhythms are those cyclic rhythms that when played using the alternating-hands method, have their onsets divided into two consecutive sets, such that the onsets of the first set are played consecutively with one hand, and subsequently the onsets of the second set are played consecutively with the other hand. Thus, playing this way feels as if one hand responds to a question posed by the other hand, analogous to the customary call-and-response method of singing existent in much of sub-Saharan Africa. The most pleasing and interesting results with this method are obtained when the left and right hands strike drums that are tuned differently, so that they produce sounds of distinct tones or timbres. However, even on a single drum, the left and right hands will almost always produce distinct sounds, since they strike the drum skin at different locations, and thus the effect will still be audible and operative. However, even if all the accented strokes sound the same,

* See Toussaint, G. T. (2010) for a more thorough description of the methods described in this chapter.

the system yields good timelines. Indeed, in some musical practices such as sub-Saharan Africa, timelines by their usual definition have the property that they do not contain accents, that is, all their onsets have equal importance.

The motion of the right and left hands in this manner of playing may be conveniently described with a notation such as RLRLRLRLRLRLRLRL, which indicates all the pulses present in the rhythmic cycle, as well as which pulses are struck with which hand, R standing for the right hand on even-numbered pulses, and L for the left hand on odd-numbered pulses. The rhythm that emerges from this process may be notated using a bold face font for the accented onsets. For example, one possible toggle rhythm with this pattern is **R**L**R**L**R**L**R**LR**L**R**L**R**L**RL. By accenting the four right-hand and three left-hand strokes, the rhythm that emerges in the form of the accented onsets may be described in box notation as [x . x . x . x . . x . x . x . .]. Note that in this example every pair of consecutive onsets played with the right hand (or left) is separated by one silent pulse. Note also that the transitions between the right-hand onsets and the left-hand onsets are separated by an interval of two silent pulses. Toggle rhythms that have this property will be called *smooth* toggle rhythms because this transition is smooth. On the other hand, a 16-pulse pattern such as **R**L**R**L**R**L**R**L**RL**R**L**R**L**R**L**, which may be expressed in box notation by [x . x . x . x . x x . x . x . x], has no silent pulses between the transition of left-hand and right-hand pulses. This transition is abrupt or sharp, and so toggle rhythms with this property are termed *sharp* toggle rhythms.

The methods for generating rhythms in this and Chapter 28 are techniques that are useful for drummers and percussion players. By fixing simple repetitive hand motion patterns that a drummer can learn to play automatically without thinking, and then applying simple repetitive attack patterns on the sequence of drums or other instruments such as bells or cymbals, the drummer can engender the emergence of several complex rhythmic patterns being played simultaneously, without thinking and with little effort other than trying not to listen too carefully to what is being produced, lest one is thrown off guard. These methods may also be used as models for how certain traditional African rhythms on instruments such as the balafon may have come into existence.* Figure 29.1 illustrates an algorithm for generating a family of smooth toggle rhythms, by starting with the simplest smooth toggle rhythm that acts as a seed pattern. This seed pattern shown in Figure 29.1 (a) consists of a cycle of eight pulses with two right-hand onsets at pulses zero and two, and one left-hand onset at pulse five. This rhythm, as we have seen earlier, has an inter-onset interval structure given by [2-3-3], and is used in the traditional music of several cultures: it is the bell rhythm used in the *nandon bawaa* music of the Dagarti people of Ghana, as well as a rhythm found in Namibia and Bulgaria. From this seed pattern, we may create new longer rhythms by repeatedly cutting the rhythm in half, and inserting a copy of the *left–right transition segment* in between the resulting two pieces, as illustrated in Figure 29.2. The top of Figure 29.2 shows the eight-pulse seed rhythm with the left–right transition segment shaded. The middle shows the original rhythm cut into two pieces of equal duration. Note that the cut is made at the midpoint that separates the

* Scherzinger, M. (2010).

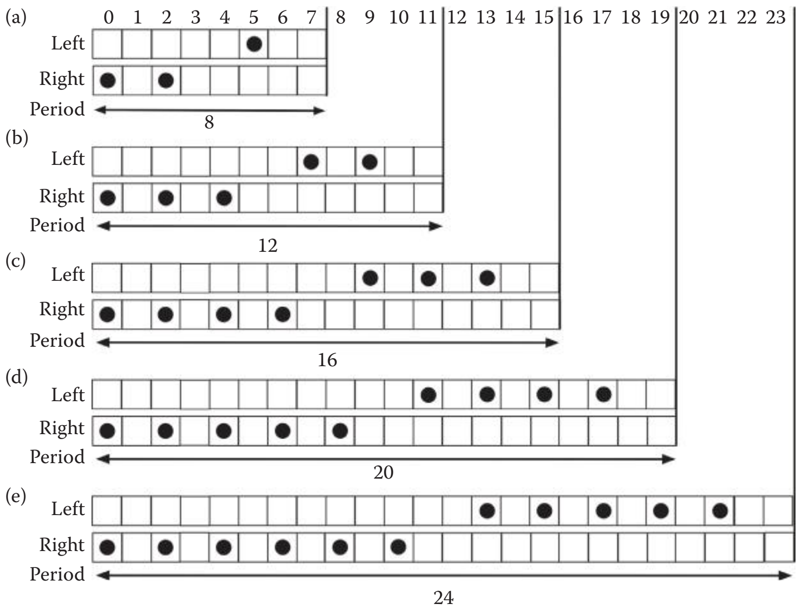


FIGURE 29.1 One method for generating *smooth* toggle rhythms. (a) Simple toggle rhythm, (b) inserting a segment in the middle of the (a) rhythm, (c) inserting a segment in the middle of the (b) rhythm, (d) inserting a segment in the middle of the (c) rhythm, and (e) inserting a segment in the middle of the (d) rhythm.

right-hand strokes from the left-hand strokes, which in this case happens between pulses three and four. The bottom shows the final 12-pulse rhythm obtained by splicing the three pieces together, which is the rhythm shown in Figure 29.1 (b). This process may be iterated by repeatedly inserting the shaded four-pulse transition segment into the preceding rhythm in the same manner. In this way, we may generate the remaining rhythms shown

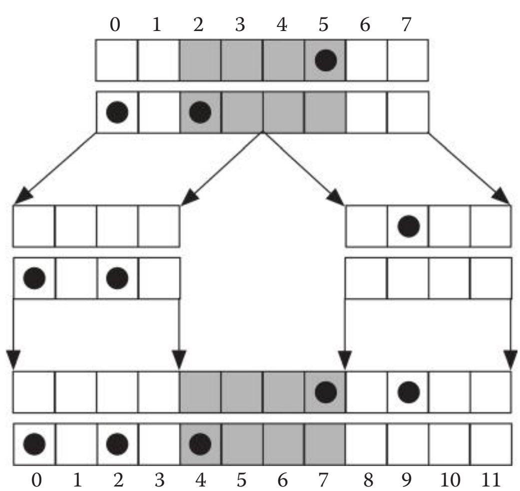


FIGURE 29.2 Splicing a *smooth* toggle rhythm by inserting the right–left transition segment (shaded).

in Figure 29.1, as well as longer ones if desired. However, a cycle of 24 pulses is almost always long enough to serve as a timeline.

The reader will by now be very familiar with the rhythm in Figure 29.1 (b) with inter-onset interval structure [2-2-3-2-3], as being the *fume-fume* bell pattern popular in West Africa, and the former Yugoslavia. The rhythm in (c) with inter-onset interval structure [2-2-2-3-2-2-3] is a hand-clapping timeline pattern from Ghana.* The rhythm in (e) consisting of 11 onsets among 24 pulses is perhaps the longest existing smooth toggle timeline, and is played by the Aka Pygmies of Central Africa.

Note that the toggle properties are of course rotation invariant, and thus all rotations of toggle rhythms are also toggle rhythms. Consider the seven-onset, 16-pulse rhythm in Figure 29.1 (c) that starts with four downbeats followed by three upbeats. Rotations of this rhythm **R**L**R**L**R**L**R**LR**L**R**L**R**L**RL that are common in Brazilian samba music include starting with three downbeats instead, as in **R**L**R**L**R**LR**L**R**L**R**L**R**L**RL, or starting and finishing the sequence with two downbeats each, as in **R**L**R**LR**L**R**L**R**L**RL**R**L**R**L. The former toggle rhythm is shown in clock notation in Figure 29.3 (left), along with a variant (right) that became well integrated in the United States in the 1950s and the 1960s. This variant has been used, for example, as a saxophone riff by Bill Haley in *Rock Around the Clock*, and as a bass riff by the Supremes in *You Can't Hurry Love*. On the surface, the change made to obtain this variant appears minimal: only the last onset at pulse 13 has been moved to its adjacent pulse number 12. The consequences of this small change, however, are significant. For one, the samba rhythm (left) has the rhythmic oddity property but the variant violates it with the antipodal pair of onsets at pulses 4 and 12. Furthermore, the samba has four strong double-off-beat onsets at pulses 7, 9, 11, and 13, whereas its variant has only three. These properties, along with the continuous forward locomotive-like motion, endow the samba rhythm with a distinctly African aesthetic. The variant on the other hand swaps the double-off-beat onset at pulse 13 with the last of the four strong downbeats at pulse

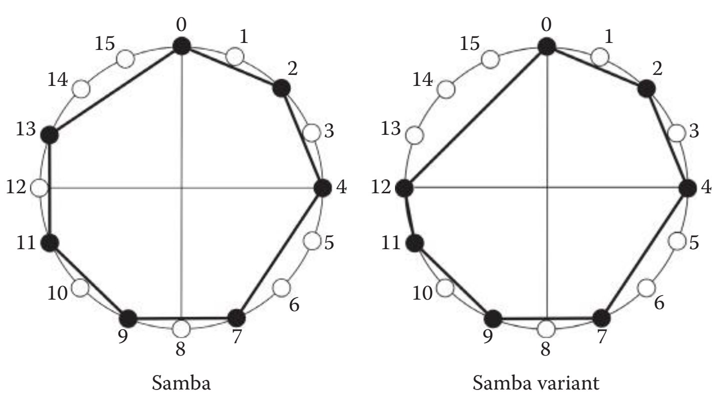


FIGURE 29.3 Brazilian samba timeline (left) and American variant (right).

* Kauffman, R. (1980), Table 1, p. 409.

12, bringing a strong sense of closure to the pattern. This perhaps explains why the variant rather than the original became so popular in American pop music.

As an aside, it is also interesting to compare the inter-onset interval histograms of the two rhythms. The samba is missing intervals of duration one and eight. On the other hand, the variant is an all-interval rhythm. Since the number of inter-onset intervals in a rhythm is one measure of the rhythm's complexity, albeit a weak one, one might be tempted to say that the samba variant is more complex than the original samba. On the other hand, the samba has the rhythmic oddity property and a higher off-beatness value, both of which also measure complexity. Therefore, it seems that the transformation from the original to the variant is exchanging one type of complexity for another.

A similar approach may be used to generate a family of sharp toggle rhythms, as illustrated in Figures 29.4 and 29.5. Figure 29.4 shows a collection of six sharp toggle rhythms ranging in time spans from 4 to 24 pulses, in increments of four pulses. The top of Figure 29.5 shows the shaded four-pulse right–left transition segment that must be inserted into a sharp toggle rhythm to create a new longer sharp toggle rhythm. The remainder of the figure details how the splicing may be done on the sharp toggle rhythm of Figure 29.4 (b). Note that here the cut made between the right-hand onsets and the left-hand onsets occurs

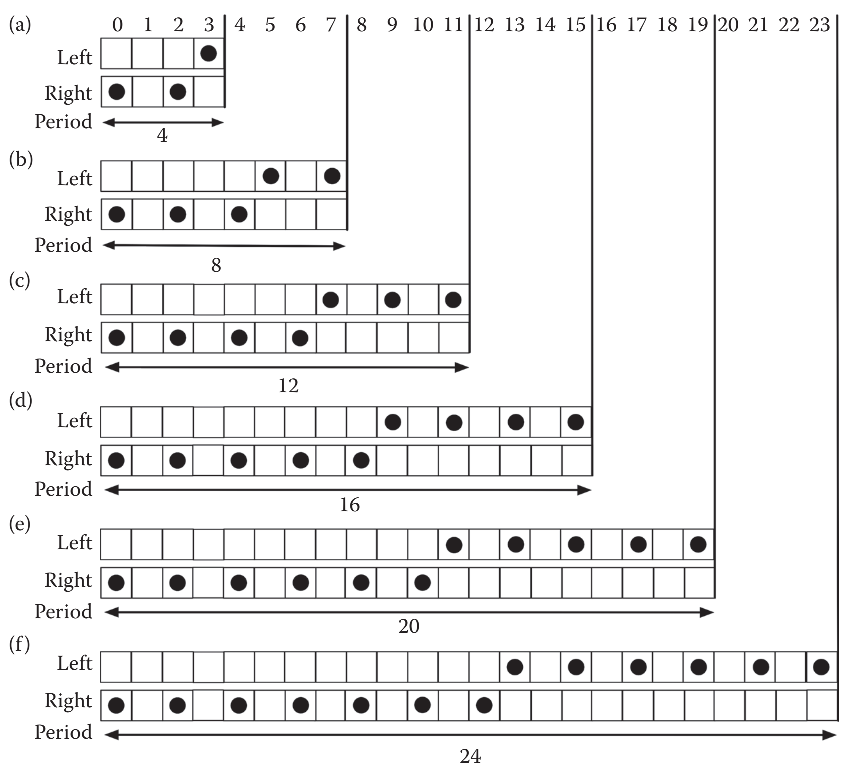


FIGURE 29.4 A method for generating sharp toggle rhythms. (a) A simple sharp toggle rhythm. In (b)–(f) onsets are repeatedly inserted in the previous rhythm at the end of the left-hand pattern and the start of the right-hand pattern.

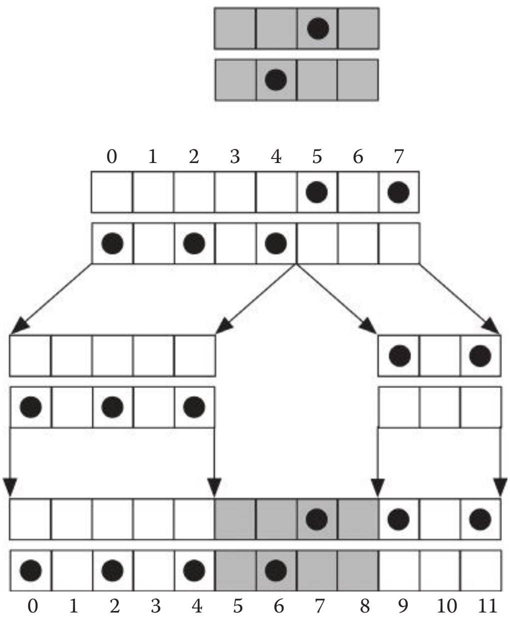


FIGURE 29.5 Splicing a *sharp* toggle rhythm by inserting the right–left transition segment (shaded).

between pulses four and five, and partitions the rhythm into two unequal pieces: one with five pulses and three onsets, and the other with three pulses and two onsets. The resulting new rhythm timeline at the bottom of Figure 29.5 is the seven-onset 12-pulse timeline of Figure 29.4 (c). Of course, another simple method of generating sharp toggle rhythms, if one already has the smooth toggle rhythms to begin with, is to convert a smooth version to a sharp version by adding at the appropriate places one onset to the patterns of each hand. For instance, the smooth toggle rhythm of Figure 29.1 (b) may be converted to the sharp toggle rhythm of Figure 29.4 (c) by adding one right-hand onset at pulse number six, and one left-hand onset at pulse number 11.

The 16-pulse toggle rhythm in Figure 29.4 (d) is a timeline played in several parts of Africa.[*] Sometimes this rhythm necklace and its complement are played simultaneously, the rhythm with the right hand, and its complement with the left.[†] One of the longest sharp toggle rhythms is the *bobanji* timeline played on a metal bell by the Aka Pygmies of Central Africa; it has 13 onsets in a cycle of 24 pulses.[‡] A rotation of this timeline is shown in Figure 29.4 (f). The bobanji timeline is actually started on the fourth onset at pulse number six.

The methods described earlier for composing smooth and sharp toggle rhythm timelines generate rhythms with the property that the first set of onsets played with the right hand has one more onset than the second set played with the left hand. Obviously this property may be easily reversed. Another set of rhythms may be determined by first interchanging

[*] Kubik, G. (2010a), p. 45.
[†] Kubik, G. (2000), p. 288.
[‡] *Ibid.*

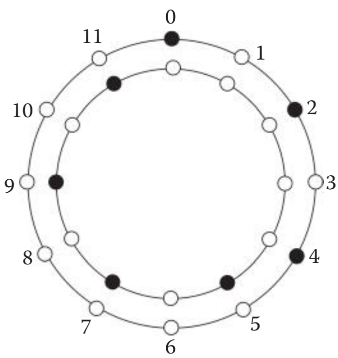


FIGURE 29.6 A double-circle portrayal of a *toggle* rhythm: the right-hand and left-hand onsets are contained on the outer and inner circles, respectively, fusing together to yield the *standard pattern*.

the onsets played with each hand, and then playing the rhythms thus obtained in reverse order. For example, such a transformation applied to the sharp toggle rhythm of Figure 29.4 (c) yields the rhythm **R**L**R**L**RL**RLRL**RL**.

Since rhythm timelines are repeated throughout a piece, and are thus cyclic, it is natural to represent toggle timelines using two concentric circles as pictured in Figure 29.6, where the outer and inner circles mark the right-hand and left-hand onsets, respectively, of a rotation of the *standard pattern*. Since the rhythm has seven onsets, seven different timelines may be obtained by starting the cycle at any of the seven onsets. Indeed, as we have seen in previous chapters, all such rhythms are used as timelines in different parts of sub-Saharan Africa.

The representation of cyclic rhythms on a circle permits an alternate definition of toggle rhythms based on the notion of *linear separability* in geometry. A rhythm (set of integer points on the circle) is a toggle rhythm if there exists a straight line that separates the left-hand onsets (on odd-numbered pulses) from the right-hand onsets (on even-numbered pulses). The rhythm in Figure 29.6, for instance, is a toggle rhythm because there exists a line passing through two points: one being the midpoint between pulses four and five, and another the midpoint between pulses zero and eleven, that leaves all the right-hand pulses on one side of this line, and the left-hand pulses on the other side. Note that although the two circles are drawn as having different sizes for the sake of visualization, they should be considered as one and the same circle for this definition to remain valid in all cases.

The toggle rhythms considered heretofore have the property that the number of right-hand onsets differs by one from the number of left-hand onsets. This is not a requirement for a rhythm to belong to the toggle family. The clave son shown in circular toggle notation in Figure 29.7 (left) has only one of its five onsets played with the left hand, and yet it is a toggle rhythm since it admits a line that separates this onset from all the others, such as, for example, the line through pulses one and five. On the other hand, the bossa-nova rhythm timeline of Figure 29.7 (right) with three right-hand onsets and two left-hand onsets is not a toggle rhythm since it does not admit any line that separates the onsets at pulses 3 and 13 from the other three.

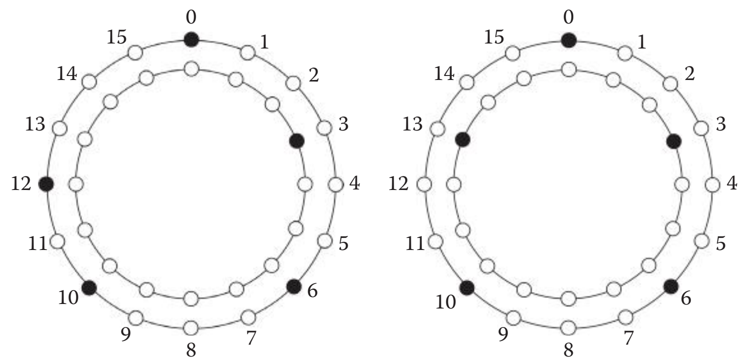


FIGURE 29.7 The clave son (left) and the bossa-nova clave (right) in circular toggle notation.

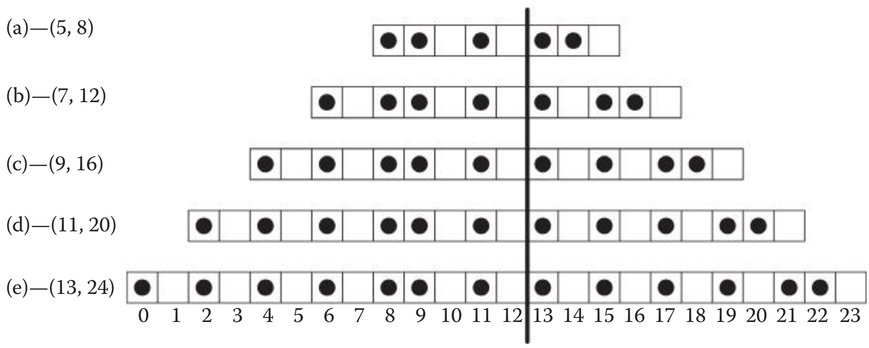


FIGURE 29.8 Kubik's *pyramid* of West and Central African asymmetric time-line patterns.

If the right-hand and left-hand patterns of the sharp toggle rhythms are combined into single sequences, then the resulting timelines bear a close resemblance to the *pyramid* structure proposed by anthropologist Gerhard Kubik to characterize West and Central African asymmetric time-line patterns.* Kubik describes the structural relationships between some common African timeline patterns using the trapezoidal shape pictured in Figure 29.8. The pyramid is built from the top down starting with the five-onset, eight-pulse rhythm in (a), which is partitioned into two pieces of five and three pulses given by [x x . x .] and [x x .], respectively. The next rhythm in the pyramid is constructed by inserting the pattern [x .] at the leftmost end of both parts to obtain the rhythm in (b). This approach is continued to create the rhythms in (c), (d), and (e). Note that putting together the left- and right-hand patterns in Figure 29.4, the rhythms in (b) through (f) are rotations of the five rhythms listed in Kubik's pyramid structure of Figure 29.8. The 24 pulse pattern (e) is the longest asymmetric timeline pattern found in sub-Saharan Africa and was discovered by Kubik and Maurice Djenda in 1966.†

* Kubik, G. (1999), p. 54.
† Kubik, G. (1998), p. 218.

CHAPTER 30

Symmetric Rhythms

Folklore has it that when Napoleon Bonaparte, having been exiled to the island of Elba, set foot there, he exclaimed the words "Able was I ere I saw Elba." Able he probably was, since he managed to escape from Elba within a year, and was soon in power again in France. Whether he knew English well enough to utter these words, and whether he truly said them are open to question. However, one thing is irrefutable: this sequence of letters (ABLEWASIEREISAWELBA) reads the same forward and backward. This sequence of letters exhibits a property called mirror symmetry, in this case about the letter R, the only letter that occurs once in the sentence. A sequence that exhibits mirror symmetry is called a *palindrome*. Palindromes have been the source of both delightful entertainment and serious exploration in a wide variety of domains for thousands of years. Mathematicians study palindromic numbers such as 37485658473.[*] Computer scientists analyze and generate palindromic sequences such as 1000001, as well as recursively palindromic sequences such as 1011101.[†] In a recursively palindromic sequence, the left and right halves of the sequence are also palindromic. Thus, in the latter sequence, the left and right halves given by 101 are also palindromic sequences but in the former sequence, the subsequences 100 and 001 are not. Writers and poets write books, stories, and poems that read the same forward or backward using, letters, words, or sentences as the units.[‡] Artists and designers use decorative frieze patterns that are visual shape palindromes.[§] The frieze patterns in Figure 30.1 are examples of geometric patterns that "read" the same from left to right and from right to left. In other words, each has reflection symmetry about a vertical line through the middle. Note that these patterns also enjoy mirror reflection about horizontal lines through their middles, and they are composed of pairs of adjacent units each of which also has local mirror vertical and horizontal mirror symmetries. The frieze at the bottom is also recursively palindromic at three levels of mirror reflection about vertical lines.

[*] Guy, R. K. (1989).
[†] Ji, K. Q. and Wilf, H. S. (2008).
[‡] Montfort, N. and Gillespie, W. (2002).
[§] Liu, Y. and Toussaint, G. T. (2011, 2010b,c). Shape palindromes have also been called inversions by Kim, S. (1996).

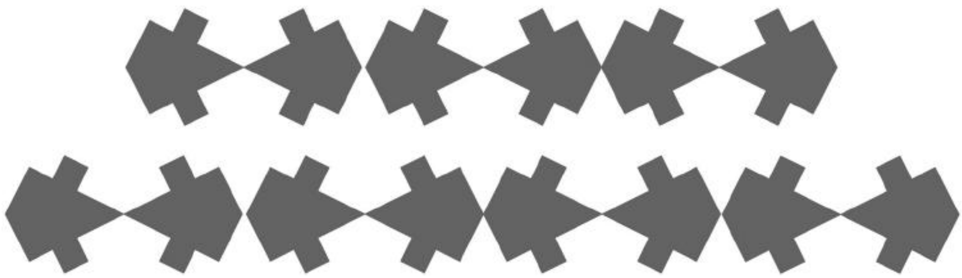


FIGURE 30.1 Palindromic (top) and recursively palindromic (bottom) geometric frieze patterns.

Symmetry is one of the most consequential features of the world we inhabit. Alexander Voloshinov refers to symmetry as "the most important principle of harmony both in the universe and in art."[*] Our frontal appearance exhibits strong mirror symmetry about a vertical line through the center of our bodies, even if our internal anatomy is less symmetric. So, it is not surprising that symmetry should play a major role in composing and perceiving music.[†] Simha Arom writes: "One of the features that is most helpful in perceiving metric structure is undoubtedly symmetry."[‡] Elsewhere, he suggests that symmetry might be a music *universal*.[§]

Many composers have taken advantage of the resources that symmetry has to offer.[¶] Johann Sebastian Bach is a prime example. In his *Contrapunctus*, his notes have mirror symmetry about a horizontal line.[**] In *A Musical Offering*, the second half of the piece is the reverse of the first half.[††] His *Goldberg Variations* "is a musical journey through the world of symmetry."[‡‡] Some composers have written pieces of music that sound the same when played forward or backward. Such compositions are also called *palindromes* or *crab canons*.[§§] A nice example is Joseph Haydn's Symphony number 47 in G major that has a minuet written in 3/4 time (movement number three) that is referred to as *The Palindrome* precisely because it has this property. The melodic contour of the upper notes of this minuet is shown in Figure 30.2, using piano-roll notation. The horizontal axis indicates time in terms of the shortest notes. The bold vertical lines indicate the bars of the 3/4 measures. The vertical axis demarcates the pitch of the notes in semitones. Observe that the section in the right diagram is the mirror image reflection of the section on the left. The two sections are shown separated for the purpose of visualization, but of course in the actual piece they are contiguous.[¶¶]

[*] Voloshinov, A. V. (1996), p. 109. Just as symmetry is much sought in composing music, so is the breaking of symmetry, leading some to characterize the relation in terms of "love–hate." See Wilson, D. (1986).

[†] Donnini, R. (1986), Liebermann, P. and Liebermann, R. (1990).

[‡] Arom, S. (1991), p. 188, Christensen, T. (2002), p. 682.

[§] Arom, S. (2001), p. 28.

[¶] Feldman, M. (1981), Hunter, D. J. and von Hippel, P. T. (2003).

[**] Hargittai, I. and Hargittai, M. (1994), p. 14.

[††] Harkleroad, L. (2006), p. 37.

[‡‡] Du Sautoy, M. (2008), p. 251.

[§§] Hodges, W. and Wilson, R. J. (2002), p. 83. Hodges, W. (2006) discusses other types of symmetries used by many composers.

[¶¶] Handel, S. (2006), p. 188, provides evidence that it is much more difficult to actually hear (temporal) symmetry, than to see it.

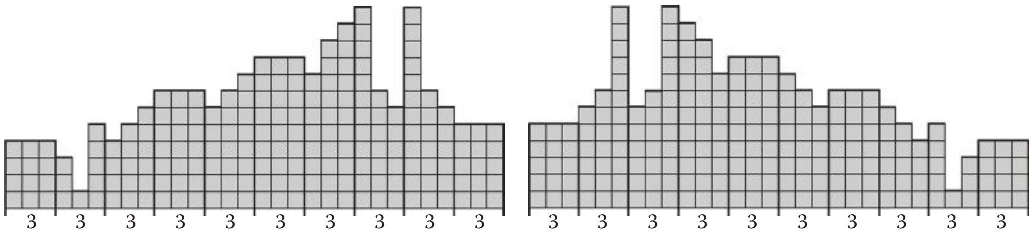


FIGURE 30.2 The palindromic minuet in Haydn's Symphony No. 47.

A *palindromic* rhythm (also called *nonretrogradable* because it cannot be used as a different new rhythm by merely reversing it in time) has the same property as Haydn's minuet: it exhibits mirror symmetry about a point in time.* Two examples of popular palindromic rhythm timelines that have mirror symmetry about their starting points are given in Figure 30.3. When the rhythms are expressed in polygon notation, this type of mirror symmetry translates to symmetry about a vertical line through the starting onset of the rhythm. The first simple but effective rhythmic ostinato (left) is an electric guitar riff used in the minimalist song titled *This Must Be the Place* (subtitled *Naïve Melody*) from the Talking Heads album *Speaking in Tongues*, released in 1983. This rhythm is used extensively in sub-Saharan African music. The unique lyrics of this song were penned by the band leader David Byrne, and the hypnotic and captivating music was composed in collaboration with his band members Chris Frantz, Jerry Harrison, and Tina Weymouth. The second, more complex palindromic rhythm, is the drum "rap" rhythmic ostinato used by Chucho Valdez in his jazz composition *Invitation* (EGREM CD0233, Havana, Cuba, 1997). Many other palindromic rhythms are found throughout this book.

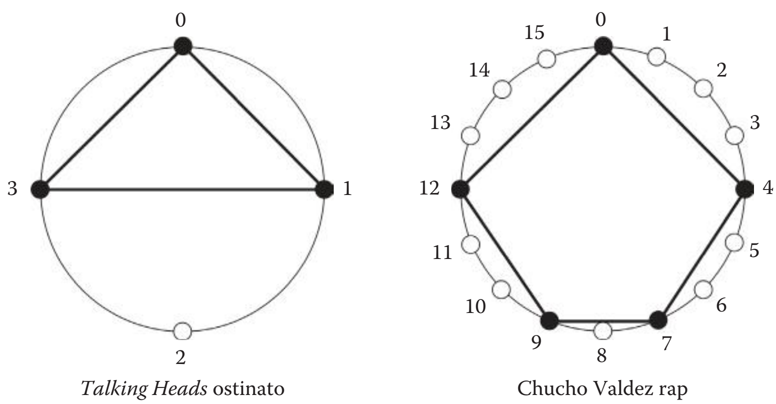


FIGURE 30.3 A pair of popular *palindromic* rhythms with vertical mirror symmetry.

* Palindromic or nonretrogradable rhythms are strongly associated with the French composer Olivier Messiaen. See Johnson, R. S. (1975) and Messiaen, O. (1956), Chapter 5.

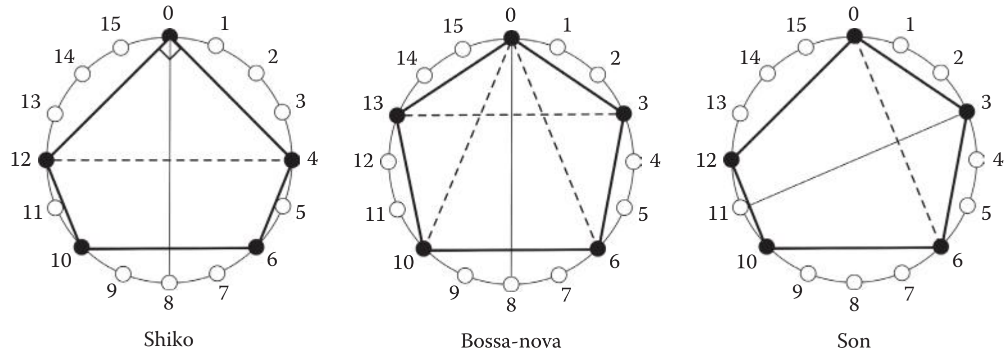


FIGURE 30.4 Three of the six distinguished timelines have reflection symmetry.

Three of the six distinguished five-onset, 16-pulse timelines, revisited throughout this book, and repeated in Figure 30.4 also have mirror symmetry. The shiko and bossa-nova have vertical mirror symmetry, whereas the clave son exhibits a more unusual mirror symmetry with respect to the *diagonal* line through pulses 3 and 11. It should be noted in passing that the term "symmetric rhythm" is sometimes reserved for rhythms that exhibit only vertical symmetry, and the clave son would be described as *asymmetric*. According to the terminology used here however, the clave son is symmetrical, whereas the clave rumba with interval vector [3-4-3-2-4] is asymmetric. The distinction reflects whether the cyclic pattern is viewed as a rhythm with a fixed starting point or as a necklace independent of the starting point.

If the Chucho Valdez rap rhythm of Figure 30.3 (right) is rotated clockwise by 90°, one obtains the rhythm in Figure 30.5 (left), which is employed in Beijing-Opera music, and has horizontal mirror symmetry. Furthermore, if onsets at pulses two and six are inserted in this rhythm, a sub-Saharan African timeline called *kpatsa* results, as shown in Figure 30.5 (right).*

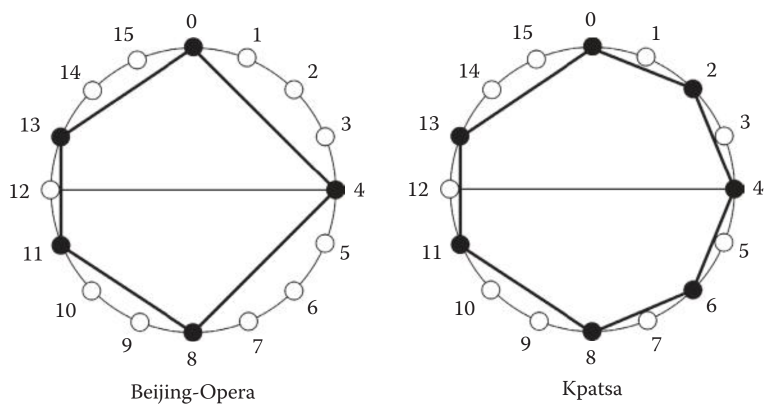


FIGURE 30.5 Two palindromic rhythms with horizontal mirror symmetry.

* Eckardt, A. (2008).

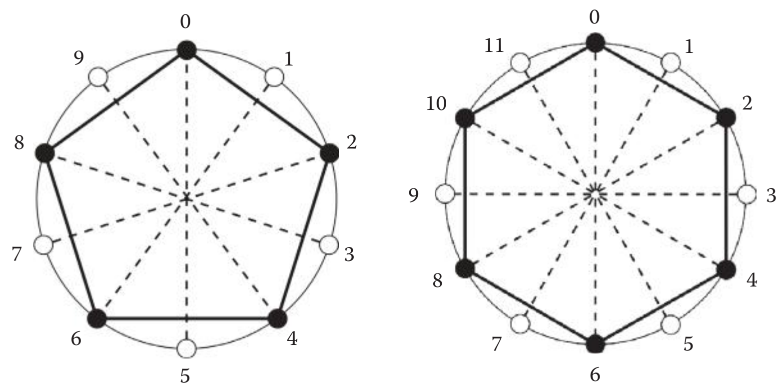


FIGURE 30.6 Regular rhythms possess an axis of mirror symmetry through every onset.

All these mirror-symmetric rhythms contain only one line of reflection. However, rhythms may possess many more such symmetries. Rhythm polygons that possess many axes of reflection will tend to resemble more like regular polygons. Indeed, regular rhythms with k onsets have k distinct lines of reflection symmetry, as illustrated in Figure 30.6. If a rhythm has an odd number of onsets such as the pentagon on the left, then there is an axis of reflection through every one of its onsets. On the other hand, if the number of onsets is even, as in the hexagon on the right, then there is an axis of reflection through every pair of antipodal onsets as well as through the midpoints of every pair of antipodal edges. In these two cases, there is an axis of mirror symmetry through every pulse.

It is useful for the purpose of rhythm classification, as well as for measuring the similarity between rhythms, to distinguish between several coarse types of reflection symmetries that may serve as features for characterizing families of rhythms. In addition to the vertical and horizontal reflection lines, as well as the lines with a positive slope (as in the clave son), we may include lines with a negative slope, as well as a variety of combinations of these four categories.* These combinations permit the distinction between the nine classes of rhythms pictured in Figure 30.7. Traversing from left to right and top to bottom, the first rhythm has all four types of mirror symmetry. The second has vertical and horizontal lines of symmetry. The third has diagonal reflection lines of positive and negative slope. Diagrams four through seven have a single diagonal line of symmetry in all four orientations. The eighth rhythm has one vertical and two diagonal lines of symmetry, and the ninth has one horizontal and two diagonal lines of symmetry.

* Since we are dealing here with rhythms, and not necklaces, the "horizontal," "vertical," and "skew" directions make musical sense because they determine distinct and separate temporal symmetries. Of course the simplicity of recognizing the visual symmetries present in these polygonal representations of the rhythms does not imply that their corresponding temporal symmetries are easy to perceive. It would be interesting to determine the relative perceptual importance of these symmetries in the temporal domain, as well as the correlation with their visual counterparts.

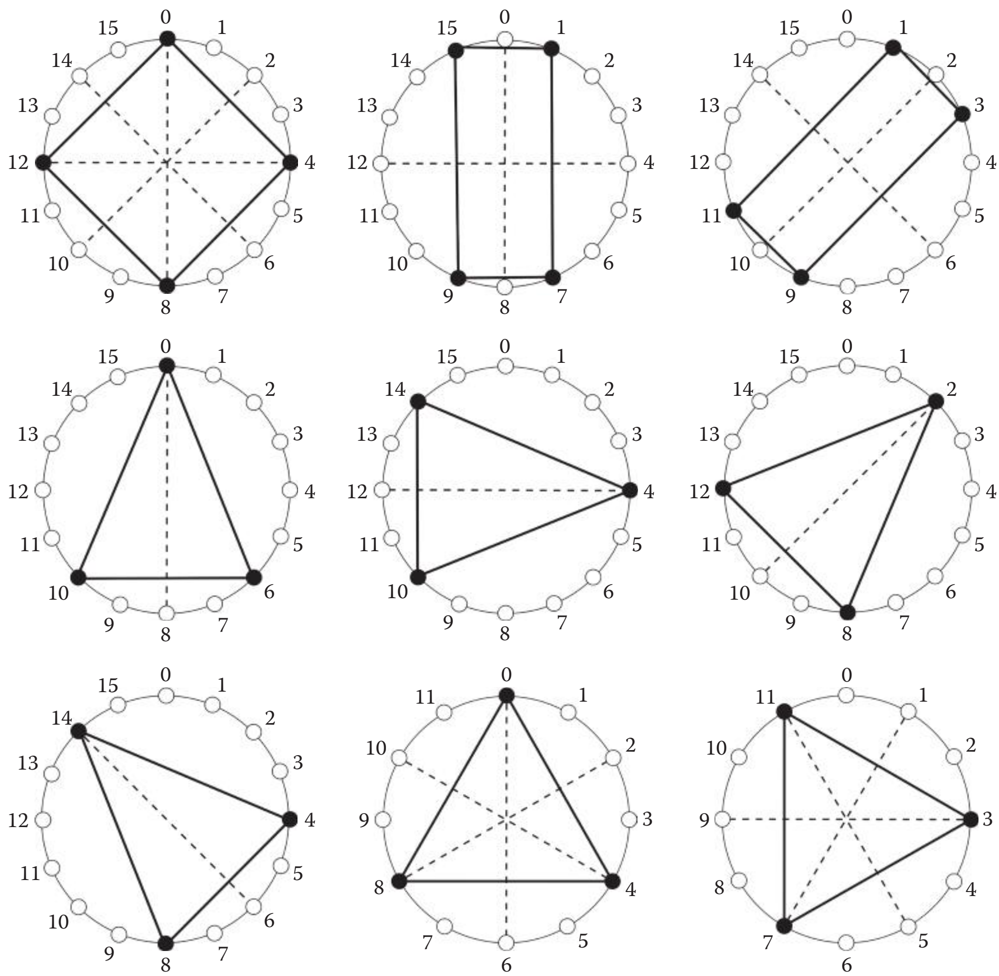


FIGURE 30.7 Some categories of reflection symmetries.

The reader may have observed that Figure 30.7 does not contain the class of rhythms that have exactly two lines of symmetry such that one be either vertical or horizontal, and the other have negative or positive slope. In short, no rhythms are shown that have exactly two reflection lines that are not orthogonal to each other. There is a good reason for this absence: it turns out that such rhythms do not exist. To see this, let us first follow one onset of a rhythm as it is reflected across its lines of symmetry.* As a warm-up exercise, consider the case of two reflection lines, one vertical and the other horizontal, for a cycle of 12 pulses as illustrated in Figure 30.8. Assume that the rhythm we desire has an onset at pulse one. Since the rhythm has a vertical reflection line (left), it must also contain an onset at pulse 11. Furthermore, since the rhythm has a horizontal line of reflection (right), it must also have onsets at pulses five and seven. If we reflect these two new onsets about the vertical

* This is a gentle introduction to group theory. For a more advanced treatment on the application of group theory to music, the reader is referred to Noll, T. (2007).

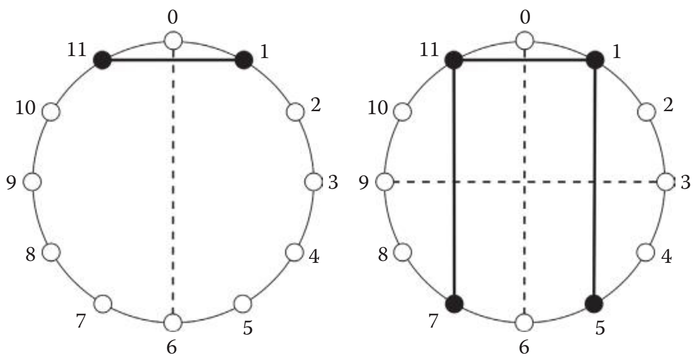


FIGURE 30.8 Generating a symmetric rhythm with two *orthogonal* reflection lines.

or horizontal lines of reflection, no new onsets are created, and thus, the final rhythm has onsets at pulses 1, 5, 7, and 11, and has only these two lines of reflection.

Now, consider applying the same method to create a rhythm that has exactly two lines of reflection such that one is vertical and the other diagonal (refer to Figure 30.9). Assume that the rhythm contains an onset at pulse one and a diagonal reflection line through pulses one and seven. Owing to the vertical reflection line, there must be an onset at pulse 11, and because of the diagonal reflection line, this onset is reflected to pulse three. The onset at pulse three is then reflected by the vertical line to pulse nine, which by the diagonal

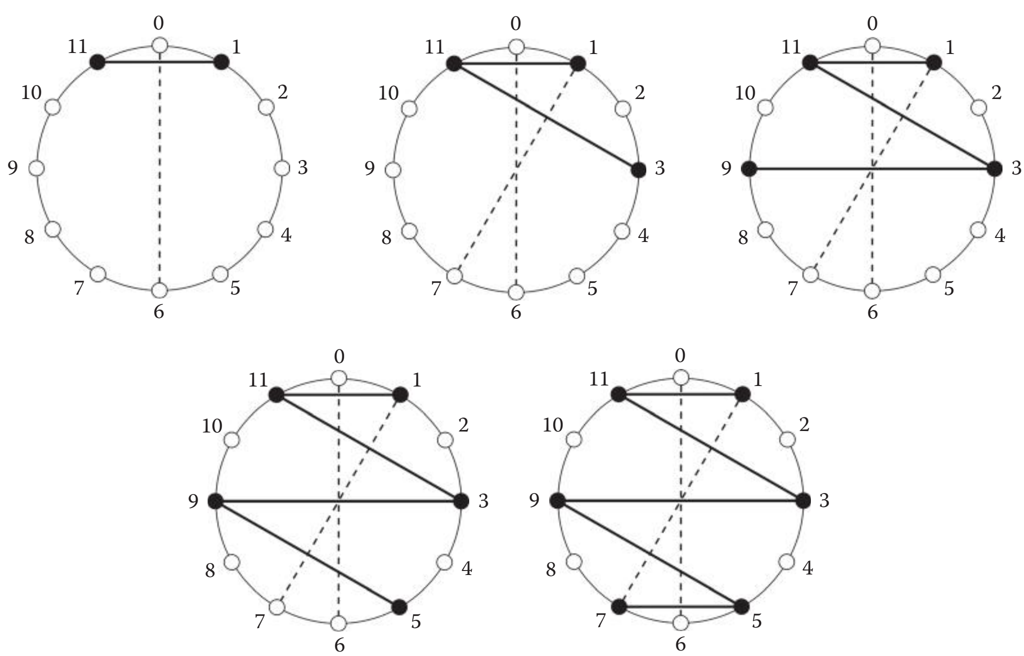


FIGURE 30.9 Attempting to generate a symmetric rhythm with *exactly two nonorthogonal* reflection lines by starting with an onset at pulse one.

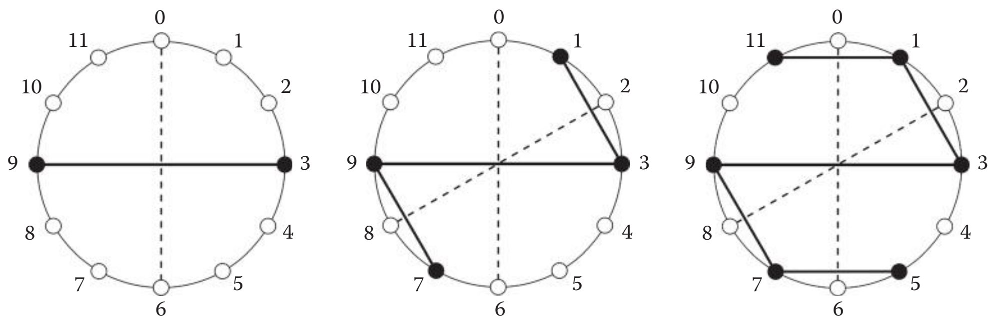


FIGURE 30.10 Attempting to generate a symmetric rhythm with *exactly two nonorthogonal* reflection lines by starting with an onset at pulse three.

reflection line, goes to pulse five, finally ending up at pulse seven, by virtue of the vertical reflection line. The rhythm thus created is a regular hexagon with onsets at pulses 1, 3, 5, 7, 9, and 11. However, this rhythm (being regular) has many other reflection lines: one through every pair of antipodal pulses.

As a second example (Figure 30.10), consider starting with an onset at pulse three and assume that in addition to vertical symmetry, it is desired that the rhythm contains mirror symmetry about the line through pulses two and eight. By vertical symmetry, there must be an onset at pulse nine (left). Because of the diagonal symmetry, there must be onsets at pulses one and seven (middle), and again by vertical symmetry, there must also be onsets at pulses 11 and 5 (right). At this stage, no more onsets are induced by these six onsets and two symmetries. Again, the resulting rhythm is a regular hexagon, which has additional mirror symmetries through every pair of antipodal pulses.

As a final example, consider generating a rhythm with symmetry about the vertical line and the line through pulses two and eight by starting with an onset at pulse two, as in Figure 30.11 (left). Owing to the vertical symmetry, there must be an onset at pulse 10 (left), and by the diagonal symmetry, there must be an onset at pulse six (right), yielding

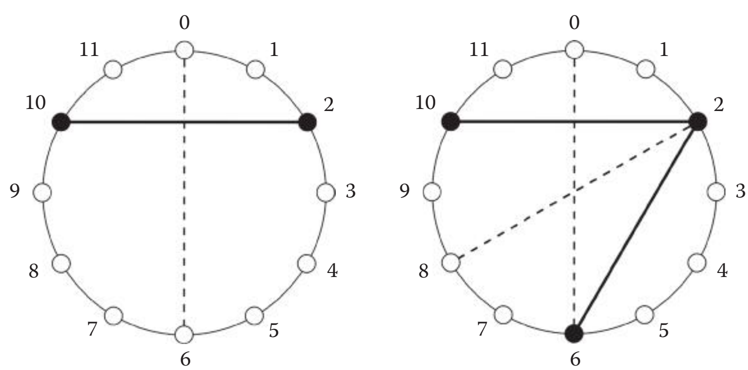


FIGURE 30.11 Attempting to generate a symmetric rhythm with exactly two nonorthogonal reflection lines by starting with an onset at pulse two.

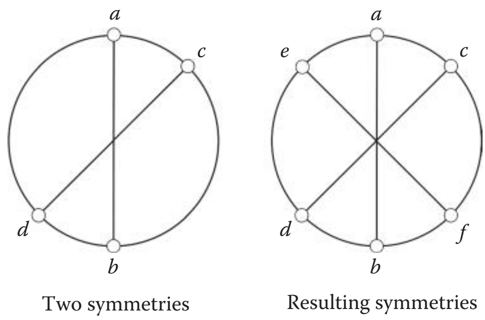


FIGURE 30.12 Two nonorthogonal axes of mirror symmetry (left) always induce a third axis of mirror symmetry (right).

a triangular regular rhythm with additional diagonal symmetry about the line through pulse pair (4,10).

As these concrete examples suggest, no matter how one chooses the nonorthogonal pair of mirror symmetry axes, and no matter which onset one starts with in the iterative process outlined above, one will always generate a rhythm that contains a third axis of mirror symmetry. To see this in full generality, let the pair of points (a,b) and (c,d) in Figure 30.12 (left) denote two arbitrary nonorthogonal axes of symmetry. The symmetry about (a,b) implies that we may think of rotating the diagram out of the page about the axis (a,b) by 180° to obtain the same image as before rotation. But this rotation brings the axis of symmetry (c,d) into the position (e,f) on the right. Therefore, the rhythms must also have mirror symmetry about this axis. Furthermore, axis (c,d) will coincide with axis (e,f) only if the first two symmetry axes are orthogonal. Therefore, there must in fact be three distinct axes of mirror symmetry in such a situation.

HOURGLASS DRUMS AND HOURGLASS RHYTHMS

One special case of vertically symmetric rhythms deserves to be highlighted. Here, these are called symmetric *hourglass* rhythms. Consider the Chinese drum illustrated in Figure 30.13. It consists of a hollow cylindrical body with skins mounted on both sides tied together with string. The main feature of this drum is that the shape of the body resembles an hourglass: the diameter of the drum first becomes smaller until the middle of the drum is reached, after which it becomes larger again.* An hourglass rhythm has the similar property that its consecutive durations first become smaller and then larger. Furthermore, like the symmetric hourglass drum, symmetric hourglass rhythms exhibit vertical mirror symmetry when viewed as rhythms, and other mirror symmetries when considered as necklaces. The examples below are a further special class of hourglass polygons: they are all isosceles triangles. Olivier Messiaen begins his discussion of nonretrogradable rhythms by focusing on such triangles thus: "Outer values identical, middle value free. All rhythms of three values thus disposed are nonretrogradable."†

* A similar hourglass drum called *dhāk* is used in southern India, where the name also refers to a spirit-possession ritual that incorporates the drum (see Roche, D. (2000), p. 63).

† Messiaen, O. (1956), p. 20.

FIGURE 30.13 A Chinese *hourglass* drum. (Courtesy of Yang Liu.)

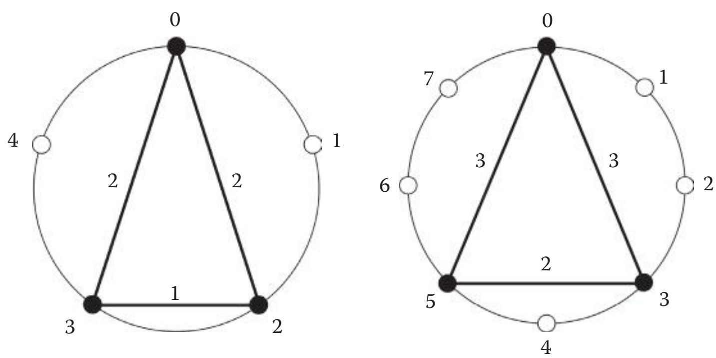


FIGURE 30.14 Two *hourglass* rhythms used in Beijing-Opera.

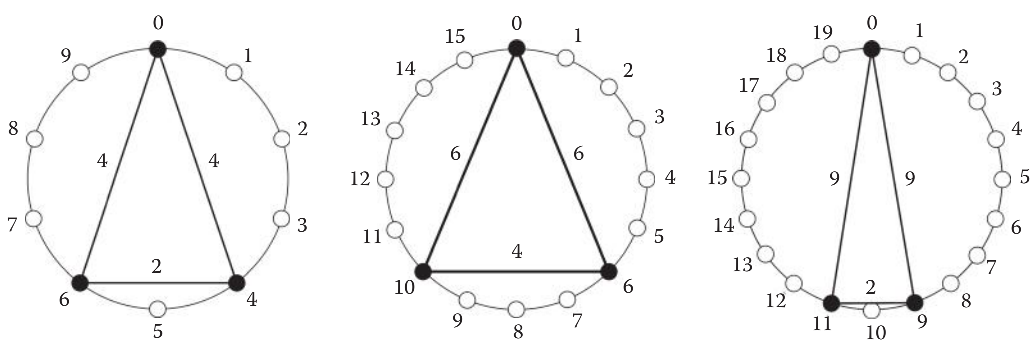


FIGURE 30.15 Three *hourglass* rhythms used in India: *dhenki* (left), *vijaya* (center), and *mathya-samkirna* (right).

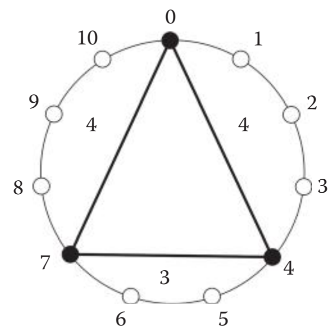


FIGURE 30.16 An *hourglass* Balkan folk rhythm with $n = 11$.

Hourglass rhythms have appeared in many places in this book. For example, the rap rhythm used by Chucho Valdez with durational pattern [4-3-2-3-4] is an hourglass rhythm. Here, we give some examples of notable three-attack hourglass rhythms. Such hourglass rhythms are quite common in Chinese,* Korean,† and Indian‡ music. Two hourglass rhythms with duration interval vectors [2-1-2] and [3-2-3], used in the Beijing Opera, are shown in Figure 30.14. These rhythms are also found in India: [2-1-2] on the left is the *denkhî* and [3-2-3] is the *mathya-tisra* decitala. Three additional Indian hourglass rhythms with $n = 10$, 16, and 20 are shown in Figure 30.15. An hourglass rhythm with 11 pulses used in Balkan folk music is pictured in Figure 30.16.

* Wells, M. St. J. (1991), p. 149.
† Lee, H.-K. (1981), p. 123.
‡ Roche, D. (2000).

CHAPTER 31

Odd Rhythms

In the latter half of the nineteenth century, a group of French archaeologists in North Africa and Greece were busy searching for ancient lost cities that had been buried over time by the geological forces of nature. In Greece, they had been keenly interested in excavating the ancient city of Delphi since 1861, but bureaucratic stumbling blocks prevented them from doing, so until 1893, at which point they promptly made world headlines by discovering a fascinating marble slab, on which they found clearly inscribed characters denoting both the text and the musical notes of a hymn in praise of the God Apollo, one of the most distinguished gods in the Greek Pantheon, known as the God of light, truth, medicine, music, and the arts, just to name a few of his laurels. Although there may have been some controversy about the exact pitch of the notes carved on this marble slab, there was little doubt about the rhythm of the hymn.

Within a year of this important discovery, an international athletic conference was held at the Sorbonne in Paris, where Baron Pierre de Coubertin organized a performance of the hymn, in an effort to kindle interest in reviving the Olympic Games that in ancient times were held every 4 years in the ancient city of Delphi. It seems that this hymn, or perhaps its rhythm, released its magical powers at the conference, because soon thereafter, in the year 1896, the first modern Olympic Games were held at the *Panathenaic* stadium in Athens.*

The rhythms used in the hymn to Apollo struck their discoverers, for they were rather unusual to the European ear of the late nineteenth century. The first five measures of the hymn are pictured in box notation in Figure 31.1.†

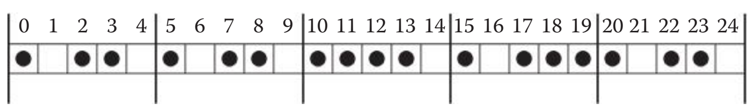


FIGURE 31.1 The rhythms in the first five measures of the *Delphic Hymn* to Apollo.

* Guttmann, A. (1992).

† Abdy Williams, C. F. (2009), p. 39.

What distinguishes these rhythms from most of the common practice in Europe a little over one hundred years ago is that they are composed of measures consisting of five pulses.* In ancient Greece, such rhythms were common but utilized primarily for the more serious music practiced in religious ceremonies, such as sacred dances and hymns, where the expression of intensity of emotion was a desired outcome. Rhythm was extremely important to the ancient Greeks, as it was considered to have the power to produce certain specific emotional effects upon the listener. For instance, the rhythm [x . x .] was considered appropriate in religious contexts because it inspired awe, whereas the rhythm [x x x .] was suitable for military marches because it produced energy. C. F. Abdy Williams wrote in 1911, “To the Greek musician the laws of rhythm were as important as those of harmony and counterpoint are to the modern student.”†

In a tradition that is believed to be hundreds of years old, the Tuareg people, who now live mainly in West Africa, play two rhythms in the performance of their heroic ballads, called *n-geru* and *yalli*, that use the quintuple durational pattern [x . x . .].‡ This rhythm is the same as the second ancient Greek pæonic rhythm shown in Figure 31.2. It is also

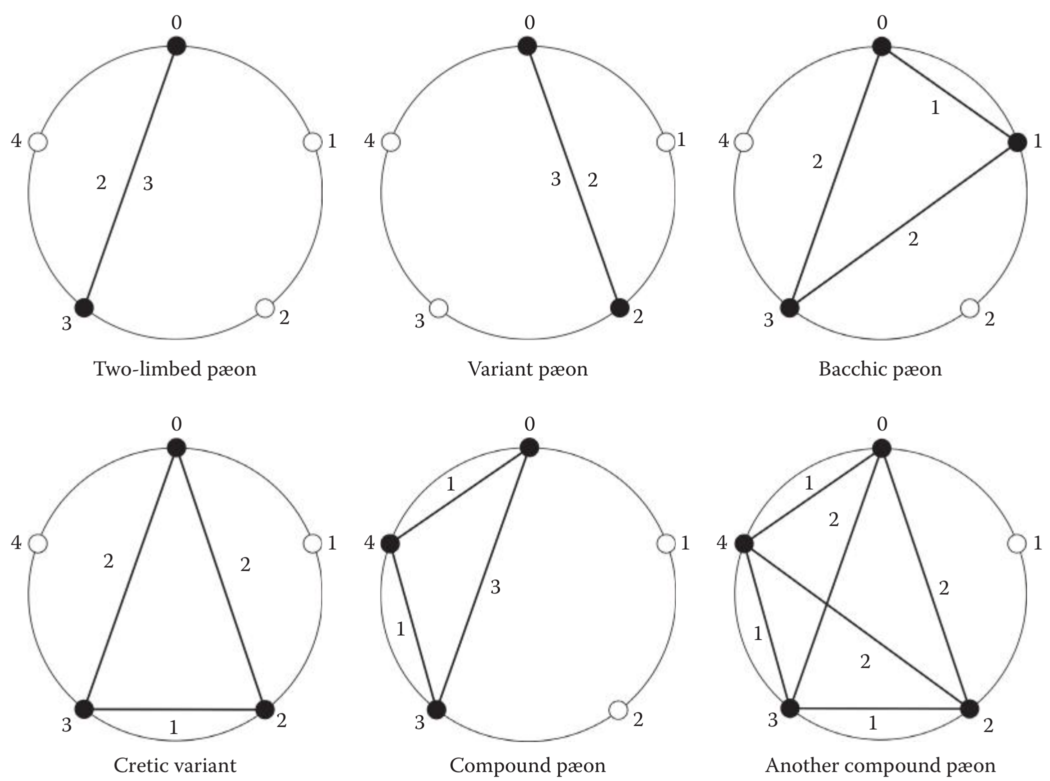


FIGURE 31.2 The *pæonic* or *quintuple* rhythms of ancient Greece.

* This is not to imply measures of five pulses were absent from classical music of the nineteenth century. A noteworthy example is the second movement of Tchaikovski’s Symphony No. 6 in B minor, Op. 74, titled *Pathétique,* composed in 1893. I am indebted to the Harvard University mathematician Noam Elkies for this example.

† Abdy Williams, *op. cit.*, p. 1.

‡ Wendt, C. C. (2000), p. 221.

the metric pattern of the *paiduska* Macedonian dance.* Since the Tuareg people migrated during the past 2000 years to West Africa from a region closer to Greece in North Africa, it is possible that this rhythm was passed on from Greece to Africa, and perhaps it forms part of a tradition of the Tuareg people that is older than previously believed. However, this is mere speculation. Indeed, this rhythm is also an ancient Arab-Persian rhythm dating back to at least thirteenth-century Baghdad called *khafif al-ramal*.† The quintuple meter [2-1-2] (the Cretic variant in Figure 31.2) is also used in Korean instrumental music.‡

The jazz pianist Dave Brubeck was fond of using odd meters in some of his best-selling compositions.§ One nice example of odd rhythmic ostinatos in his music is used in the composition titled *Unsquare Dance*. This piece uses a cycle of seven pulses, and is characterized by a three-onset bass line and a four-onset complementary hand-clapping pattern shown in Figure 31.3. The bass line with durational pattern [2-2-3] is the metric pattern of a Bulgarian Easter dance.¶ When started on the third onset, the meter [3-2-2] is obtained, which is the most common rhythm in Macedonia,** and used, for example, in the *makedonsko horo*.†† Note that both rhythms are *deep*. In the bass rhythm, the interval of duration three occurs once and the interval of duration two twice. In the hand-clapping pattern, the intervals of durations one, three, and two occur once, twice, and three times, respectively. If one rotates the bass rhythm by two pulses in a counterclockwise direction, the pattern [2-3-2] results, which is also the meter of the third movement (*Precipitato*) of Prokofieff's *Piano Sonata Number 7*.‡‡

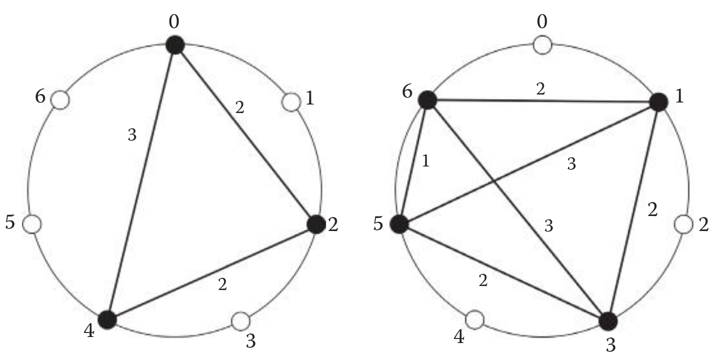


FIGURE 31.3 Seven-pulse rhythms: the bass pattern (left) and the clap pattern (right) in Dave Brubeck's *Unsquare Dance*.

* Singer, S. (1974), p. 386.
† Wright, O. (1978), p. 216. The cycle of the *khafif al-ramal* actually has 10 pulses and so the complete rhythm is [2-3-2-3].
‡ Ku, L. H. (1981), p. 123.
§ Tracey, A. (1961), p. 113. In spite of his legendary successful career and his many fans, some critics were not too sympathetic with his rhythmic explorations. Andrew Tracey criticizes Brubek for not being African enough in the choice of his rhythms, while admitting that time signatures such as [2-2-2-3] look promising. However, it appears that Tracey missed the fact that this time signature is not African, but Turkish.
¶ Rice, T. (2004), p. 51, and Fracile, N. (2003), p. 199, list both [2-2-3] and [3-2-2] in their compilation of Balkan rhythms.
** Singer, *op. cit.*, p. 386.
†† *Ibid.*, p. 43.
‡‡ Sicsic, H.-P. (1993).

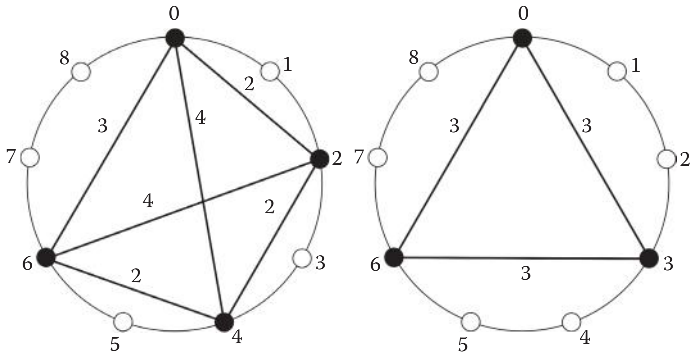


FIGURE 31.4 Nine-pulse rhythms: the two alternating variations in the rhythm timeline of Dave Brubeck's *Blue Rondo à la Turk*.

In Dave Brubeck's *Blue Rondo à la Turk*, one of his biggest hits inspired by listening to street musicians while traveling in Turkey, the rhythm alternates between the two nine-pulse rhythmic ostinatos shown in Figure 31.4. The first rhythm with durational pattern [2-2-2-3] is the metric pattern of the Thracian wedding dance *daichovo horo*,[*] and the Turkish *dum-tek* rhythm *tchifty sofian*, in which case the *dum* is sounded on the first and third onsets.[†] The other onset-rotations of this rhythm, [2-2-3-2], [2-3-2-2], and [3-2-2-2], are also used in Turkey,[‡] and the latter meter is used by Don Ellis in *Strawberry Soup*.[§] These duration patterns are also *deep* rhythms: the intervals of durations three, four, and two occur once, twice, and three times, respectively. In spite of the similarities in titles, Brubeck's piece should not be confused with Mozart's *Rondo Alla Turca—Turkish March,* the Allegretto in his Piano Sonata No. 11, which has a simple 2/4 time signature. Although timelines with nine pulses are rare in sub-Saharan Africa, Robert Kauffman lists this as a common rhythmic pattern.[¶] Another unlikely geographical location in which rhythms with nine pulses are found is Ireland, famous for its *jigs* since at least the seventeenth century.[**] The slip jig (also called hop jig) has a nine-pulse meter with durational pattern [2-1-2-1-3].

Figure 31.5 depicts three 11-pulse metric patterns. The first (left) with durational pattern [3-2-3-3] is the meter in Dave Brubeck's *Countdown*. If it is started on pulse five, it is the meter of Frank Zappa's *Outside Now*.[††] The intervals of durations two, five, and three occur once, twice, and three times, respectively, and hence this rhythm is also deep. The second meter (center) with intervals [2-2-2-3-2] comes from the chorus of *I Say a Little Prayer* composed in 1967 by Burt Bacharach and Hal David for singer Dionne Warwick. Bacharach is a prolific music producer as well as a composer and pianist known for his syncopated rhythms and odd time signatures. The rhythm he used here is also deep: the

[*] *Ibid*., p. 33.
[†] Hagoel, K. (2003), p. 115.
[‡] Cler, J. (1994), p. 194.
[§] Keith, M. (1991), p. 126.
[¶] Kauffman, R. (1980), Table 1, p. 409.
[**] Hast, D. E. and Scott, S. (2004), p. 66.
[††] Keith, *op. cit*., p. 126.

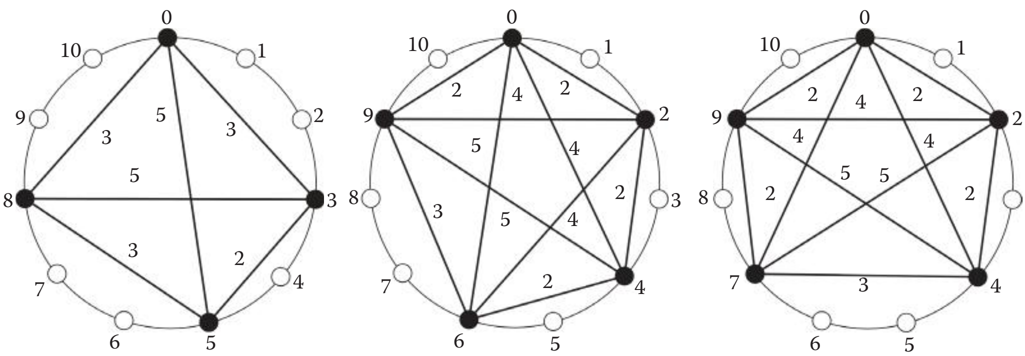


FIGURE 31.5 Eleven-pulse rhythms: the meters of Dave Brubeck's *Countdown* (left), Burt Bacharach and Hal David's chorus of *I Say a Little Prayer* (center), and the meter of the Bulgarian folk dance *krivo horvo*.

intervals of durations three, five, four, and two occur once, twice, three times, and four times, respectively. The third meter (right) is from the Bulgarian folk dance *krivo horvo*.* Since it is a rotation of the second meter, it is of course also deep. This rhythm as well as [2-2-2-2-3], [2-3-3-3], and [4-3-4] are all meters used in Balkan folk songs.†

Three 13-pulse rhythms are pictured in Figure 31.6. The first rhythm (left) with durational pattern [2-3-3-2-3] is a Balkan folk rhythm that was used in Olivier Messiaen's movement No. 20 of his composition *Vingt Regards sur l'Enfant-Jesus* titled *Regard de l'Église d'Amour* (or *Gaze of the Church of Love*).‡ It is another example of a deep rhythm: the interval durations of six, two, three, and five pulses occur once, twice, three times, and four times, respectively. Messiaen was well known for his use of complex rhythms, strongly influenced by the music of India and ancient Greece. He was fond of rhythms with

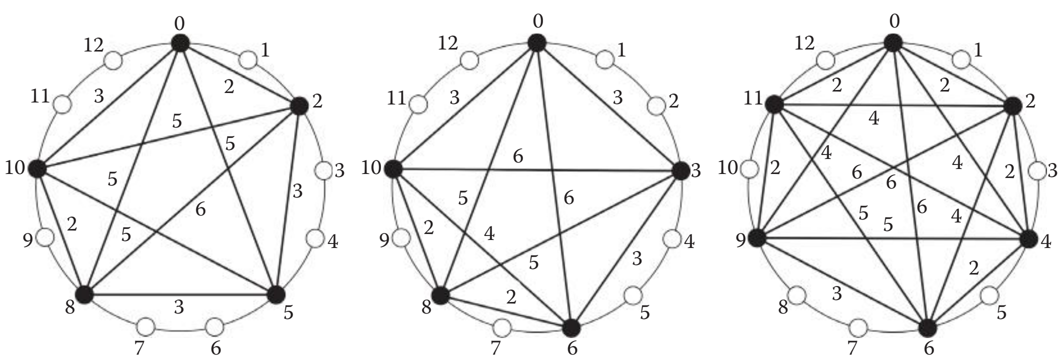


FIGURE 31.6 Thirteen-pulse rhythms: (left) *Vingt Regards sur l'Enfant-Jesus: movement 20*, Messiaen; (center) *Connection* by Don Ellis; and (right) Bulgarian meter *krivo plovdivsko horo*.

* London, J. (1995), p. 68. The generic name for Bulgarian dances that use eleven-beat meters is *kopanitsa*. See Rice, T. (2004), p. 82.

† Fracile, N. (2003), p. 199.

‡ Walt, van der S. (2007).

five-pulse cycles, especially the palindromic (nonretrogradable) rhythms such as the Greek *amphimacer* [4-2-4] and the Indian *denkhî* [2-1-2].[*]

The second rhythm in Figure 31.6 (center) with durational pattern [3-3-2-2-3] is the meter in the composition *Connection* by Don Ellis, a drummer and trumpet player, a composer, and a leader of a big band during the 1960s and 1970s. However, like Dave Brubeck, Ellis is most famous for his compositions with odd meters such as 5, 7, 9, 11, 19, 25, and 33 or more, as well as his polyrhythmic compositions within more conventional time signatures. Some musicologists have used the term "exotic" to describe his rhythms. Don Ellis called them simply "new rhythms" and divided them into the two categories for which he is best remembered: (1) complex rhythms within regular meters, and (2) "odd meters." The latter category, he eagerly points out, does not refer to "strange" or "weird" rhythms, but rather to those with an odd (as opposed to even) number of pulses in their rhythmic cycle, and is the inspiration for the title of this chapter.

In his PhD thesis, Sean Fenlon points out that Ellis divided the exotic meters into two other categories that he called "straight ahead" meters and the more complex "additive" meters. He reserved the term "straight ahead" for those meters that had no accent on any pulse, or had an accent only on the first pulse of the cycle. Thus, [x . . . .] is an example of a "straight ahead" five-pulse meter, whereas [x . x . .] is not. By "additive" meters, Ellis meant the construction of meters by the concatenation of groups with at least one group of two pulses and one of three. Among the many noteworthy examples of such complex additive meters used by Ellis are the odd 19-pulse [3-3-2-2-2-1-2-2-2], the odd 25-pulse [2-2-3-2-3-2-2-3-2-2-2] in *How's This for Openers?* as well as the even 18-pulse [3-2-2-2-3-2-2-2] employed in *Strawberry Soup*, one of his most well-liked and sophisticated compositions.[†]

The third rhythm in Figure 31.6 (right) with interval sequence [2-2-2-3-2-2] is a Bulgarian dance rhythm called *krivo plovdivsko horo*, meaning *Crooked dance from the Plovdiv region*.[‡] A rotation of this rhythm with durational pattern [2-2-2-2-2-3] is the meter of a Balkan song.[§] The characteristic feature of Bulgarian dance music, and Balkan music in general, is the use of asymmetric meters composed of groups of two-pulse, three-pulse, and four-pulse beats. This particular 13-pulse rhythm is also deep: the intervals of durations three, five, six, four, and two occur once, twice, three times, four times, and five times, respectively.

[*] Šimundža, M. (1987, 1988).

[†] Fenlon, S. P. (2002).

[‡] Rice, *op. cit.*, p. 67.

[§] Fracile, *op. cit.*, p. 199.

CHAPTER 32

Other Representations of Rhythm

In this book, I have up to this point used predominantly three notation systems that are most convenient for the types of analyses undertaken: the box notation (in several variations), the convex polygon notation, and the inter-onset interval vector notation (durational patterns). The first two approaches are geometric and emphasize visualization. The third method is a numerical system that indicates duration with numbers. Another noteworthy numerical system is *gongche* notation, the traditional Chinese system that uses numbers to indicate pitches, and dots and lines for rhythm.* The musical information and the notation systems that encode this information are interdependent. They have their unique advantages and drawbacks, and different applications may benefit more from notation systems that are tailored to them.† Music performers, ethnomusicologists, and music psychologists may also benefit more from notation systems that are tailored to their particular needs.‡ In this chapter, several additional notation systems are reviewed, and the contexts in which they are used are indicated. Standard Western music notation is notably left out of this discussion because descriptions of it are ubiquitous.

ALTERNATING-HANDS BOX NOTATION

In Chapter 29, a technique using alternating hands was described for generating toggle rhythms. The technique is also ideal for teaching and learning hand drumming. Recall that it consists of continuously, and gently, striking the drum by alternating with the right

* Lau, F. (2008), p. 52.

† Cohen, D. and Katz, R. (1979). Ellington, T. (1992) focuses on notation systems developed in several cultures. See also Ellington, T. (1992) and Kaufman Shelemay, K. (2000). Koetting, J. and Knight, R. (1986), p. 60, find that box notation is convenient for their "fastest pulse analysis" of rhythm. Brinkman, A. R. (1986) describes a binomial notation system for pitch that codes additional information. For a comparison of several mathematical notation systems, see Liu, Y. and Toussaint, G. T. (2010a). Benadon, F. (2009a) provides tools for notating microtiming in jazz while minimizing the complexity of notation.

‡ For instance, Perkins, D. N. and Howard, V. A. (1976), p. 76, propose a "psychological" notation system that visually reflects the "clustering" or figural perceptual grouping of the attack points of a rhythm.

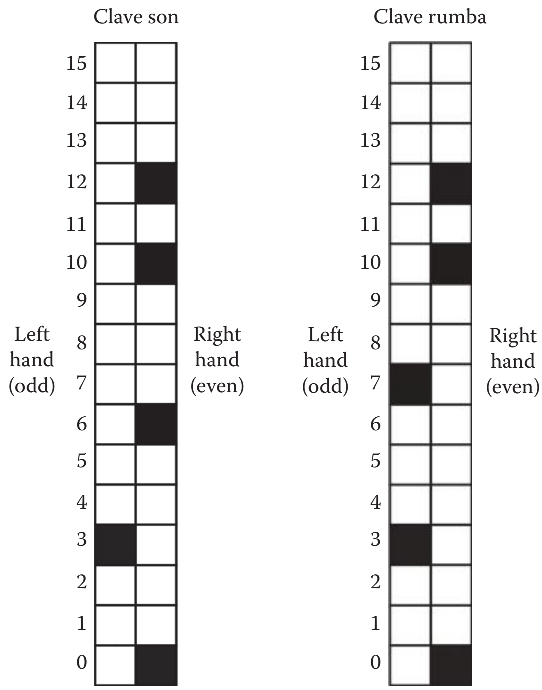


FIGURE 32.1 The *alternating-hands* box notation.

and left hand, much like walking, making sure that all the durations between two consecutive strikes are equal, and that all the sounds made are identical to each other, much like a metronome.* The box notation system can be modified so that it provides a clearer visualization of the alternating hands process, and so that it is easy to read, as illustrated in Figure 32.1. This notation also helps the student to understand and remember the structure of the rhythms more easily. In addition, it helps the student acquire coordination, left–right independence, and timing precision, by embodying a metronomic sense of pulse that is essential for mastering rhythmic performance.

The idea is simple: use two side-by-side sequences of boxes, one for each hand, and rotate them to a vertical position so that the left and right hands play the left and right columns, respectively. In Figure 32.1, time flows from bottom to top. On the left is the clave son, and on the right, the clave rumba. The differences between the two rhythms are highlighted by the onsets played by the left hand. The clave son contains only one left-hand onset at pulse three, whereas the clave rumba contains an additional left-hand onset at pulse seven. In order to develop proper left–right independence, playing with the reversal of the roles of the left and right hands should also be mastered.

SPECTRAL NOTATION

Consider the inter-onset interval durations of the clave son, [3-3-4-2-4], and plot these numbers, in the order in which they occur, as heights of bars in a bar graph such as that

* Konaté, F. and Ott, T. (2000), p. 20. With this technique, also called "hand-by-hand" drumming, the performer playing a beat pattern also feels all the pulses in between the beats, which aids the acquisition of a metronomic precision.

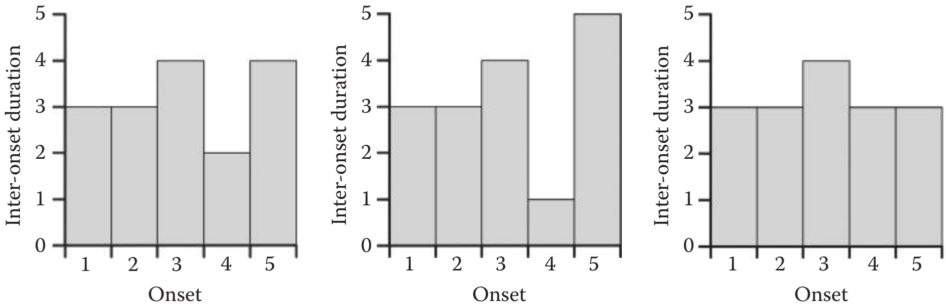


FIGURE 32.2 From left to right, respectively, the *son*, *soukous*, and *bossa-nova* timelines in spectral notation.

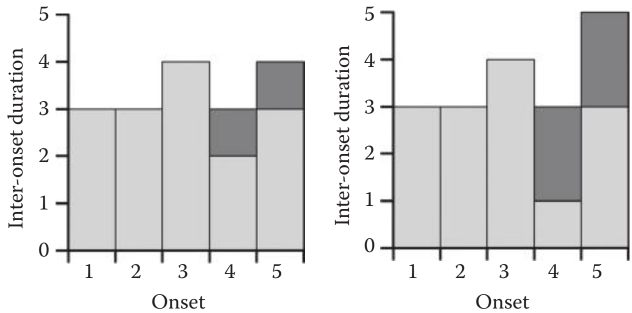


FIGURE 32.3 The area distance between the *bossa-nova* and the *son* (left) and between the *bossa-nova* and the *soukous* (right).

illustrated in Figure 32.2 (left). This type of notation has been used by researchers in the linguistics field interested in the analysis of speech rhythms. This notation provides a compact view of the *spectrum* of the inter-onset intervals in the order in which they appear in time, and it accentuates the irregularity (or regularity) of the rhythms. Hence, we call it spectral notation. In Figure 32.2, it may be perceived quite compellingly that the bossa-nova (right) is much more regular than the soukous (middle).

Any rhythm notation system suggests a variety of new methods for measuring the distance (dissimilarity) between rhythms, and spectral notation is no exception. A natural way to measure distance in this case is by overlaying one spectrum over the other and calculating the area in between the two curves, as shown in Figure 32.3.

On the left, the spectra of the bossa-nova and clave son are superimposed over each other, and the difference between the two curves, highlighted in dark gray, is two. On the right, the bossa-nova spectrum is superimposed over the soukous spectrum to reveal an area between the curves equal to four. Thus, we conclude that the bossa-nova is more similar to the son than the soukous.

TEDAS AND CHRONOTONIC NOTATION

A disadvantage of the spectral notation elucidated in Figure 32.3 is that the veracity of the relative time durations of the inter-onset intervals along the time axis is lost because these intervals are displayed in the vertical direction. In an attempt to recover this valuable lost visual information, while at the same time maintaining some of the advantages of the spectral notation, in 1987, Kjell Gustafson, a researcher at the Phonetics Laboratory of the University of Oxford, interested in displaying speech rhythm, proposed an original and simple method that places time on both the vertical and horizontal axes.* Since an element that displays the durations along two orthogonal directions determines a square, Gustafson called his notation TEDAS notation, an acronym for *Time Elements Displayed as Squares*. Figure 32.4 shows the son, soukous, and bossa-nova timelines in TEDAS notation.

A variant but equivalent notation was later rediscovered by the music psychologist Ludger Hofmann-Engl who christened it *chronotonic* notation, and proposed it for designing a distance measure that was shown by means of psychological experiments to exhibit considerable agreement with human perception of rhythmic dissimilarity.†

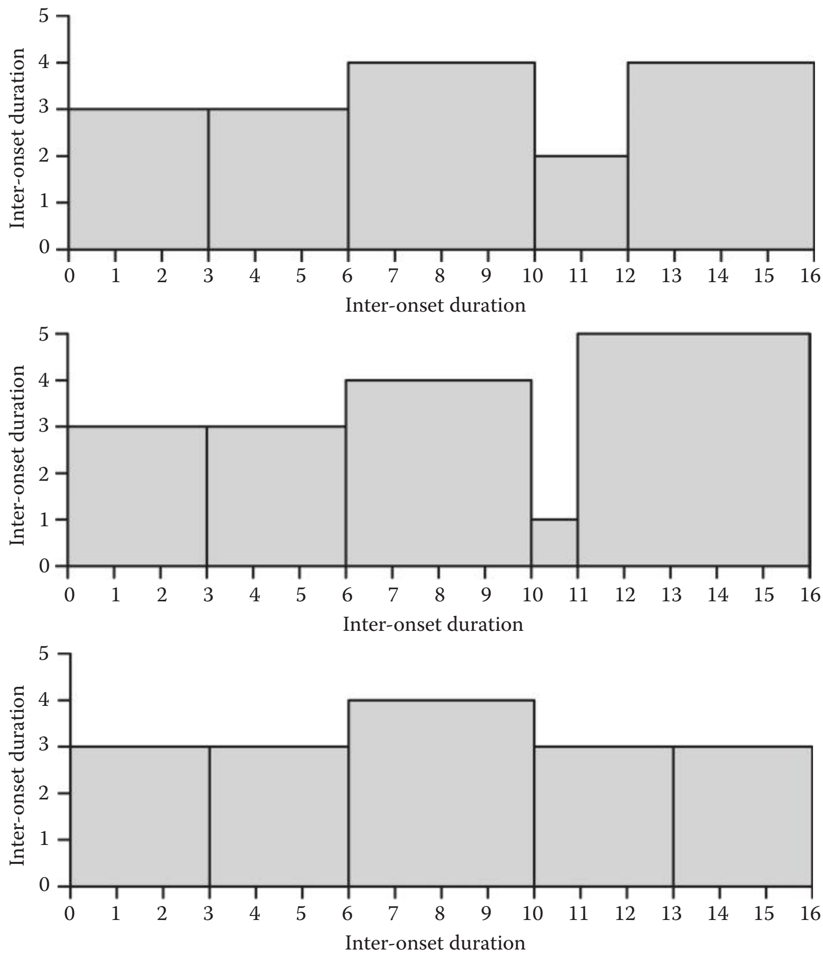


FIGURE 32.4 The *son*, *soukous*, and *bossa-nova* timelines in TEDAS notation.

* Gustafson, K. (1987, 1988).
† Hofmann-Engl, L. (2002).

CHAPTER 33

Rhythmic Similarity and Dissimilarity

In several places in this book, the topic of rhythm similarity has been mentioned briefly in different contexts. In this chapter, this topic is considered in greater detail. Before examining the complex problem of measuring sequence similarity, let us look at the simpler problem of comparing only two shapes. Consider the three pairs of shapes ordered left to right in Figure 33.1. Everyone would without doubt agree that the two shapes on the left are very hard to distinguish from each other, and that the two in the center are more similar to each other than the pair on the right. We are all experts at judging shape similarity. Even newly born infants can distinguish between squares and circles, and they can tell the difference between the faces of their mother and father. The similarity between two objects is one of the most important features for distinguishing between them and for pattern recognition in general. Indeed, the survival of the human species depends crucially on this skill, and therefore it is not surprising that humans are masters at telling the difference between the myriad of shapes they encounter in the world, even when their differences are quite subtle. Can we construct a mathematical formula for measuring the similarity between two shapes, and more importantly for the topic of this book, between two musical rhythms? The artificial intelligence pioneer Marvin Minsky (1981) frames the question thus: "What are the rules of musical resemblance? I am sure that this depends a lot on how melodies are 'represented' in each individual mind."* To give a more complete account of this field of research, and because no single method works well in all applications, in this chapter, several popular methods used to measure rhythm similarity are compared and illustrated with examples. For pedagogical reasons, we begin with the simplest possible measure (the Hamming distance) and then move on to more complex measures, highlighting their strengths and weaknesses along the way.

Hamming distance: The Hamming distance is perhaps the most natural way to measure the dissimilarity between two rhythms represented as sequences of symbols. Consider

* Minsky, M. (1981), p. 35.

FIGURE 33.1 Pairs of (from left to right) identical, similar, and dissimilar shapes.

the two horizontal sequences of eight symbols each in Figure 33.2. They differ in their third, fourth, sixth, and eighth positions. The Hamming distance between two sequences is defined as the number of corresponding locations where the two sequences differ.* In this example, the Hamming distance is equal to four.

This distance measure begets its name from the mathematician and computer pioneer Richard Hamming. During World War II in 1945, Hamming was programming early versions of computers to calculate equations that physicists were using to determine whether or not exploding an atomic bomb would ignite the whole planet's atmosphere and thus possibly destroy the entire human race. After the war, he worked at Bell Laboratories using punched card readers. One of the problems with these cumbersome machines was that they introduced many errors when reading the sequences of input bits (ones and zeros). As a result of this frustration, he invented several coding methods to detect and correct errors, and published them in a seminal paper in which he introduced the Hamming distance. The Hamming distance may be computed very efficiently,† and works well in computer applications where one is interested only in detecting whether two sequences are identical or not, and in correcting the errors when they are not the same.

Measuring the similarity between two symphonies, songs, melodies, or rhythms is a challenging problem in music analysis and technology, and has many applications

FIGURE 33.2 The Hamming distance between these two symbol sequences is four.

* Hamming, R. W. (1986).

† Toussaint, G. T. (2004b). Given two binary sequences with n symbols in each, the Hamming distance may be computed using a number of simple operations that is proportional to n.

ranging from generating playlists to copyright infringement resolution, music information retrieval,[*] phylogenetic analysis, and discovering the evolution of rhythmic patterns and motifs in a style of music.[†] There exist two broad types of approaches to measuring the similarity between objects: *feature*-based methods and *transformation*-based methods. In feature-based methods, objects are compared in terms of the number of traits they have in common.[‡] In transformation-based approaches, similarity is measured by how little effort is required to transform one object to another.[§]

The Hamming distance described above has been used in musicology,[¶] but for some applications, it has a serious drawback. Consider the two pairs of sequences in Figure 33.3. In (a), the two sequences differ in their fourth and fifth positions. In (b), the sequences differ in their fourth and eighth positions. Thus, in both (a) and (b), the Hamming distance is equal to two. Yet, almost everyone will agree that the pair of sequences in (a) are more similar to each other than those in (b) because the symbols that do not match are much further away from each other in (b) than in (a). If the sequences are considered as mere collections of objects, then this separation may not matter much. However, if the symbols make up temporal sequences, then their position in the sequence may be crucial.

Swap distance: The *swap* distance does not suffer from the problem inherent in the Hamming distance outlined above. To define the swap distance, we first need a definition of the swap operation. One of the oldest and most fundamental operations in computer science is the interchange of elements in strings of symbols made up of numbers and

FIGURE 33.3 A weakness of the Hamming distance for musical rhythm analysis. (a) The symbols that differ in the two sequences are close to each other, and (b) the symbols that differ in the two sequences are far from each other.

[*] Bello, J.P. (2011), Antonopoulos et al. (2007).
[†] Crawford et al. (2001), p. 56, Mont-Reynaud, B. and Goldstein, M. (1985).
[‡] Tversky, A. (1977).
[§] Hahn et al. (2003).
[¶] Paiement et al. (2008).

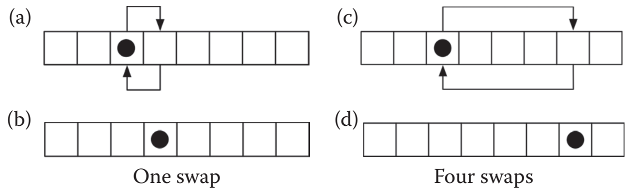


FIGURE 33.4 Illustrating the definition of the *swap* operation.

letters, in the context of the design of algorithms for sorting, dating back to 1974.* When the elements in a sequence that are interchanged are required to be adjacent to each other, the swap operation is called a *mini-swap* or *primitive-swap*.† Here, on the other hand, the shorter term, *swap* is used to mean an interchange of two *adjacent* elements. For rhythms represented as binary sequences in box notation, this means that the operation interchanges an empty box with a filled box when the two boxes lie next to each other. Thus, in Figure 33.4, the sequence in (a) may be converted to that in (b) with one swap. On the other hand, to convert the sequence in (c) to that in (d) requires four swaps. Note that this number of swaps is equivalent to the distance that the source attack in (c) must travel to reach the target position in (d). The *swap distance* between two rhythms is defined as the *minimum* total number of swaps necessary to transform one rhythm to the other, and is equivalent to the minimum value of the sum of distances (measured in number of pulses) traveled by all the attacks (onsets) during this transformation.

Figure 33.5 illustrates the calculation of the swap distance between the shiko and gahu rhythm timelines. Since each rhythm has five attacks, it follows that the first attack of shiko must move to the first attack of gahu, the second to the second, and so on. The arrows indicate the positions to which each attack of the shiko must move in order for it to match its corresponding position of the gahu. The bottom row indicates the cost in the number of swaps that each attack incurs in reaching its goal. The swap distance between the two sequences is simply the sum of all the individual costs. Thus, the swap distance between the shiko and gahu timelines is three.

The swap distance is a special case of the Minkowski metric described earlier in Chapter 12 on the topic of binarization and quantization of rhythms. To see this, represent the rhythms as d-dimensional vectors, where d is the number of attacks, and each element

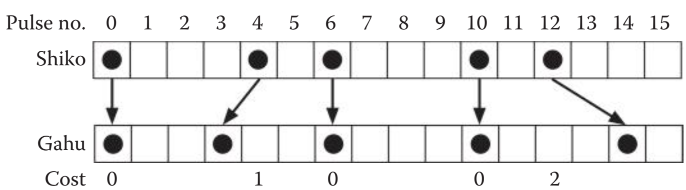


FIGURE 33.5 The swap distance between the *shiko* and *gahu* timelines.

* de Bruijn, N. G. (1974).
† Biedl, T. et al. (2001).

in the vector is the *x*-coordinate of the attack. Consider two rhythms X and Y consisting of *d* attacks each. Let the coordinates of the attacks of rhythm X be $x_1, x_2, \ldots, x_d$, and of rhythm Y be $y_1, y_2, \ldots, y_d$. Recall that the Minkowski metric between rhythms X and Y is given by

$$d_p(X,Y) = (|x_1 - y_1|^p + |x_2 - y_2|^p + \cdots + |x_d - y_d|^p)^{1/p}$$

where $1 \le p \le \infty$. In our example $d = 5$, shiko = X has the vector (0, 4, 6, 10, 12) and gahu = Y has the vector (0, 3, 6, 10, 14). Now, letting $p = 1$, the swap distance is equal to

$$\begin{aligned} d_1(X,Y) &= |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| + |x_4 - y_4| + |x_5 - y_5| \\ &= |0 - 0| + |4 - 3| + |6 - 6| + |10 - 10| + |12 - 14| \\ &= 0 + 1 + 0 + 0 + 2 \\ &= 3 \end{aligned}$$

This first-order Minkowski metric, also called the L_1 norm, because $p = 1$, has been used frequently in pattern recognitition applications since the 1950s, and in 1970, it was shown to be optimal for some applications with suitable probability density functions governing the data.* More recently, it has been applied to voice leading.† One issue that comes up when using *p*-Minkowski metrics for either voice leading or measuring similarity in general is the selection of the value of *p*. Hall and Tymoczko (2012) have developed a novel and powerful approach to deal with this issue by means of the concept of submajorization.‡ Submajorization is a tool that permits the avoidance of total commitment to a particular metric (even beyond Minkowski metrics), while still being able to make similarity judgments that remain true regardless of the metric used.

Directed swap distance: The swap distance described in the preceding works well when both rhythms have the same number of attacks. However, if the two rhythms have a different number of attacks, then it is impossible to convert the sequence with fewer attacks to the sequence with a higher number of attacks, since there are not enough of them. To solve this more general problem, the swap distance may be modified in the following manner. Let us call the rhythm that has a higher number of attacks the *denser* rhythm, and the other the *sparser* rhythm. The *directed swap distance* is then defined as the *minimum* total number of swaps needed by all the attacks of the denser rhythm to convert it to the sparser rhythm, with the constraints that every attack of the denser rhythm must move to an attack position of the sparser rhythm, and every attack of the sparser rhythm must receive at least one attack of the denser rhythm. The term "directed" comes from the fact that the mapping used here is directional, from the denser rhythm to the sparser rhythm.

* Toussaint, G. T. (1970).
† Callender, C., Quinn, I., and Tymoczko, D. (2008).
‡ Hall, R. W. and Tymoczko, D. (2012).

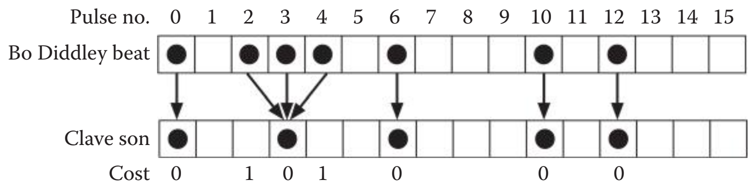


FIGURE 33.6 The directed swap distance between the *Bo Diddley beat* and the *clave son*.

Note that now the basic operation is not a swap between an occupied box and an empty box, but simply a *shift* of an attack by a distance of one pulse. In other words, the contents of an occupied box may be moved to another occupied box, leaving the previous box empty. An example calculation of the directed swap distance between the renowned *Bo Diddley beat* and the *clave son* is illustrated in Figure 33.6. Here, the three attacks at pulses two, three, and four of the Bo Diddley beat all move to the second attack of the clave son at pulse three, yielding a distance of two, between this pair of rhythms.

This distance measure between sequences is also used in computational biology to measure the similarity between long molecules such as DNA, which are modeled as symbol sequences, where it is called the *restriction scaffold assignment* problem.* Several variants and generalizations of the directed swap distance have been proposed in the literature on information retrieval and combinatorial pattern matching, including the *fuzzy* Hamming distance† and the *generalized* Hamming distance.‡ More recently, this distance measure has also been applied to voice leading.§

Many-to-many assignment distance: Although the swap and directed swap distances may work well in some situations where the rhythms are not too different from each other, in some cases, they yield unsatisfactory results in the sense that they may not agree with the judgments of musicologists. It is reasonable to expect that a more realistic distance measure should allow fission as well as fusion of attacks, in the transformation process of one rhythm to another. The *many-to-many assignment* distance serves this function well. Consider the comparison of two popular Baroque Siciliani rhythms in Figure 33.7 given by duration interval vectors [3-1-2-3-1-2] and [3-1-1-1-4-2]. Both rhythms have six attacks

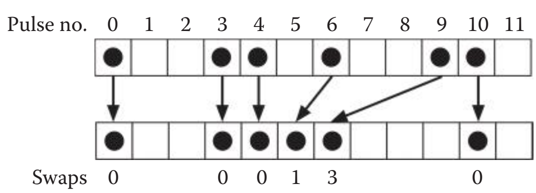


FIGURE 33.7 The swap distance between two Baroque Siciliani rhythms.

* Ben-Dor, A. et al. (2003), Colannino, J. and Toussaint, G. (2005a,b). For an efficient algorithm to compute the directed swap distance, see Colannino, J., Damian, M., Hurtado, F., Iacono, J., Meijer, H., Ramaswami, S., and Toussaint, G. T. (2006). See also Gusfield, D. (1997).

† Bookstein, A., Klein, S. T., and Raita, T. (2001).

‡ Bookstein, A., Kulyukin, V. A., and Raita, T. (2002).

§ Callender et al., *op. cit.*

and so the swap distance assigns the attack at pulse nine of the first rhythm to the distant attack at pulse six of the second rhythm, yielding a rather large distance value equal to 4. However, the attack at pulse nine is a preparatory (anticipation syncopation) attack for the attack at pulse 10. Therefore, a better assignment would match the attack at pulse nine of the first rhythm to the attack at pulse 10 of the second rhythm.

A many-to-many assignment between these two rhythms is shown in Figure 33.8. Note that in this assignment, there is a *fission* operation in which the attack at pulse four in the first rhythm is assigned to two attacks at pulses four and five, and there is a *fusion* operation in which the two attacks at pulses 9 and 10 of the first rhythm are mapped to the attack at pulse 10 of the second rhythm. In this assignment, the resulting distance is two. An efficient algorithm for computing this distance measure also exists.* This distance measure has also been recently applied to voice leading.†

Edit distance: The swap distance described in the preceding works well when both rhythms have the same number of pulses. However, in general, it is desired to compare two rhythms that have different numbers of pulses. The *edit* distance between two sequences of symbols, also called Levenshtein distance after its inventor Vladimir Levenshtein, the father of Russian information theory, is defined as the minimum number of *edits* (or *mutations*) necessary to convert one sequence to the other.‡ The edit operations allowed are *insertions*, *deletions*, and *substitutions*.§ An insertion inserts a symbol into a sequence, thus making it longer. A deletion deletes a symbol from a sequence, making it shorter. A substitution replaces one symbol by another, thus not changing the length of the sequence. These operations allow the comparison of sequences of different lengths. For example, one way to convert the sequence **WAITER** to **WINE** is by means of two deletions (**A** and **R**) to obtain **WITE**, followed by one substitution of **T** by **N**, resulting in a total of three edit operations. In the context of musical rhythms represented as binary sequences, consider converting the 16-pulse clave son to the 12-pulse fume-fume, illustrated in Figure 33.9. One rather

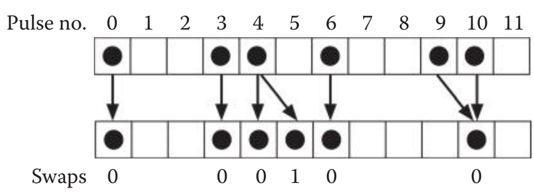


FIGURE 33.8 The many-to-many assignment distance between the rhythms of Figure 33.7.

* Colannino et al (2007). See also Mohamad et al. (2011).

† Callender et al., *op. cit*.

‡ Levenshtein, V. I. (1966). Mutation operations are commonly employed to compare molecular sequences in computational biology. In the music domain, similar operations are called rhythmic or *metric modulation*. Arlin, M. I. (2000) provides an excellent discussion of a variety of metric modulation operations. See also Mongeau, M. and Sankoff, D. (1990), p. 165.

§ Other operations may also be added to generalized versions of the edit distance. For example, Lowrance, R. and Wagner, R. A. (1975) added the *swap* operation to the three standard operations. See also Takeda, M. (2001), Orpen, K. S. and Huron, D. (1992), and Oomen, B. J. (1995).

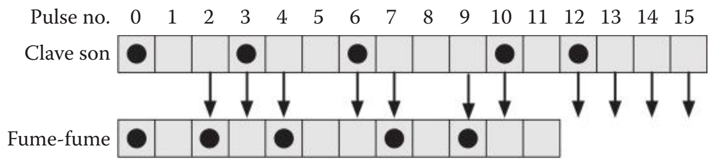


FIGURE 33.9 Eleven edit operations for converting the *clave son* to the *fume-fume* rhythm.

inefficient set of edit operations is obtained by aligning one sequence over the other and performing the required edits in left to right order. This computation yields four substitutions of silent pulses by attacks at pulses two, four, seven, and nine, three substitutions of attacks by silent pulses at pulses 3, 6, and 10, one deletion of an attack at pulse 12, and three deletions of silent pulses at pulses 13, 14, and 15, for a grand total of 11 edits. However, the edit distance between these two rhythms is four. Since the fume-fume is four pulses shorter than the clave son, the minimum possible number of edits is four. Furthermore, this minimum is realizable if four silent pulses are deleted judiciously. In particular, as shown in Figure 33.10, deletion of silent pulses 2, 5, 9, and 15 will accomplish the task.

It may be observed that if one rhythm has many more pulses than another, then the edit distance grows as a function of the *difference* in lengths between the two rhythms being compared, since this many pulses from the longer rhythm must be removed. It may be argued that this difference should contribute to rhythm dissimilarity. However, if this is deemed undesirable, then the rhythms may be mapped to a common number of pulses before applying the edit distance. For example, in the case of the 16-pulse clave son and the 12-pulse fume-fume, they may both be mapped to a 48-pulse cycle. This is akin to a discrete simulation of placing both rhythms on a single continuous circle.

Geometric distance: Just as the spectral notation may be employed to define an area distance between two rhythms by overlaying their spectral representations, the TEDAS and chronotonic notations may be used in the same way to obtain a geometric area distance between two rhythms that is called *chronotonic distance*.* Figure 33.11 illustrates the chronotonic distances (shaded areas) between the son and soukous (distance = 8 at the top), and between the bossa-nova and soukous (distance = 12 at the bottom).

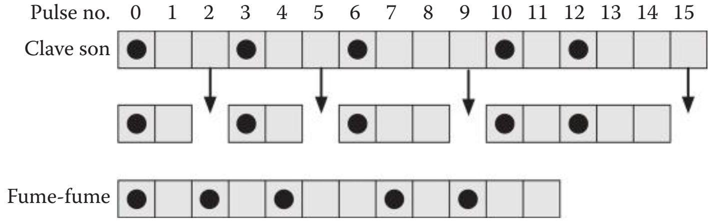


FIGURE 33.10 The edit distance between the *clave son* and the *fume-fume* rhythms is four.

* ÓMaidín, D. (1998) proposed a similar measure, and Aloupis et al. (2006) showed that it could be computed efficiently.

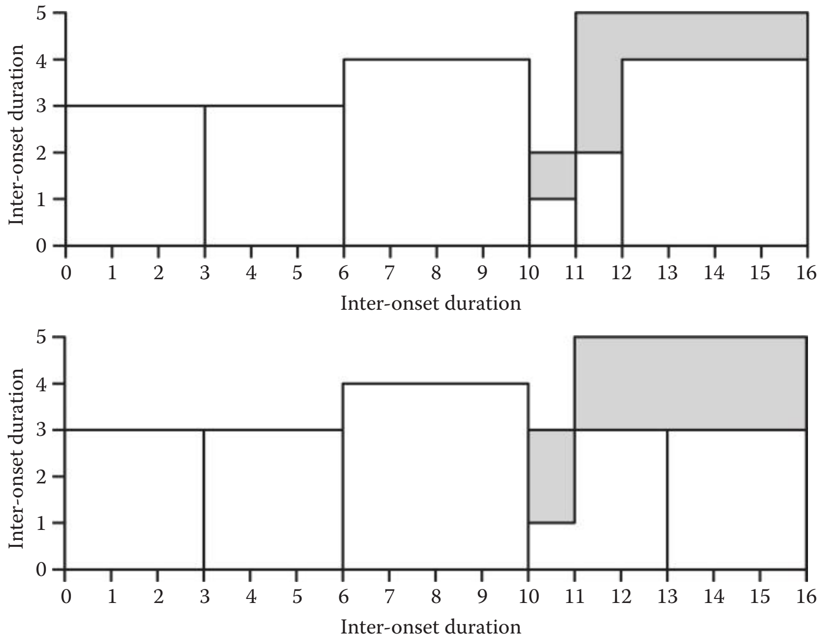


FIGURE 33.11 The area distance (shaded) in TEDAS notation.

There exist many additional mathematical methods for measuring rhythm similarity, including measures designed for comparing time series,[*] histograms,[†] and chords and scales,[‡] which may be directly applied to rhythm,[§] raising the question of which measure one should use in practice. The answer is of course that it depends on the application at hand. There are numerous considerations to be taken into account before choosing a measure, such as the computational complexity of the algorithm available for computing it, how well it correlates with human judgments, and how well it performs in phylogenetic analyses of rhythms.[¶] The Hamming distance may be computed very efficiently in time linearly proportional to the length of the rhythms, but it has musicological limitations. A generalization of the Hamming distance that allows shifting operations, circumvents some of its drawbacks, and may also be computed efficiently.[**] On the other hand, the edit distance may be calculated using dynamic programming,[††] which takes time proportional to the square of the lengths of the rhythms.[‡‡] There is experimental evidence that the edit

[*] Kulp, C. W. and Schlingmann, D. (2009). Berenzweig et al. (2004) provide a survey of acoustic music-similarity measures.
[†] Cha, S.-H. and Srihari, S. N. (2002).
[‡] See Morris, R. (1979–1980), Chrisman, R. (1971), and the references therein.
[§] See for example Scott, D. and Isaacson, E. J. (1998).
[¶] Toussaint, G. T. (2006b).
[**] Jiang, M. (2009).
[††] Smith et al. (1998). See also Bradley, D. W. and Bradley, R. A. (1999), p. 196, for the application of dynamic programming to the comparison of bird songs.
[‡‡] Wagner, R. A. and Fischer, M. J. (1974), Wagner, R. A. (1999).

distance correlates well with human judgments of rhythm similarity.[*] For the specific problem of content-based polyphonic music retrieval, there exist generalizations of the edit distance, such as the earth-mover's distance and the transportation distance,[†] with some evidence that for such applications the geometric methods are superior to the edit-distance approach.[‡] And of course there is Larry Polansky's survey, already mentioned in a previous chapter, for an in-depth comparative analysis of a plethora of metric equations for measuring distance and similarity between a variety of musical objects such as rhythms, scales, contours, tuning systems, and so on.[§]

[*] Post, O. and Toussaint, G. T. (2011), and Toussaint et al. (2011).

[†] Typke et al. (2003).

[‡] Lemström, K. and Pienimäki, A. (2007), p. 148.

[§] Polansky, L. (1996).

CHAPTER 34

Regular and Irregular Rhythms

Once upon a time, a Chinese Buddhist monk traveled to India in search of religious scriptures to take back to China. On his way there, he came upon a flooded river where a large fish offered to help him get across, in return for a favor. The fish requested that while in India, the monk must find out how to achieve salvation. On his return trip to China, having spent many years in India collecting the scriptures, the monk encountered the same flooded river and the same fish to help him across. In the middle of the river, the fish asked the monk about his request, but the monk had forgotten about it. In anger, the fish dumped the monk into the river. Luckily, a fisherman passing by saved the monk, but the scriptures were completely destroyed. Back in China, the monk was extremely angry at the fish. Therefore, he carved a wooden sculpture in the form of a fish and beat its head using a wooden stick. To the monk's surprise, the fish disgorged one character from the lost scriptures he had collected during his travels in India. The monk continued hammering the fish head every day until several years later he had recovered the entire scriptures.

This story describes how the hollow wooden sculpture of the fish head with a slit representing the mouth became a percussion instrument called *muyü*, which is used in traditional Chinese music. The muyü comes in many different sizes and shapes. The one illustrated in Figure 34.1 looks more like a true fish than the more common designs.

Traditionally, the muyü blocks were used in Buddhist temples to accompany the chanting of the monks. However, today, they are employed widely in the Beijing Opera. The rhythms played on this instrument are mostly *regular* rhythms.* Regular rhythms have been imbued with emotional power. K. M. Wilson writes: "If we estimate emotion by its

* Regular rhythms are called *plain* rhythms by Jay Rahn (1983), p. 245.

FIGURE 34.1 Chinese *muyü* temple block. (Courtesy of Yang Liu.)

dynamic quality, the regular rhythms are the stronger."* Regular rhythms may be described on the rhythm circle as regular polygons, that is, polygons with all their sides equal and all their angles equal. The rhythm most frequently played on the muyü has durational pattern [4-4-4-4] shown as a polygon in Figure 34.2 (left). However, the pieces usually end with a more syncopated rhythm such as that shown in the middle, which is quite irregular, has durational pattern [4-5-2-1-4], and uses seven of the eight possible inter-onset intervals, as can be seen from its interval histogram on the right.

A. M. Jones writes: "The simplest rhythmic background to a song is a steady succession of regular claps."† This is probably why regular rhythms are present in almost all the music of the world, especially if they are used for dancing or marching. In modern electronic dance music, the mixing of regular and irregular (asymmetric) rhythms is one of its key

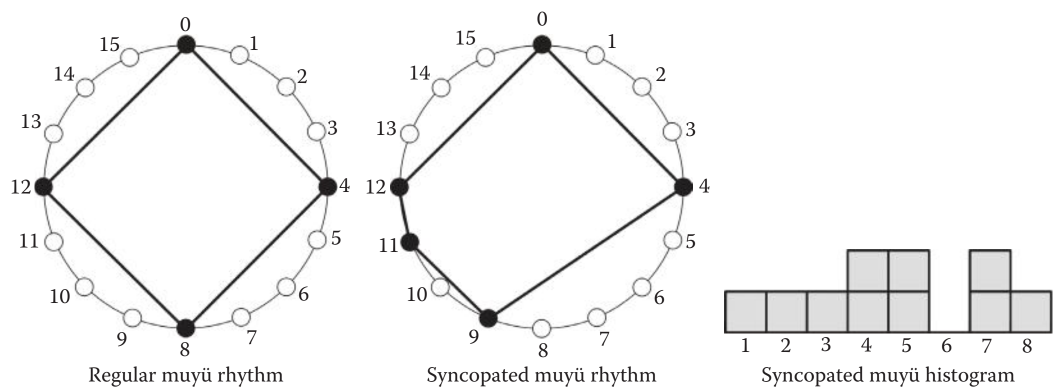


FIGURE 34.2 Typical *muyü* rhythms.

* Wilson, K. M. (1927), p. 4. Concerning irregular rhythms, Wilson writes: "We may say that the more irregular is the rhythm of music, the more intellectual it will be." I think, however, that to characterize regular and irregular rhythms as "merely emotional" and "intellectual" appears to be a rather facile dichotomy. The world is replete with irregular rhythms that are laden with emotional content. The distinction between emotional and intellectual rhythms is perhaps a useful one, but its discriminating feature must surely be more complex than irregularity in isolation.

† Jones, A. M. (1954a), p. 27 and Jones, A. M. (1973/1974), p. 45.

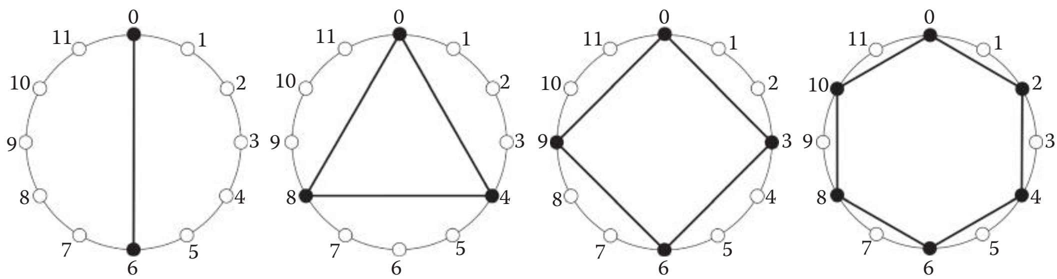


FIGURE 34.3 Four common regular rhythms played simultaneously, on different instruments, in sub-Saharan African music.

features.* In the 12-pulse polyrhythmic music of sub-Saharan Africa, it is common to play simultaneously the four regular rhythms shown in Figure 34.3, each on a different drum, a percussion instrument, or with hand clapping. These four rhythms represent all the possible divisions of the 12-pulse cycle into regular polygons, saving the 12-sided polygon obtained by playing every pulse.

All the rhythms of the world may thus be classified into two categories: *regular* and *irregular*.† Since the latter category is much broader than the former, it is useful to refine it by measuring the degree of irregularity possessed by a rhythm. The *Oxford Dictionary* definition of the term *irregular* is clear enough: "occurring at uneven or varying rates or intervals." Irregular is the opposite of *regular*, which in the context of time in musical rhythm means that onsets occur with the same duration between individual adjacent instances. What we need then to quantify rhythm irregularity is a measure of the amount of unevenness present in the adjacent inter-onset interval durations. It is easy enough to distinguish visually at a glance between a regular and an irregular polygon, at least when the polygon has a relatively small number of sides. Similarly, for a sounded rhythm, it is not difficult to tell whether or not it consists of regularly spaced beats. Thus, there is little doubt in any listener's mind that the seven rhythms in Figure 34.4 are all irregular.

What is more difficult to gauge is which of the two given polygons is more irregular than the other. Consider, for example, the polygons labeled one and seven in Figure 34.4, with inter-onset intervals [1-1-1-1-1-1-2-2-2] and [1-1-2-1-1-2-1-1-2], respectively. At first glance, polygon seven appears more regular because the long edges are evenly spaced out among the short edges.‡ Furthermore, the overall shape of the polygon appears to be a regular triangle with clipped-off corners. However, polygon one is composed of two perfectly regular

* Butler, M. J. (2001).

† In the real world of acoustic rhythms played by performers, there are of course no regular rhythms, if regularity is defined in precise mathematical terms, as is done here. In real performances, rhythms live in a continuous space, no two inter-onset intervals have exactly the same duration, and hence *all* rhythms are irregular in this strict sense. This is further exacerbated in real performances that exhibit expressive variations. Regularity here is analyzed in an abstract music-theoretical setting much as pitches are studied with set theory. In practice, with acoustic input, the concept of regularity should be replaced with *approximate* regularity. In any case, rhythms that are almost regular are often perceived by a listener to be perfectly regular, and that is the important point for this discussion.

‡ This rhythm is the bell pattern of the Afro-Peruvian *Samba Malató*. See Feldman, H. C. (2006), p. 112.

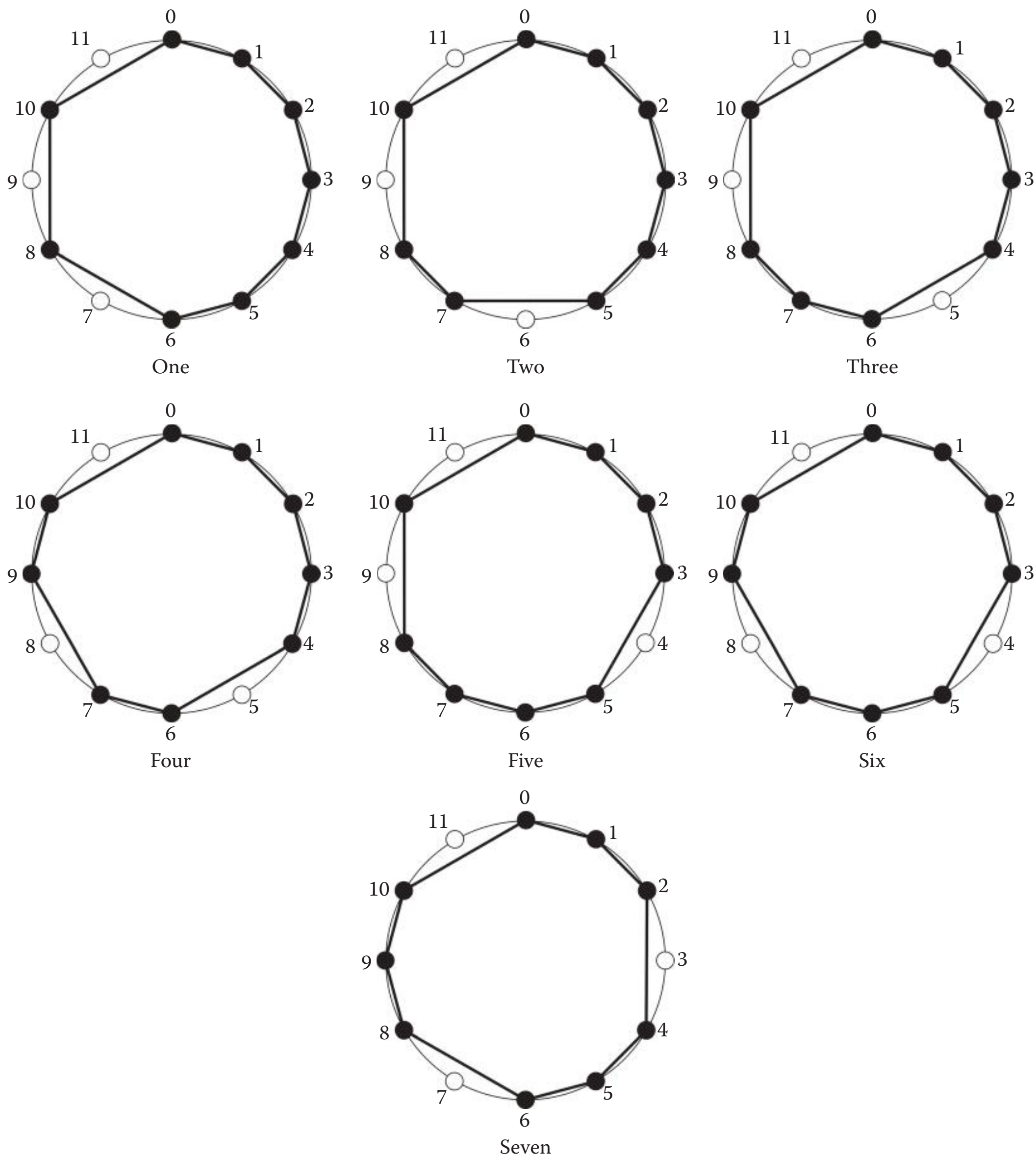


FIGURE 34.4 The seven irregular rhythm *bracelets* containing six adjacent inter-onset intervals of duration one, and three of duration two.

polygonal chains: the left half composed of three long edges, and the right half made up of six short edges. It is as if a regular six-sided polygon and a regular 12-sided polygon were each cut vertically in half, and the left side of the hexagon was glued to the right side of the dodecagon. How does one weigh one of these properties against the other? A psychological solution to the problem would be to ask human subjects which of the two polygons appears to be more regular. On the other hand, a mathematical approach might either measure the irregularity with a local contrast measure such as the nPVI used to measure language rhythm irregularity,* or produce a list of geometric properties possessed by the family of regular polygons, and then count how many of these properties are present in each of the two polygons being compared. However, finding such properties is not a trivial task. In addition to being able to distinguish among the different polygons, such properties should correlate well with human perception of the irregularity of the corresponding rhythms. Interestingly, the nPVI values for rhythms one and seven are 8.33 and 57.8, respectively, and thus, by this measure, rhythm one is considered to be much more regular than rhythm seven.

As for a list of possible geometric properties that one might hope to use to measure regularity, consider a fundamental geometric property of a polygon: its area. One of the earliest theorems proved in geometry by the ancient Greeks states that of all polygons with a given number of sides and total length of its edges, the regular polygons maximize the area. Unfortunately, this theorem does not help to produce a ranking of the seven polygons because they all have the same area. To see this, consider the pair of polygons labeled one and seven in Figure 34.5. Connect each vertex of the polygon to the center of the circle creating nine triangles: three shaded equilateral triangles and six white smaller isosceles triangles. The three shaded triangles are equal and all six small triangles are equal. Since all the polygons may be so partitioned into three of the shaded triangles and six of the smaller triangles, it follows that the two polygons have the same area. Indeed, all seven polygons consist of cyclic permutations of these same nine triangles.

Another geometric property of regular polygons is that they exhibit a large number of mirror symmetries. This suggests that perhaps the more regular a polygon is, the greater the number of symmetries it will possess. Polygon one has one axis of symmetry about the line through pulses three and nine. However, polygon seven has three axes of symmetry about the lines through pulse pairs (1,7), (5,11), and (9,3). This suggests that what seems quite plausible perceptually, that is, that polygon seven is more regular than polygon one, appears to be also true from the point of view of geometric symmetry. Furthermore, rhythm seven is a Euclidean necklace, and thus, maximally evenly distributes nine onsets among 12 pulses.

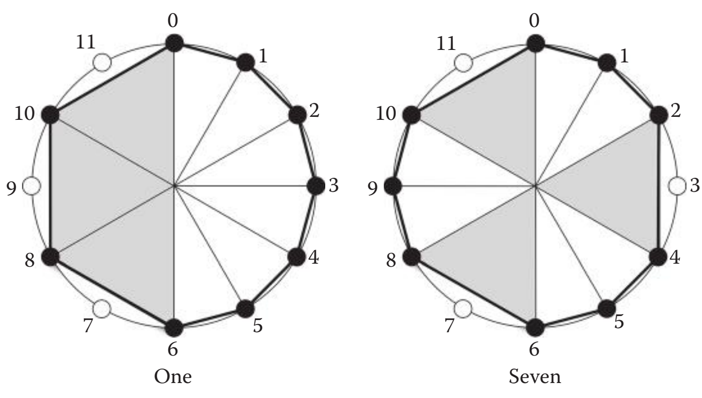


FIGURE 34.5 The areas of all seven polygons in Figure 34.4 are the same.

* See Chapter 17 on rhythm irregularity for a discussion of the nPVI.

The reader may have observed upon viewing the seven polygons in Figure 34.4 that some of them, such as polygons one, four, five, and seven, contain axes of mirror symmetry at various inclinations, whereas the others do not. Now, it is known that in the domain of visual perception, vertical and horizontal symmetries are detected more easily than their counterparts, by the human perceptual system. Therefore, one may wonder if the problem of ranking these polygons by the amount of regularity they possess is made easier when the polygons are rotated so as to make their symmetries more evident. Of course, the number of these symmetries possessed by the polygons is not affected by rotating them.

Figure 34.6 shows the seven bracelets of Figure 34.4 where each is rotated so as to reveal its regularity better. This has been accomplished by rotating the polygons that contain an axis of symmetry, so that the axis is vertical, and rotating the remaining polygons so as to maximize the number of their edges that are vertical and horizontal. The perceptual difference between the two sets of figures is striking, and it appears easier to say from Figure 34.6, which polygons are more regular than the others. It is fair to say that polygon seven is the most regular of them all. Furthermore, polygons two, three, and six are the most irregular since they possess no axes of symmetry. To rank these three among themselves, we may resort to the arrangement of their three long edges. Polygon six may be considered more regular than the other two since all three long edges are separated by short edges, that is, the long edges are more regularly spaced. Finally, we may classify polygon three as being more regular than polygon two because in the former, the two adjacent long edges are more evenly separated from the third long edge (four and two) than in the latter (five and one). In this way, we may conclude that polygon two is the most irregular among this collection.

The preceding discussion focused on measuring the irregularity of polygons. However, as already emphasized, one must be careful in carrying over this visual analysis from the domain of polygons to the aural perception of musical rhythms. For example, in the case of polygons, a geometric feature described in the preceding, such as the presence of axes of mirror symmetry, is rotation invariant. On the other hand, the visual perception of these polygons may change markedly when the polygons are rotated. The same is true for the aural perception of rhythms when they are rotated. Therefore, when measuring rhythm irregularity, it is important to stick to the rhythms themselves rather than their more general necklace or bracelet representations, which ignore the starting onset of the rhythmic cycles.

Several chapters in this book have already dealt with concepts that may be interpreted as implicit or indirect measures of irregularity. They include the measures of syncopation, metrical complexity, off-beatness, and rhythmic oddity. Here, we turn our attention to measuring irregularity directly. One rather obvious way to quantify the irregularity of a rhythm is simply to calculate some suitable distance measure between the given rhythm and a suitable perfectly regular rhythm. This sounds simple enough, but how should the regular rhythms be placed in the cycle? One natural solution is to place the regular rhythms to align with the meter. For example, for some applications, a square in a 16-pulse cycle could be placed so as to create the rhythm [x . . . x . . . x . . . x . . .], rather than say [. x . . . x . . . x . . . x . .], or [. . x . . . x . . . x . . . x .]. For other applications, one may want to calculate

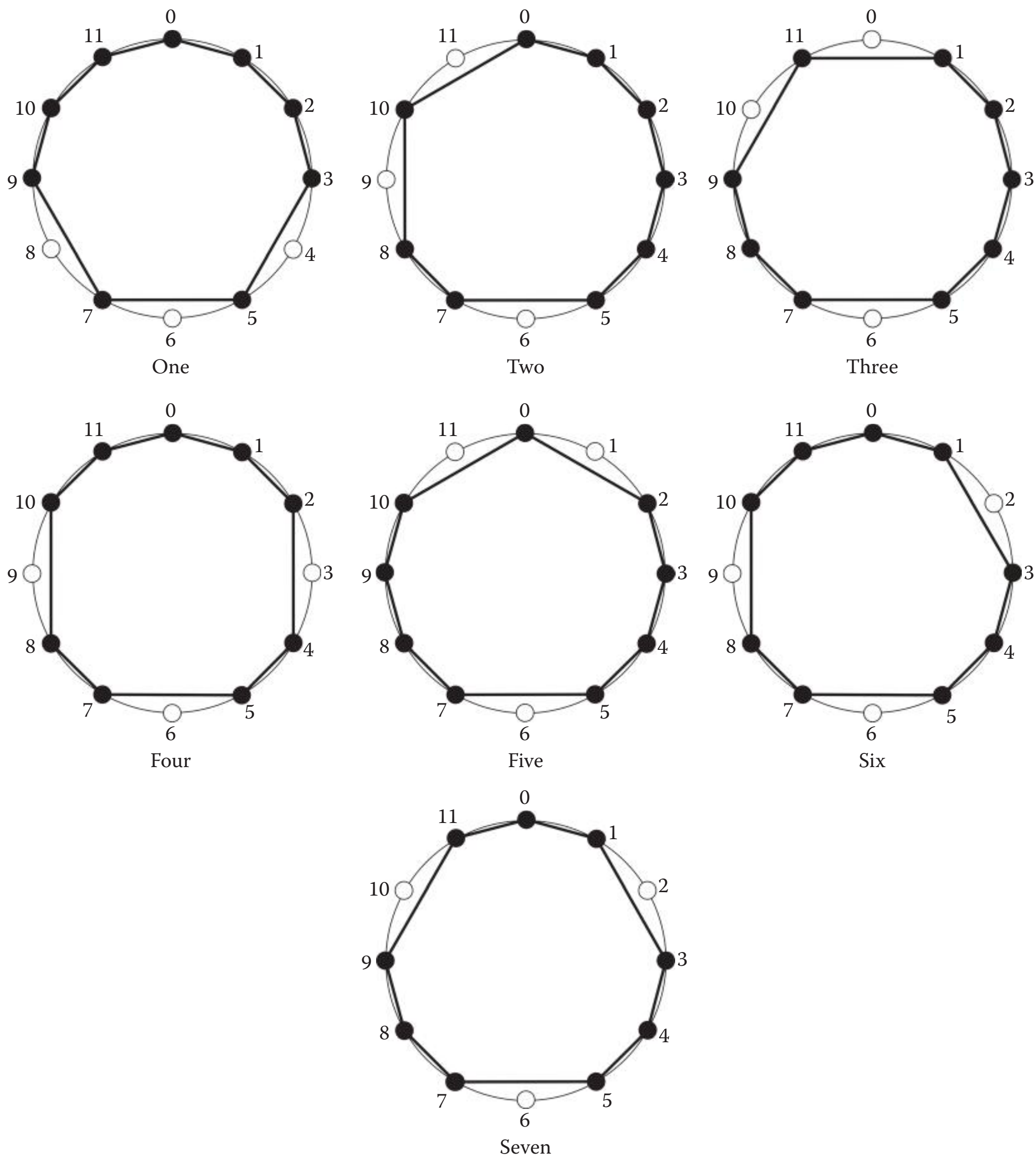


FIGURE 34.6 The seven bracelets of Figure 34.4 rotated to reveal their regularity better.

the minimum distance over all rotations of this pattern. One may even consider rotations of the rhythm in the continuous circle.

Another question is what should be done if a rhythmic cycle admits more than one perfectly regular rhythm. Consider for example the 16-pulse clave son timeline. It admits three regular rhythms with two, four, and eight beats, determined by the durational patterns [8-8], [4-4-4-4], and [2-2-2-2-2-2-2-2], respectively. This scenario suggests several possibilities. If it is desired to measure the irregularity relative to one of these specific underlying meters, then that is the regular rhythm that should be used in the calculation. Figure 34.7 illustrates the calculations for the clave son timeline with eight, four, and two beats per

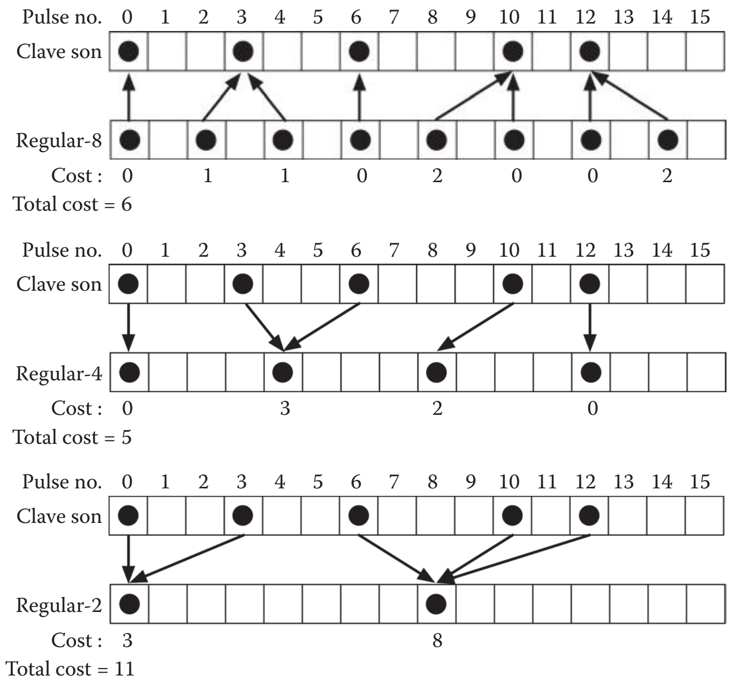


FIGURE 34.7 Converting the directed swap distance to a measure of irregularity.

cycle using the *directed swap* distance measure,* giving costs equal to 6, 5, and 11 (top, middle, and bottom, respectively). If a measure of irregularity is desired that is independent of these three meters, or in the context of polyrhythmic music in which all three meters are present simultaneously, then we may take the average of these three values to obtain $(6 + 5 + 11)/3 = 22/3$. For a comparison, consider the clave rumba given by [x . . x . . . x . . x . x . . .], in which the third onset of the clave son is moved forward by one pulse. The reader may verify that the distances of the clave rumba to the three regular meters are 6, 4, and 10, respectively. Averaging over the three meters yields a value of $(6 + 4 + 10)/3 = 20/3$. Thus, according to this mathematical measure, the clave son is more irregular than the clave rumba. By contrast, most musicologists consider the clave rumba to be more syncopated than the clave son, thus highlighting the difference between the two related notions of syncopation and irregularity. The clave son more clearly separates the "call" or first half of the measure [x . . x . . x .] from the "response" or second half of the measure [. . x . x . . .], than the rumba with counterparts [x . . x . . . x] and [. . x . x . . .]. Thus, by this reckoning, the clave son is indeed more irregular than the clave rumba.

This chapter highlights the variety of methods for measuring the irregularity of a musical rhythm. Intuitively, regularity is related to concepts such as evenness and symmetry. Clearly, a perfectly regular rhythm is also a perfectly even rhythm. What is more difficult is to compare rhythms that are not regular in terms of the amount of regularity they possess, and to determine the role that symmetry plays in the creation of regularity.

* Another distance measure that has been explored recently in this context is the edit distance (Toussaint, G. T. 2012b).

CHAPTER 35

Evolution and Phylogenesis of Musical Rhythm

Anyone who has seen flamenco dancing, or heard classical music favorites such as Georges Bizet's *Carmen*, Claude Debussy's *Iberia Suite*, or Carl Orff's *Carmina Burana* has surely been enchanted by the marked, fresh, high-pitched sound of *castanets*, such as those pictured in Figure 35.1. Castanets are clam-shaped clappers traditionally made of hard woods such as olive trees, which are hung on the thumbs of each hand to allow the fingers to create rhythms by causing the two halves to strike against each other. Although the castanets are visually and aurally compelling, and add spice to the music and dance, the rhythms produced on the castanets play a decorative rather than a central structuring role. The most important rhythm in nearly all flamenco music is the underlying metric pattern used, or *compás* as it is called in Spain, which is often played by hand clapping, and is always felt in an embodied manner by the musician or dancer, even when

FIGURE 35.1 Spanish *castanets*. (Courtesy of Yang Liu.)

it is not sounded. These metric patterns play a similar role in flamenco music as timelines do in sub-Saharan African music, and *talas* play in Indian music.

By some accounts, there are more than 70 different styles of flamenco music in Spain, determined by factors such as the location where the music is popular, whether it is instrumental or voice only, what kinds of instruments are used, in what festivals it is employed, what scales are used, what kinds of stories do they tell (love, war, poverty, persecution, etc.), and what types of rhythms or compás accompany the song. All these different styles use predominantly a ternary cycle of 12 pulses that is typically executed by clapping all the pulses, and marking the compás by accenting certain claps by means of loudness, timbre, or a combination of the two. In the case of timbre, the clap is usually performed either high-pitched and crisp, or low-pitched and muffled. To be sure, other flamenco styles exist that use exclusively binary cycles of four or eight pulses. These include notables such as the *tango* and its variants that include the *tanguillo*, *rumba*, *farruca*, *garrotín*, *zambra*, and *mariana*. However, all these binary styles use one and the same metric pattern given by [. x x x], where "." denotes in this context not a silent pulse (as in the rest of the book), but rather a soft clap, and "x" denotes a loud clap. This chapter is concerned with the phylogenetic analysis of the more characteristic ternary 12-pulse metric patterns employed in flamenco music.

At this point, the reader may be wondering if there is any connection between the 12-pulse rhythms used in flamenco music, and the 16-pulse clave son timeline frequently highlighted in this book. A slight digression here is in order. First, recall that there is a great deal of mathematical, musicological, and perceptual similarity between the binary clave son given by [3-3-4-2-4] and its ternary counterpart with interval structure [2-2-3-2-3]. Furthermore, as described in Chapter 10, there exists some possible evidence that the ternary version may have mutated to the binary version by a process that ethnomusicologist Rolando Pérez Fernández calls binarization, in the context of intercultural transplantation. As for the relation between flamenco music and Afro-Cuban music, historically, there has been a great deal of cultural transplantation between the geographical triangle consisting of Cuba, southern Spain, and sub-Saharan Africa. More than 1000 years ago, the Arabs who lived in southern Spain and North Africa traveled south, exploring much of the African continent, as far south as Mali, Ghana, and beyond, and have written accounts about the music they encountered in their travels. A common metric pattern used in flamenco music such as the guajira given by [x . . x . . x . x . x .] is a rhythm played on the musical bow by the *San* people of Botswana, Namibia, and Angola.* After the Spanish conquest of Cuba, the slave trade created strong cultural exchanges between all three regions. The popular *punto cubano* timeline in Cuba uses the same metric pattern as the guajira, and is often accompanied by the claves that play the derived version [x . x x . x x . x . x .]. In addition, musicologists have certain beliefs based on both historical knowledge and oral tradition, about the evolution of flamenco music. Therefore, flamenco music provides a convenient data set on which to test phylogenetic analysis tools to determine what can be garnered from analyzing the metric patterns. At the same time, such an analysis may shed light on

* Kaemmer, J. E. (2000), p. 314.

the evolution of the clave son and its migrations between Ghana and Baghdad, keeping in mind that the simpler a rhythmic timeline is, the higher is the probability that it was born independently in different places, without necessarily migrating from one place to another.

In the analysis carried out in this chapter, all dimensions of flamenco music other than the time patterns of the meters are ignored. These drastic assumptions are made to simplify the analysis, and at the same time, determine how much information can be obtained from such a minimalist skeletal approach. The approach illustrated is akin to the methods used in bioinformatics that study human evolution by focusing on the DNA molecules, which are sequences of symbols in an alphabet of four letters, while ignoring all the other rich biological and cultural dimensions that characterize human beings.

There is consensus among flamenco musicologists that the fountain of flamenco music is the fandango, which uses the meter [x . . x . . x . . x . .]. This rhythm is symmetric and periodic, repeating the waltz-like metric pattern [x . .] four times. There are in flamenco music the four additional asymmetric 12-pulse meters shown below:*

[. . x . . x . x . x . x]—soleá

[. . x . . . x x . x . x]—bulería

[x . x . x . . x . . x .]—seguiriya

[x . . x . . x . x . x .]—guajira

These four patterns along with the fandango are depicted as convex polygons in Figure 35.2, where the “0” marks the time at which the meter starts, but which may be different from the position at which the rhythm is sometimes “launched” for the convenience of the dancers. Before proceeding further, a caveat should be noted concerning the labels attached to these rhythms. The soleá and bulería patterns are used in a large variety of flamenco styles. Furthermore, the pattern here referred to as soleá is sometimes used in a style named *bulería tradicional*, and the pattern called here bulería may sometimes be employed in the so-called *bulería moderna*. The guajira rhythm is used in fewer styles, but is also most notably used in the petenera. The names adopted here for these rhythmic timelines are used mainly for convenience, but they also reflect the most notable styles that use the meters.

It is known that the bulería pattern is a relatively recent mutation that evolved from the soleá pattern in the twentieth century. It is also interesting to note that of all the five rhythms, the bulería is the only one that has the rhythmic oddity property, a geometric measure of the rhythm’s complexity. This accords with the theory that as patterns evolve, their complexity increases,† and highlights the importance of the rhythmic oddity property in isolating the bulería from the older patterns. The more obvious difference between the bulería and all the other rhythms is that it is the only rhythm that contains intervals

* See Cano, D. M. (1983), Keyser, C. H. (1993), Fernández-Marin, L. (2001), Gamboa, J. M. (2002), Rossy, H. (1966), and Manuel, P. (2004, 2006), p. 102. Parra, J. M. (1999).

† Crofts, A. R. (2007), p. 25.

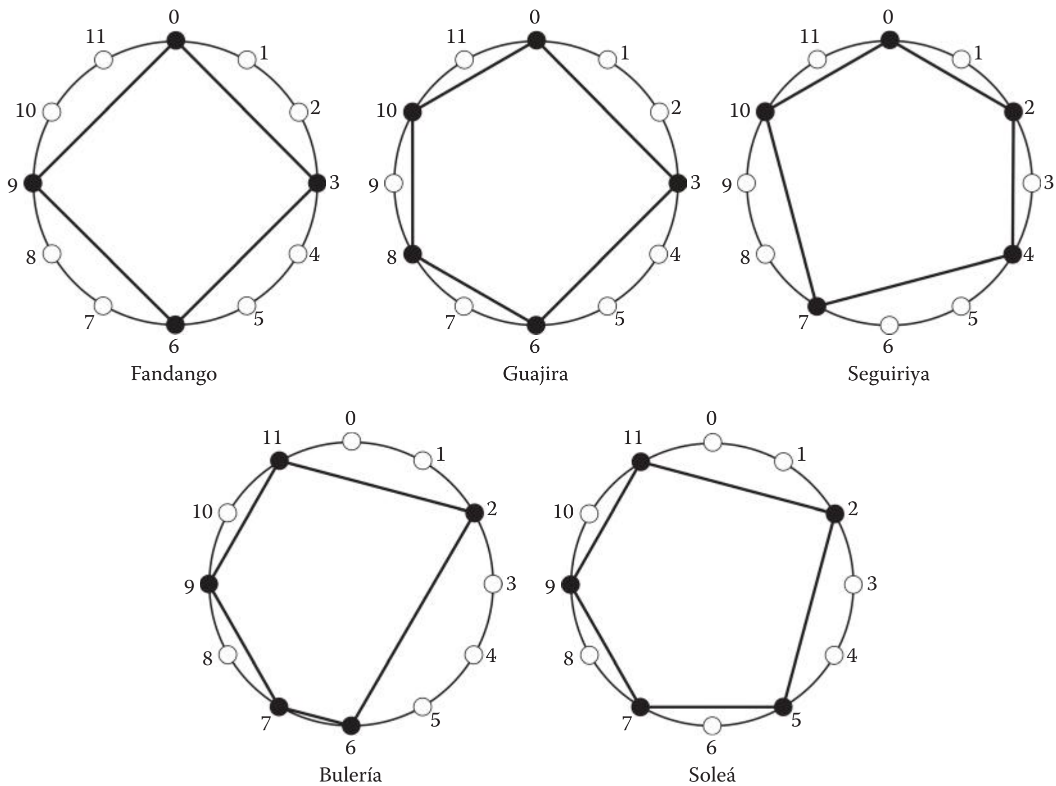


FIGURE 35.2 The flamenco ternary (12-pulse) meters as polygons.

of lengths one, two, three, and four. The other rhythms have intervals of length four only, or of lengths two and three only. There are several measures of rhythm complexity that depend on the variability of these inter-onset durations, the most noteworthy being the standard deviation (SD) and the nPVI. The soleá has SD = 0.55 and nPVI = 10 whereas the bulería has SD = 1.14 and nPVI = 53.8. According to both measures, the bulería is more complex than the soleá, again supporting the complexity theory of evolution.

The guajira, often described as having a significant Cuban influence, is also distinguished from the other four rhythms in that it is the only five-onset rhythm with an off-beatness value of zero.

Since the fandango has four onsets and the other rhythms have five, the directed-swap distance discussed earlier was used to compute the distance matrix shown in Figure 35.3.* The bottom row in this figure, indicated with the Greek uppercase sigma symbol, gives the sum of the distances of the rhythm indicated at the top of the column, to all the other rhythms. This figure in itself already reveals some interesting information. It indicates, for example, that the seguiriya meter with ($\Sigma = 31$) is the most different from the collection, and the fandango and guajira meters with ($\Sigma = 21$) are the most similar to the rest of the group.

* See Guastavino et al. (2009) for a comparison of the directed-swap distance with other distance measures.

	Soleá	Bulería	Seguiriya	Guajira	Fandango
Soleá	0	1	11	7	7
Bulería		0	12	8	8
Seguiriya			0	4	4
Guajira				0	2
Fandango					0
Σ	26	29	31	21	21

FIGURE 35.3 The directed-swap distance matrix for the five flamenco ternary meters.

Ethnomusicologists have not yet fully embraced mathematical and computational methods in their analyses, although this is slowly changing. Stone describes a method of reconstruction of historic patterns in a survey of theoretical approaches to ethnomusicology.* Phylogenetic analysis techniques have been used for quite some time now to study the evolution of cultural objects.† The time seems ripe to add rhythms to the collection, and apply phylogenetic analysis tools to musicology in general.‡

Phylogenetic trees belong to the family of *proximity graphs*§ that provide not only an effective method for visualizing the distance matrix, but yield additional structures such as clustering relationships and information for the reconstruction of possible ancestral rhythms. They may also be used to discover "evolutionary chains" of minimal mutations in motifs frequently used in a particular style or period of music.¶ A variety of techniques have been developed for generating phylogenetic trees from distance matrices.** Some of these methods yield *graphs* containing cycles rather than trees, when the underlying proximity structure between the rhythms is not tree-like. One notable example in this class is a program called *SplitsTree*.†† Like the more traditional phylogenetic tree programs, *SplitsTree* computes a drawing in the plane with the property that the distance between any two nodes reflects as closely as possible the true distance, between the corresponding two rhythms, that is contained in the distance matrix. If the tree structure does not match the data perfectly, then new nodes in the graph may be introduced. Such is the case, for example, in the *SplitsTree* of Figure 35.4, which contains two new nodes indicated with white circles.

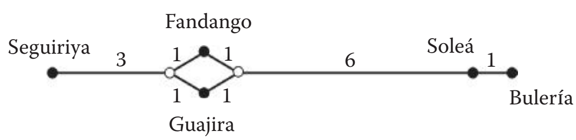


FIGURE 35.4 The *SplitsTree* obtained with the directed-swap distance matrix.

* Stone, R, M. (2008), p. 68.

† Lipo, C. P., O'Brien, M. J., Collard, M., and Shennan, S. J., Eds. (2006), see also Collard, M. and Tehrani, J. (2005), Mace, R., Holden, C. J., and Shennan, S., Eds. (2005).

‡ See Toussaint, G. T. (2012) for a survey of computational tools for phylogenetic analysis of rhythms.

§ There are many other types of proximity graphs that may be used to visualize data. See Toussaint, G. T. (1980, 1988) as well as Jaromczyk, J. W. and Toussaint, G. T. (1992).

¶ Crawford, T., Iliopoulos, C. S., Winder, R., and Yu, H. (2001), Miranda, E. R. (2004).

** Gonnet, G. H. (1994).

†† Dress, A., Huson, D., and Moulton, V. (1996), Huson, D. H. (1998), Bryant, D. and Moulton, V. (2004), and Huson, D. H. and Bryant, D. (2006).

Such nodes may suggest implied "ancestral" rhythms from which their "off-spring" may be derived. In addition, edges may split to form parallelograms, as is also evident in Figure 35.4. The relative sizes of these parallelograms are proportional to an *isolation* index that indicates the significance of the clustering relationships inherent in the distance matrix. *SplitsTree* also computes the *splitability* index, a measure of the goodness-of-fit of the entire splits graph. This fitness value is obtained by dividing the sum of all the approximate distances in the splits graph by the sum of all the original distances in the distance matrix.

The *SplitsTree* of Figure 35.4 suggests a clear clustering structure in which the bulería and soleá form one tight cluster, the fandango and guajira form a second tight cluster, and the seguiriya is off by itself. The phylogenetic tree supports the historical knowledge that the bulería pattern evolved from the soleá pattern. Furthermore, if we believe that the rhythms situated near the center of the tree correspond to earlier ancestral rhythms, this result lends support to the musicological tenet that the fandango is the fountain from which spring all flamenco rhythms. Indeed, in the genealogical trees that musicologists constructed based on historical evidence, the fandango is located low and at the center of the main trunk of such trees. That these trees do not place the fandango at the bottom of the trunk of the tree is not surprising, given that these genealogical trees were constructed with the entire music in mind, including voice and instrumentation, and not the metric rhythm in isolation.

One of the two white nodes on either side of the fandango and guajira nodes is closer to the *center* of the tree than are the fandango and guajira. The center of a tree may be considered to be the point that minimizes its maximum distance to any node. This suggests that the rhythm corresponding to this white node is an ancestor of all these rhythms. However, we do not know what this rhythm is, and therefore, it must be reconstructed. Since the output of the *SplitsTree* program provides the lengths (distances) of all the edges in the graph, from these lengths, the following graph distances, illustrated in Figure 35.5, between the unknown ancestral rhythm and all other rhythms may be inferred:

d(ancestor, guajira) = 1

d(ancestor, fandango) = 1

d(ancestor, seguiriya) = 5

d(ancestor, soleá) = 6

d(ancestor, bulería) = 7.

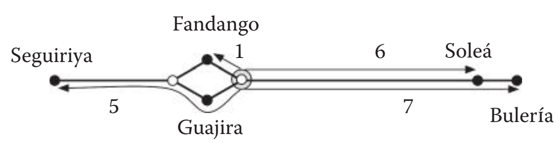


FIGURE 35.5 Reconstructing the ancestral rhythm using distance information.

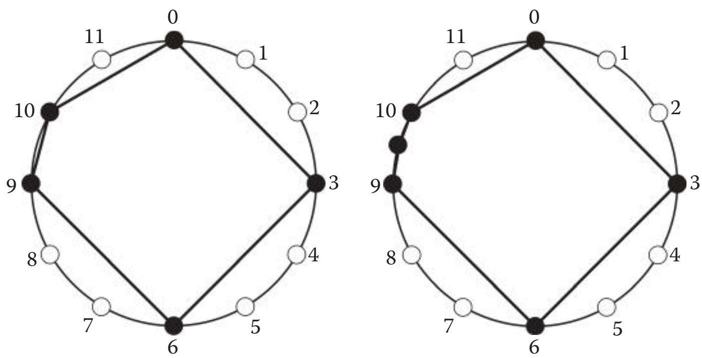


FIGURE 35.6 The "ancestral" rhythm of flamenco (left) and the *fandango de Huelva* (right).

Examining the distances in the graph leads one to suspect that the ancestral rhythm should be quite similar to both the guajira and the fandango, and would therefore consist of either four or five onsets. A computer search leads to the five-onset rhythm given by [x . . x . . x . . x x .] illustrated in polygon notation in Figure 35.6 (left). The reader may verify that this rhythm satisfies the required five distance constraints using the directed-swap distance. This five-onset rhythm is the composition of the fandango and one additional onset at pulse number 10.

A search of the literature to determine if this "ancestral" rhythm makes an appearance anywhere provided the following two observations worth noting. Chuck Keyser mentions in his book that this pattern is used often in flamenco music as a closure or resolution phrase.* Furthermore, it is almost identical, and has the same feel, as a rhythm Nan Mercader calls the *fandango de Huelva*. This rhythm is the composition of the "ancestral" rhythm of Figure 35.6 (left) with an additional onset halfway between onset number 9 and onset number 10. In other words, the *fandango de Huelva* may be described in a time span of 24 units as: [x . . . . . x . . . . . x . . . . . x x x . . .]. Removing the middle decorative onset from the triplet in the *fandango de Huelva* yields the "ancestral" rhythm obtained from the phylogenetic tree.

It is also worth pointing out the close relationship that exists between the seguiriya and an ancient thirteenth-century Arab-Persian rhythm called *al-hazaj*. The latter has a durational pattern [4-3-3-2]† and results when one silences the second onset of the seguiriya.

GUAJIRA

According to Nan Mercader, among others, the guajira rhythm was strongly influenced by Cuban music (the punto cubano in particular) and has the flavor of nineteenth-century colonial Spain.‡ The pattern is also the underlying rhythm of the Mexican

* Keyser, C. H. (1993).
† Wright, O. (1978), p. 216.
‡ Mercader, N. (2001), p. 26.

son.[*] It was notably used by Leonard Bernstein as the metric pattern for his piece "America" in the hit movie *West Side Story*.[†] Supposedly, the rhythm traveled from Cuba to Spain via the Canary Islands where it mixed with some indigenous elements.[‡] The name comes from the word *guajiro* for a creole campesino. The guajira pattern, like the fandango, is located near the center of the graph, and the two have a twin relationship with each other. They both have the same set of directed-swap distances to the other three rhythms. Does this suggest that the guajira, like the fandango, plays a significant ancestral role in the evolution of flamenco? There is evidence to suggest that it does. For almost 800 years, from 711 to 1492, Andalusia, the motherland of flamenco music, was ruled by the Arabs, and North African Moors, who traveled extensively in the African continent, going as far as South Africa. In the Tuareg region of Africa, there exist several 12-pulse patterns that are used for dancing and informal entertainment. In particular, an old rhythm given by [x . . x . . x . x . x .] with the name of *abakkabuk* with inter-onset intervals [3-3-2-2-2] is the same as the guajira rhythm.[§] Furthermore, the *San* people of southern Africa play the same rhythm on a musical bow. Thus, it is possible that the guajira rhythm employed in flamenco music traveled from sub-Saharan Africa to Andalusia by way of the Moors. Conversely, there is no doubt that sub-Saharan African music has been influenced by Arabic music.[¶] There is also evidence that before the Arabs explored West Africa, in the fourth century BC, the Carthaginians set sail across the Straights of Gibraltar and traveled down the African coast.[**] So, perhaps this rhythm traveled to Europe a thousand years before the Arabs conquered North Africa.

This is all well for the ternary rhythms, but is there any connection between the ternary 12-pulse guajira rhythm and the binary 16-pulse clave son? If the guajira pattern on a 16-hour clock is binarized by snapping its onsets to the nearest onsets of a superimposed 16-hour clock, as shown in Figure 35.7, then it becomes the *kpatsa* rhythm of Ghana with durational pattern [4-4-3-2-3].[††] Figure 35.7 employs a 48-hour clock to accommodate both 16-pulse and 12-pulse cycles. The clave son is a rotation and *permutation* of these intervals, in which the interval of duration two flanked by intervals of duration three is moved to the location in between the intervals of duration four. This type of operation is the topic of the following chapter.

[*] Stanford, E. T. (1972), p. 77.

[†] See London, J. (2004), p. 129, and London, J. (1995), p. 65, for an analysis of Bernstein's "America," and a discussion on other complex meters.

[‡] See Manuel, P. (2004) for an account of the complex musical interactions between Cuba and Spain, relevant to the *guajira*, over the past few centuries.

[§] Wendt, C. C. (2000), p. 222. Rotating the guajira by a half-cycle yields the durational pattern [2-2-2-3-3], which is a common African rhythmic pattern. See Kauffman, R. (1980), Table 1, p. 409, and Stone, R. M. (2005), p. 82. Kwabena Nketia, J. H. (1962), p. 132, identifies this rhythm as a handclapping pattern used in *Kwasi Dente*, a recreational maiden song of the Akan people of Ghana. It is also the rhythmic structure of the drumming and handclapping music that accompanies the *sōt silām* dance from *Mirbāt* in the south of Oman. See El-Mallah, I. and Fikentscher, K. (1990).

[¶] Jones, A. M. (1954b).

[**] Fryer, P. (2003), p. 106, and Migeod, F. W. H. and Johnston, H. (1915), p. 414.

[††] Eckardt, A. (2008), p. 64.

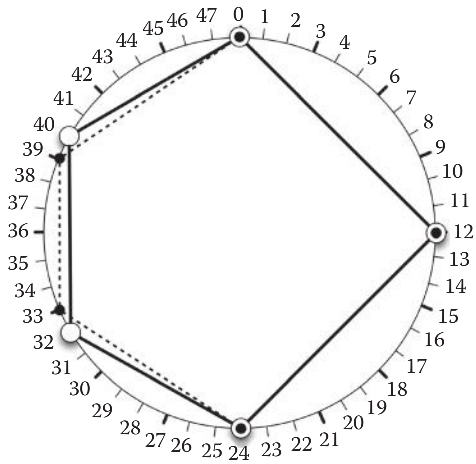


FIGURE 35.7 The binarization of the guajira metric pattern.

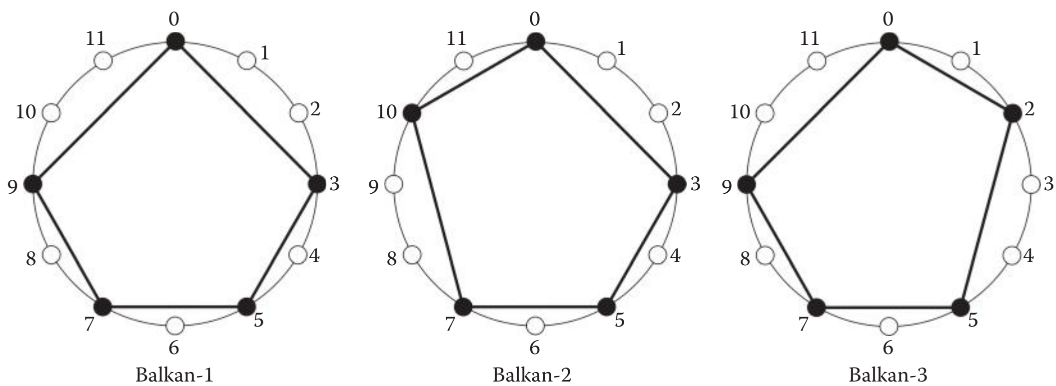


FIGURE 35.8 Three Balkan folk rhythms with five onsets among 12 pulses.

As a final note, it is worth highlighting the close relationships that exist between the Balkan folk rhythms shown in Figure 35.8 and the flamenco as well as sub-Saharan African rhythms. The first Balkan rhythm (left) is a rotation of the guajira. The other two are rotations of the fume-fume or ternary version of the clave son, and have the same rhythmic contour as the clave son. Since the Balkan region lies geographically in between Baghdad and Spain, it is thus conceivable that these rhythms migrated intact and in mutated forms from one place to the other.

CHAPTER 36

Rhythmic Combinatorics

As we have seen several times already, the clave son pattern in binary notation consists of five onsets among 16 pulses. It has also been pointed out early on in this book that the number of ways one may select five items from among a collection 16 items is given by the formula (16!)/(5!)(11!) that yields the number 4368. This is a large number of patterns. However, most of these are unsuitable for use as timeline rhythms. So far in this book, many systematic methods have been described for reducing the size of this large set by discarding rhythms that do not possess certain properties deemed desirable. A different approach, and one that proceeds in the opposite direction, starts out with one excellent rhythm as a *seed* rhythm from which it generates a family of close relatives that hopefully inherit goodness by virtue of their proximity to the seed rhythm. In fact, an example of this approach has already been discussed in the case in which a maximally even rhythm was used to generate a family of almost maximally even rhythms. In that approach, the rhythm was viewed as a binary sequence, and each onset was swapped with its neighboring silent pulse on either side. In this chapter, the rhythm is viewed instead as a durational pattern, or a sequence of inter-onset intervals, and a family of rhythms is obtained from one good rhythm by swapping the positions of the inter-onset intervals of the good rhythm, that is, by generating all the permutations of these intervals.* As an example of this approach, take as the seed rhythm, the clave son.

The clave son pattern in binary sequence representation is [x . . x . . x . . . x . x . . .], which yields the durational pattern [3-3-4-2-4]. The first question is how many permutations of this pattern of five numbers exist? Note that these numbers are *multisets* since repetitions of the elements are permitted. We have five intervals that belong to three different classes: one of class one, two of class two, and two of class three. Therefore, the total number

* In describing his principal method for creating variation in compositions, Joseph Schillinger (2004), p. xi, called these types of operations *general permutations* to distingish them from *circular* permutations. Combinatorial methods for generating and analyzing permutations and combinations of elements in sets is a technique that has been used frequently in music theory. Although Schillinger had many useful ideas, some of his terminology and notation were not well received at the time of his writing. See Backus, J. (1960) for a critique.

of permutations of the interval set [3-3-2-4-2] is (5!)/(1!)(2!)(2!) = 30. All 30 patterns are shown in box notation in Figures 36.1 and 36.2 with the clave son placed first in the list.

The reader is encouraged to play all 30 rhythms for comparison. Some sound better than others, and some are more difficult to play than others, but none are bad, and all of them sound interesting. Among these 30 permutations of the clave son, we find the rumba, the gahu, and the rap timelines, as well as, of course, their onset rotations. If one rhythm may be obtained from another by a permutation of its interval vector, the two rhythms will be said to belong to the same *interval combinatorial class*. Thus, the clave son [3-3-4-2-4], clave rumba [3-4-3-2-4], gahu [3-3-4-4-2], and rap [4-3-2-3-4] belong to the same interval combinatorial class, whereas shiko [4-2-4-2-4], soukous [3-3-4-1-5], and bossa-nova [3-3-4-3-3] each belong to their own distinct classes. Returning to the 30 rhythms of the son–rumba–gahu–rap interval combinatorial class, the remaining 26 rhythms resemble each other to one degree or another but tend to sound more modern, more jazzy. Indeed, any one of them could be successfully incorporated in new music.

Combinatorial methods such as those described above, for generating and analyzing permutations and combinations of elements in sets, are a technique that has been used frequently by composers,* as well as by church bell ringers to create interesting

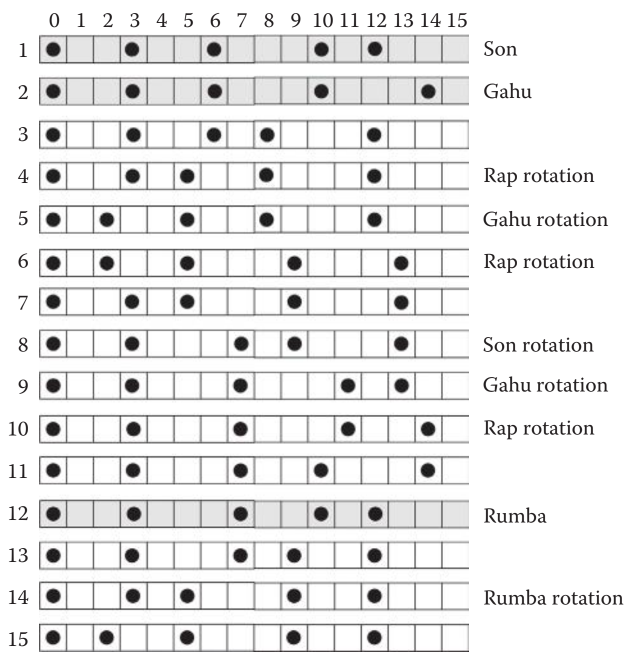


FIGURE 36.1 The first 15 interval permutations of the clave son.

* Babbitt, M. (1962) used combinatorial methods as a means to obtain maximal diversity in his compositions. See also Coxeter, H. S. M. (1968), Forte, A. (1973), Reiner, D. L. (1985), Duncan, A. (1991), Read, R. C. (1997), Dabby, D. S. (2008), and Benson, D. J. (2007), Chapter 9. Knobloch, E. (2002) is a nice survey of combinatorial methods popular in the seventeenth century. Toussaint, G. T. (2002, 2003) has applied combinatorial methods to generate "good" rhythm timelines.

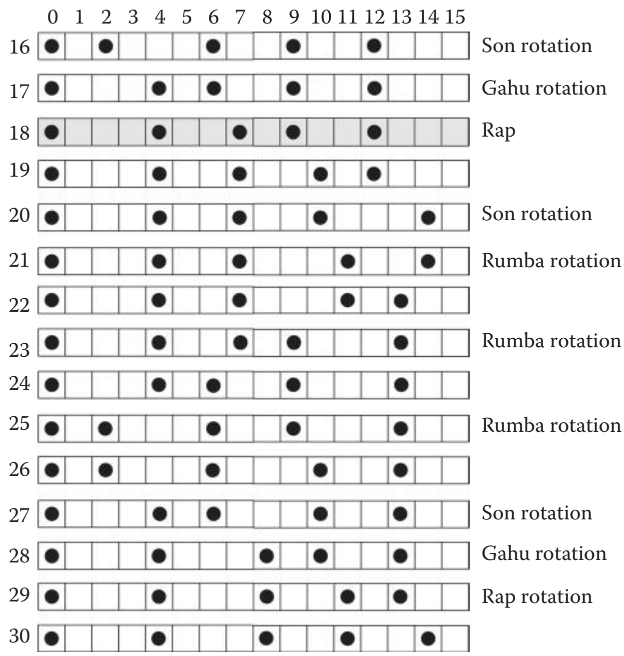


FIGURE 36.2 The second 15 interval permutations of the clave son.

nonrepetitive patterns.* It is a tool that facilitates composition. Furthermore, the various permutations of a rhythm can be selected either at random or according to some musicological rules. A recent example of this approach is the recombination of rhythm midi loops by Sioros and Guedes (2011), who first ordered the rhythms by decreasing complexity.† Indeed, one might say that the application of combinatorics to music shifts the creative process from the ineffable domain of invention to the more worldly mechanical realm of choice.

* Roaf, D. and White, A. (2006), p. 123.

† Sioros, G. and Guedes, C. (2011), p. 381.

CHAPTER 37

What Makes the Clave Son Such a Good Rhythm?

THE PREFERENCES FOR PROPERTIES OF MUSIC in general and rhythm in particular may be explored at many levels of generality, and at any point in the objective–subjective continuum. In Plato's *Republic*, Socrates tells Glaucon that "Good rhythm depends on simplicity."[*] For Jeff Pressing, "The effectiveness of a groove from the standpoint of reception is assessed by its ability to engage human movement and attention."[†] Bruce F. Katz proposes measuring musical preference at the neurological level: "A musical passage will be preferred to the extent that it creates synchrony in the neurons that are responsible for processing the passage."[‡] The approach taken in this book to measure rhythmic preference follows and extends the footsteps of Socrates. In this chapter, stock is taken of the variety of properties that are possessed by the *clave son* rhythm in particular, and "good" rhythms in general. The role that these properties play is examined, individually and as a whole, in contributing to the salience of this particular rhythm. Any one of these properties, by itself, is, perhaps, not sufficient to establish a rhythm as a candidate for a good timeline, let alone characterize the clave son uniquely. However, a group of them in combination is quite suggestive. In the final analysis, it is a question of striking the right balance between these properties. Which properties make one timeline more successful than another? In short, a good rhythm should be as simple as possible, but not simpler.

MAXIMAL EVENNESS

Maximal evenness, discussed in detail in Chapter 19, is obtained when all the adjacent inter-onset intervals of a rhythm have equal duration. This happens only when a rhythm is perfectly regular, such as, for instance, the rhythm [x . x . x . x . x . x . x . x .]. Perfectly even rhythms contain too much regularity and symmetry to be musically interesting. This

[*] Impagliazzo, J. (1989), p. 16.
[†] Pressing, J. (2002), p. 290.
[‡] Katz, B. F. (2004), p. 30.

is not to say that they have no psychological power or social usefulness. It is well known that perfectly regular rhythms have been used for thousands of years to lead soldiers into battle. Indeed, according to the musicologist Curt Sachs, humanity's sensation of regular rhythms is acquired from walking and marching.[*] Furthermore, he points out that the rhythmic tempo of much music matches the rhythm of a person's stride. To succeed as a musical timeline, however, a rhythm must be somewhat asymmetrical; it must possess some irregularity; and it must contain an element of surprise. In the words of Alexander Voloshinov, "Exact symmetry in art is perceived as static and frigid, and only the introduction of asymmetrical elements into symmetry brings about a kind of enlivening."[†] Recent scientific discoveries suggest that such "broken" symmetries are relevant to more than art, and have "deep cosmological implications."[‡]

As we have already seen in Chapter 6, there are 4368 different ways of arranging five onsets among 16 pulses, thus yielding as many rhythms. However, only six noteworthy members of this class have been used extensively as timelines in traditional music in different cultures around the world, and it is fair to say that only the clave son has overshadowed the others to take center stage by capturing the human imagination. Even though 4368 is a large number, the fraction of uninteresting patterns in this family of rhythms is also large. For instance, the timelines that have all their five onsets occupying adjacent pulses such as [x x x x x . . . .] or [. . . . x x x x x . . . .] or [. . . . . x x x x x .] and so on, are not very interesting. For a rhythm to be a good timeline, the five onsets should be distributed *almost* as evenly as possible within the 16-pulse time span. A perfectly even rhythm with five onsets among 16 pulses is not possible since five does not divide evenly into 16 on a digital clock. Of course, it does so on a continuous clock, but then the rhythm becomes regular and loses all its uniqueness. The only numbers on either side of five to divide evenly into 16 are four and eight. Four is the closest to five, and yields the rhythm [x . . . x . . . x . . . x . . .]. Superimposing this rhythm on the clave son yields [$\underline{\text{x}}$ . . x . . x . . . x . $\underline{\text{x}}$ . . .], where the underscore sign indicates the positions of four perfectly evenly distributed beats. A distance measure, such as the swap distance, between a given rhythm and the perfectly even rhythm may serve as a measure of evenness of the given rhythm. The reader may verify that the swap distance between the clave son and the four-beat regular rhythm is five. Similarly, the swap distances between the four-beat rhythm and the five other distinguished timelines are given by shiko = 4, rumba = 4, soukous = 6, bossa-nova = 6, and gahu = 7. Thus, according to this measure of maximal evenness, the son has a relatively low score of five, and is thus a fairly regular rhythm. Furthermore, the score is not an extreme value in either direction, indicating that the clave son strikes a balance between too much and too little regularity, but leaning closer to regularity than irregularity.

RHYTHMIC ODDITY

As pointed out in Chapter 15, a rhythm has the rhythmic oddity property if no two of its onsets are located diametrically opposite to each other on the circle. As already pointed out,

[*] Sachs, C. (1953). See also De Leew, T. (2005), p. 41, and Bonus, A. E. (2010), p. 371.

[†] Voloshinov, A. V. (1996), p. 111.

[‡] Wade, D. (2006), p. 38, Anderson, P. W. (1972).

this property by itself is not sufficient to establish the saliency of a timeline, as is clearly evident from a rhythm such as [x x x x x . . . .], which is not a good timeline but nevertheless does exhibit the property. Figure 37.1 illustrates the presence or absence of the rhythmic oddity property in the six distinguished timelines. It can be seen that gahu, soukous, and shiko do not have the property. For rhythms with attacks that are fairly evenly spaced, such as these six timelines, adding this property appears to contribute to their saliency. However, by itself, the property is not sufficient to highlight the clave son uniquely from among the group of six timelines, since the rumba and the bossa-nova also possess this property.

As we have seen earlier, one way to generalize the binary-valued rhythmic oddity property to a multivalued one, so as to discriminate between rhythms that have the property, is to measure how far a rhythm is from losing the property. Recall that the set of *antipodal* pulses of a rhythm are the pulses diametrically opposite to the rhythm's onsets. Figure 37.1 shows the six distinguished timelines with lines connecting their onsets to their antipodal pulses. One way to measure the amount of rhythmic oddity in a rhythm is with a version of the swap distance in which, for each onset of the rhythm, the minimum distance to its nearest antipodal pulse is calculated. The sum of these distances over all the onsets of a rhythm is a measure of how little oddity the rhythm possesses; a higher value indicates greater rhythmic oddity because the rhythm is then further away from losing the property.

As an illustrative example, consider the calculation of the amount of rhythmic oddity contained in the gahu rhythm, and refer to Figure 37.1. The distance from the first onset

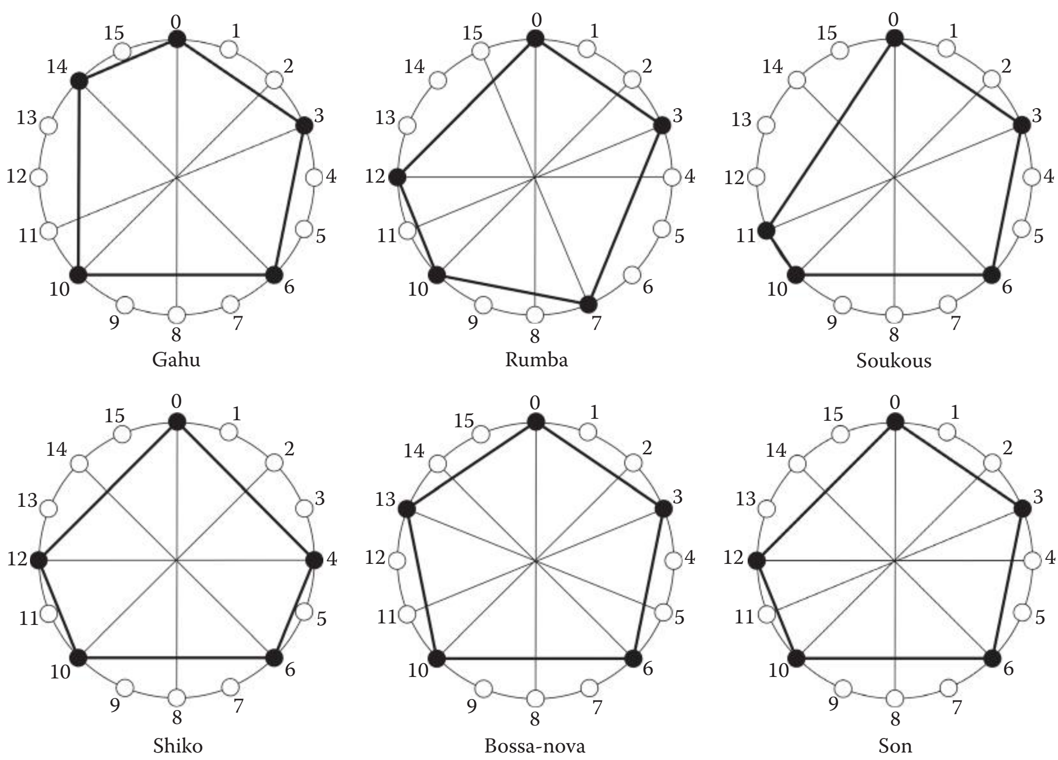


FIGURE 37.1 The amount of rhythmic oddity in the six distinguished timelines.

to its nearest antipodal pulse is two, realized by either pulse 2 or pulse 14. The distance of the second onset to its nearest antipodal pulse (pulse two) is one. The distance of the third onset to its nearest antipodal pulse is zero, since it is itself an antipodal pulse of the fifth onset. The fourth onset is at distance one from pulse 11. Finally, the fifth onset has distance zero since it lies at the antipodal pulse of the third onset. Therefore, the overall oddity score of gahu is four. The oddity values of the remaining five timelines increase from left to right with the following values: five for rumba, five for soukous, six for shiko, six for bossa-nova, and seven for son. Thus, according to this measure of rhythmic oddity, the son may be discriminated from the rumba and bossa-nova. Indeed, the clave son has the highest value of rhythmic oddity among this family of rhythms. This is one of the few properties for which the clave son takes on a complete extreme value when compared to the other five distinguished timelines.

OFF-BEATNESS

Off-beatness was the subject matter of Chapter 16. In the 16-pulse cycle in which the six distinguished timelines live, there are four main beats occurring at pulses 0, 4, 8, and 12, which divide the cycle into four equal parts. Relative to these main beats, the remaining onsets may be considered to be off the beat. There is a difference, however, between onsets that lie at pulse two, for example, rather than pulses one and three, since the former lies exactly at the midpoint between two adjacent beats, whereas the latter lies halfway between a beat and the midpoint between two beats. Indeed, Reinhard Flatischler calls the former kind of onsets *off-beats*, and the latter kind *double-time off-beats*.[*] Thus, being off the beat has something to do with the mathematical concept of divisibility of durations into the cycle time span. It was pointed out in Chapter 16 that in a cycle of 16 pulses, there is a group of special positive integers that have the property that they are not divisible by eight, four, or two. These numbers are 1, 3, 5, 7, 9, 11, 13, and 15. Onsets occurring at these locations may be considered to be strongly off the beat. The *off-beatness* property measures the number of such pulse positions occupied by onsets. Thus, off-beatness is a mathematical definition of a property related to the concept of syncopation, and a rhythm that has this property is usually considered to be more interesting or lively. The clave son in Figure 37.2 (left) has an off-beatness value of one since it has only one onset at one of these positions (pulse three).

It should be pointed out, however, that the off-beatness measure makes sense only when the rhythm is viewed in the context of an underlying regulative beat structure. The rhythms in Figure 37.2 (center and right) are composed entirely of off-beat onsets, and would thus receive a high value of off-beatness according to this measure. Yet, when viewed in isolation (without a regulative structure), they are perfectly regular rhythms that should be characterized as having no off-beatness. The six distinguished timelines have the following off-beatness values that vary in the range between zero and two: shiko = 0, gahu = 1, son = 1, soukous = 2, bossa-nova = 2, and rumba = 2. According to this measure, the clave son falls squarely in the middle of this range of values.

[*] Flatischler, R. (1992).

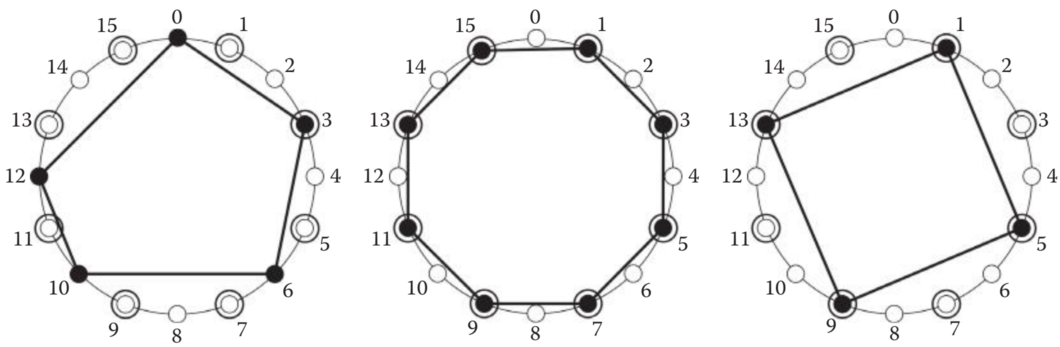


FIGURE 37.2 The clave son (left) has one *strongly off-beat* onset. Regular rhythms (center and right) may be composed entirely of off-beats onsets.

WEIGHTED OFF-BEATNESS

The off-beatness measure counts the number of onsets that lie at the odd-numbered pulses, which in this context are the onsets *strongly* off the beat, or in the terminology of Flatischler, *double-time* off-beats. The *weighted off-beatness* measure counts the total number of onsets that are both off-beats and double-time off-beats, but places different weights on each type of off-beat. In the absence of any *extra*-mathematical knowledge about the relative importance of these two kinds of off-beats, psychological, musicological, or music-theoretic, the double-time off-beat onsets may be counted as two off-beat onsets. Then, the resulting values fall in the range between two and six: shiko = 2, son = 4, rumba = 5, gahu = 5, soukous = 6, and bossa-nova = 6. Again, by this more general measure of off-beatness, the clave son value falls in the middle of the range of values.

METRICAL COMPLEXITY

It was pointed out in Chapter 13 on mathematical measures of syncopation that the Lerdahl–Jackendoff metrical hierarchy assigns to the 16 pulses of a 16-pulse rhythm, the weights (5, 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1), respectively. If for a given rhythm the weights corresponding to the onset locations are summed, we obtain a measure of the metrical expectedness (or simplicity) of the rhythm. For the clave son, the metrical simplicity is 13. The maximum value that five onsets in a cycle of 16 pulses may take is 17. Subtracting 13 from 17 yields the value four as a measure of the metrical complexity of the clave son. The six distinguished timelines in increasing the value of metrical complexity are: shiko = 2, son = 4, rumba = 5, gahu = 5, soukous = 6, and bossa-nova = 6. Once more, the range of values is between two and six, and son has a value that falls in the middle of this range.

MAIN-BEAT ONSETS AND CLOSURE

The six distinguished timelines have four main beats at pulses 0, 4, 8, and 12, as indicated in Figure 37.3 by means of double circles. The number of onsets of a 16-pulse rhythm that coincide with these four beats is a measure of the rhythm's synchronicity with the underlying beat. One might call it a *beatness* measure. Furthermore, a rhythm that has an onset at

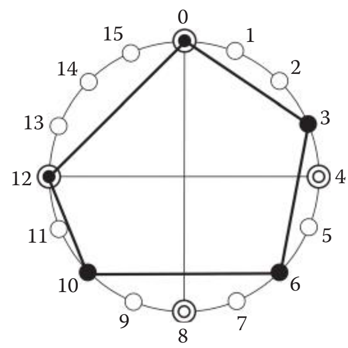


FIGURE 37.3 The clave son has two onsets at main beats on pulses 0 and 12.

the last of these main four beats (pulse 12) contains *closure*, a property that appears to be attractive to many listeners. Of the six distinguished timelines, the shiko, son, and rumba have closure, whereas the bossa-nova, soukous, and gahu do not. The number of main beats contained in each of the six distinguished timelines is: shiko = 3, son = 2, rumba = 2, soukous = 1, bossa-nova = 1, and gahu = 1. The range of values is from one to three, and the son moreover falls squarely in the middle.

DISTINCT DURATIONS

Several measures of rhythm complexity have been explored in the earlier chapters. One such measure was the entropy of the *full* inter-onset interval histograms, when the latter are viewed as discrete probability distributions. This entropy is an *implicit* measure of the number of distinct durations present in the rhythm. A higher value of entropy results from a flatter histogram, which in turn implies the presence of a wider range of distinct durations. The number of distinct durations may also be measured *explicitly* by just counting them. From the histograms in Figure 8.3, the following data may be readily observed: shiko = 4, bossa-nova = 4, son = 5, rumba = 6, soukous = 7, and gahu = 7. The values range from four to seven, and the clave son falls in the lower-middle range. Furthermore, the son and rumba values constitute the medians of the six rhythms.

DISTINCT ADJACENT DURATIONS

It has already been pointed out in Chapter 22 that the *full* inter-onset histograms sometimes fail to distinguish between different rhythms because the latter may have identical histograms. In those cases, it may help to also examine the *adjacent* inter-onset histograms of Figure 8.4. Yet another measure of rhythm complexity counts the number of distinct *adjacent* inter-onset intervals present in the rhythm. The histograms in Figure 8.4 yield the following values: shiko = 2, bossa-nova = 2, son = 3, rumba = 3, gahu = 3, and soukous = 4. The values range from two to four, with son falling midway between these two extremes.

ONSET-COMPLEXITY AND DISTINCT DISTANCES

In addition to counting the number of distinct distances between all its pairs of onsets that a rhythm may possess, one may focus on a *single* onset and measure its contribution to the

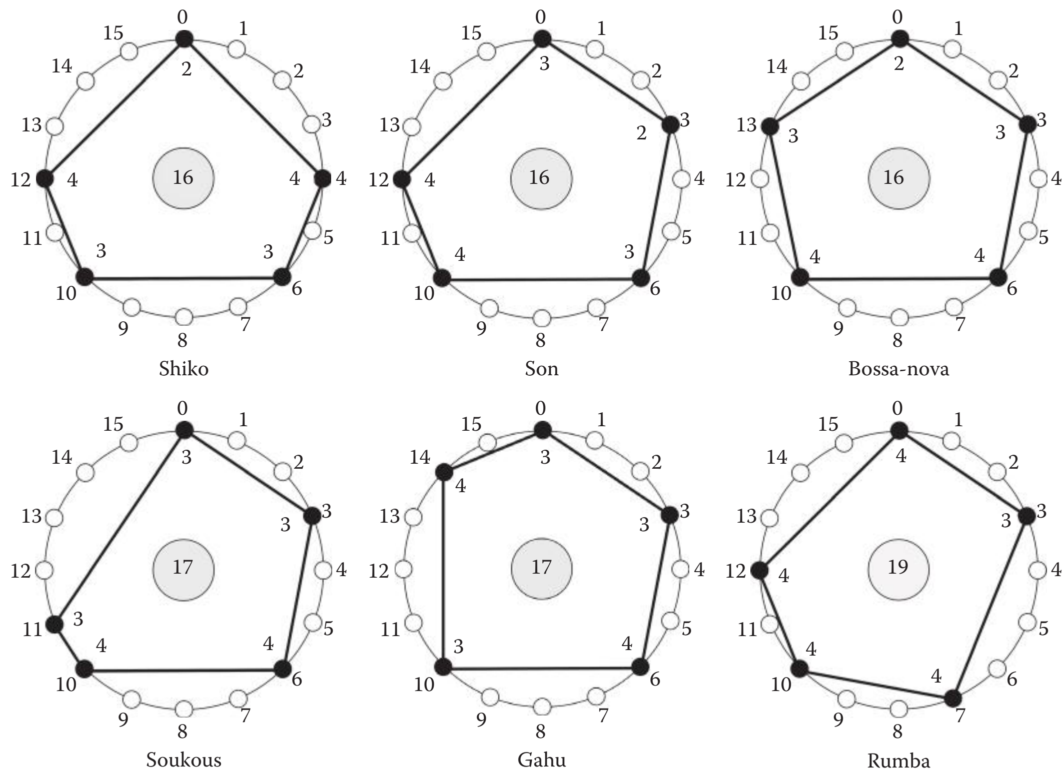


FIGURE 37.4 Onset-complexity and distinct distances.

total number of distinct distances. One might say that the number of distinct distances that are incident on a given onset is a measure of that *onset's complexity* relative to the other onsets. Then, the sum of the onset distances summed over all the onsets in a rhythm provides an onset-based measure of the rhythm's complexity. Figure 37.4 shows next to each onset the number of distinct distances to the other onsets, which it determines. The number in the shaded circle at the center of each polygon indicates the sum of all these onset values. According to this measure of complexity, the shiko, son, and bossa-nova are all considered equal, and the rumba has the highest complexity. Thus, this measure does not discriminate between the son and the shiko or bossa-nova.

DEEP RHYTHMS, DEEPNESS, AND SHALLOWNESS

Recall from Chapter 25 that a rhythm is deep if its full inter-onset interval histogram has the property that no two columns have the same height (not counting the columns of height zero). According to this binary-valued measure, among the six distinguished timelines, only the shiko and bossa-nova are deep. However, this binary measure may be converted into a *multivalued* measure of deepness by calculating the distance between the histogram of a given rhythm and that of a deep rhythm. But then, since we are trying to measure the deepness property itself, without regard to which distances actually realize the necessary unique heights of the histogram, the histogram columns should first be sorted

Shiko	0	0	0	0	1	2	3	4
Son	0	0	0	1	2	2	2	3
Rumba	0	0	1	1	1	2	2	3
Soukous	0	1	1	1	1	2	2	2
Gahu	0	1	1	1	1	2	2	2
Bossa	0	0	0	0	1	2	3	4

FIGURE 37.5 Table showing the histogram values sorted in increasing order.

by increasing height, before calculating the distances between them. Sorting the histograms in Figure 8.3 yields the histograms in Figure 37.5.

Note that the shiko and bossa-nova have perfectly deep histograms. Now, the distance between two histograms may be measured by the sum of the absolute values of the differences in heights of the corresponding columns.* The reader may readily verify that the deepness distances of the six rhythms are: shiko = 0, bossa-nova = 0, son = 4, rumba = 4, soukous = 6, and gahu = 6. A larger value of distance implies a smaller value of deepness. Therefore, this distance may also be considered as a measure of *shallowness*. Again, in terms of deepness, two rhythms are deeper than the son (bossa-nova and shiko), and two are shallower (soukous and gahu). Thus, in terms of deepness (or shallowness), the son follows the middle path.

TALLNESS

The *tallness* property measures the maximum height of the columns in the inter-onset interval histogram. A larger tallness value suggests the tendency that there is a larger concentration of inter-onset intervals, and is somewhat related to the deepness property. Also, a tall histogram has less chance of realizing many distinct distances, and so, it is also related inversely to rhythm complexity. From Figure 8.3, the following values of tallness are evident: soukous = 2, gahu = 2, son = 3, rumba = 3, shiko = 4, and bossa-nova = 4. Once more, the clave son avoids the extreme values that this property takes.

PHYLOGENETIC TREE CENTRALITY

The chapter on flamenco meters introduced the application of *phylogenetics trees* to the analysis of families of rhythms, to obtain a clustering of the rhythms as well as to infer a possible evolutionary phylogeny. Simultaneously, such an analysis sometimes constructs nodes in the tree that may determine ancestral rhythms. Applying similar techniques to the six distinguished timelines yields additional insight into the special status enjoyed by the clave son. The first step in such an analysis is the computation of a distance matrix. Computing the swap distance between all pairs of the six timelines yields the distance matrix shown in Figure 37.6.

* Distance measures between histograms have a variety of applications in music information retrieval, such as feature selection, indexing, pattern classification, and cluster analysis. Cha, S.-H. and Srihari, S. N. (2002) propose a variety of alternate techniques to measure the distance between histograms.

Rhythm	Shiko	Son	Soukous	Rumba	Bossa	Gahu
Shiko	0	1	2	2	2	3
Son		0	1	1	1	2
Soukous			0	2	2	3
Rumba				0	2	3
Bossa					0	1
Gahu						0
Σ	10	6	10	11	8	12

FIGURE 37.6 The *swap-distance matrix* for the six distinguished timelines.

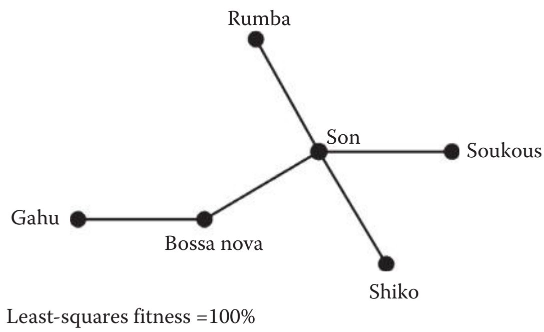


FIGURE 37.7 The phylogenetic tree *SplitsTree* computed using the swap-distance matrix.

In addition to these distances, the bottom row of Figure 37.6 shows, for each rhythm named at the top, the *sum* of all its distances to the other rhythms. This number is a measure of how dissimilar a rhythm is to the others. Note that the clave son, with the lowest value of six, is the unique rhythm most similar to all the others. This suggests that it is the central rhythm of this family of rhythms. In other words, the clave son is the rhythm that minimizes the total number of swap mutations needed from which to generate all the other rhythms. Therefore, we may consider this number as a measure of the *centrality* (or prototypicality) of a rhythm with respect to both a group of rhythms and a specific distance measure. A smaller number reflects a greater degree of centrality. Figure 37.7 shows the *SplitsTree* calculated from the distance matrix of Figure 37.6. This tree visually encapsulates at a glance all the information contained in the distance matrix.

MIRROR SYMMETRY

In Chapter 30 on symmetric rhythms, it was shown that rhythms notated on a circle may be classified according to a variety of mirror symmetries that depend on the orientation of the line of symmetry: vertical, horizontal, and diagonal being the three main categories. Furthermore, symmetry is often cited as a contributing factor to making music sound good. Indeed, as we have seen earlier, Simha Arom has suggested that symmetry might be a music universal. Only shiko, son, and bossa-nova possess mirror symmetry, and only the clave son possesses diagonal mirror symmetry. As with many other geometric properties of rhythm, symmetry by itself is no guarantee that a rhythm that possesses it will be a successful timeline. After all, regular polygons are replete with mirror symmetries, but fail to make interesting timelines. However, a rhythm that has only one line of mirror symmetry may stand a better chance. Furthermore, a line of symmetry that has a diagonal orientation, perhaps, yields rhythms that are more interesting or surprising than those possessing a vertical line of mirror symmetry. At present, this is mere speculation, and psychological experiments would have to be carried out to determine whether such symmetric preferences exist.

SHADOW CONTOUR ISOMORPHISM

Of the six distinguished five-onset, 16-pulse timelines, the clave son is the only one that has a cyclic rhythmic contour that is the same as the contour of its shadow rhythm. Figure 37.8 shows the six timelines (black pulses) along with their shadow rhythms (gray pulses) and the inter-onset durations of the shadow rhythms. The rhythmic contours of both the rhythms and their shadows are shown in Figure 37.9 for comparison. The rhythmic contour of the clave son is [0 + – + –] and that of its shadow is [+ – 0 + –], which is a rotation of the former. Interestingly, the shadow of the shiko rhythm is a rotation of the bossa-nova rhythm.

Although this property succeeds in uniquely selecting the clave son from among the six distinguished timelines, like the other properties, it falls short of characterizing good timelines in a more general setting, as the example in Figure 37.10 illustrates. Here, the

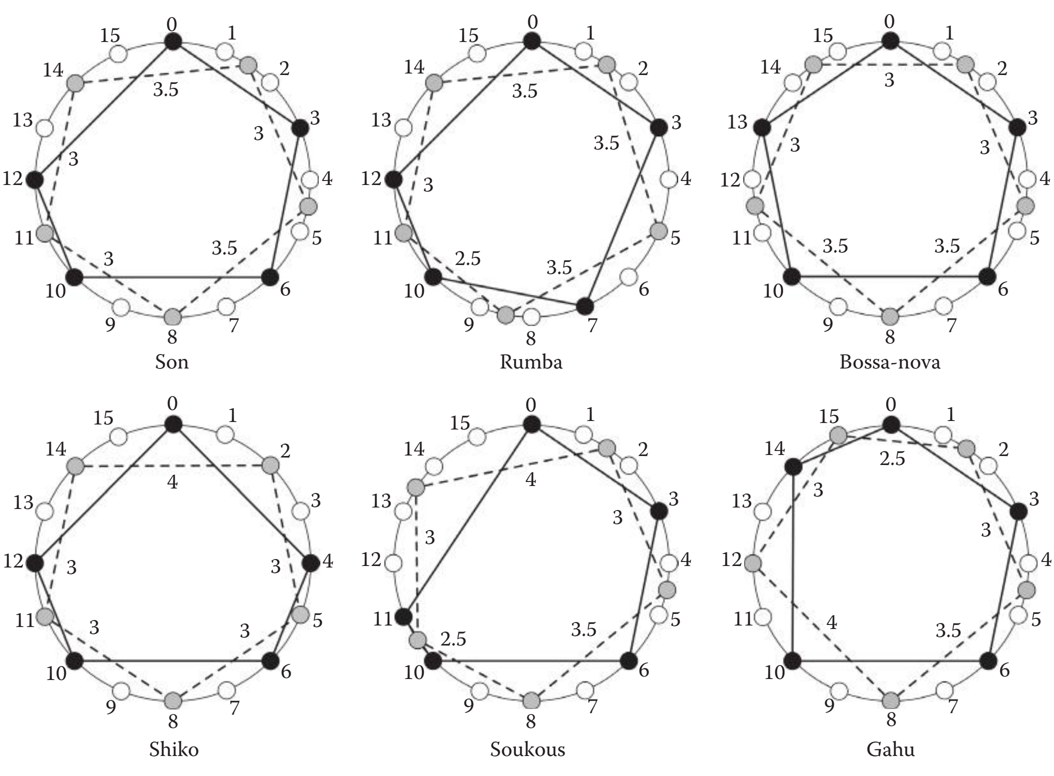


FIGURE 37.8 The six distinguished timelines and their *shadow* rhythms.

Timeline	Rhythm IOIs	Rhythm Contour	Shadow IOIs	Shadow Contour
Son	3, 3, 4, 2, 4	0 + − + −	3, 3.5, 3, 3, 3.5	+ − 0 + −
Rumba	3, 4, 3, 2, 4	+ − − + −	3.5, 3.5, 2.5, 3, 3.5	0 − + + 0
Bossa-nova	3, 3, 4, 3, 3	0 + − 0 0	3, 3.5, 3.5, 3, 3	+ 0 − 0 0
Shiko	4, 2, 4, 2, 4	− + − + 0	3, 3, 3, 3, 4	0 0 0 + −
Soukous	3, 3, 4, 1, 5	0 + − + −	3, 3.5, 2.5, 3, 4	+ − + + −
Gahu	3, 3, 4, 4, 2	0 + 0 − +	3, 3.5, 4, 3, 2.5	+ + − − +

FIGURE 37.9 The rhythmic contours, their shadows, and the contours of their shadows.

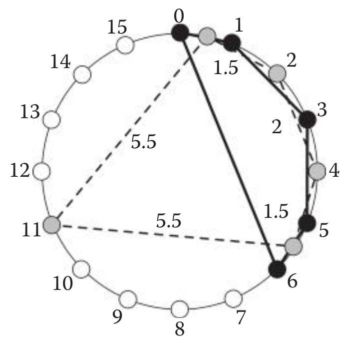


FIGURE 37.10 A timeline (solid lines) with a rotated contour of its shadow (dashed lines).

timeline with interval vector [1-2-2-1-10] is too skewed to be a successful timeline, and yet it possesses a rhythmic contour that is a rotation of the contour of its shadow (dashed line). Whether this property has any psychological or neurological weight in restricted contexts has yet to be determined experimentally. However, as a mathematical property, it is clearly useful for characterizing rhythms in general, and the clave son in particular.

For convenience, all the properties discussed in this chapter are listed together for the six distinguished timelines in the table of Figure 37.11. A property that a rhythm does not possess is indicated by the dash (–) for easy visualization. Of the 12 properties that have numerical values, nine of them yield values that for the clave son fall in the middle of the range spanned by the six rhythms. This list provides strong evidence that at least for the clave son, and perhaps for other rhythms to be good as well, these properties should not take on extreme values, but rather follow the "golden mean."

Although the shadow contour isomorphism property is obtained by only the clave son, among the six distinguished timelines, like the other properties for which the clave son is preferred over the other five timelines on the basis of *rotation invariant* properties, the question arises as to why the particular rotation of the clave son necklace is preferred over any of its other four rotations. To answer this question, refer to Figure 37.12 that shows the clave son (labeled as son-1) and its four rotations that start on the other onsets.

Two desirable musicological properties of rhythm timelines that have not yet been discussed are the *call–response* property and *metric ambiguity* (see Figure 37.13). Call–response refers to the property that the rhythm evokes the feeling that it consists of two discernible parts, the first of which plays the role of "asking a question," followed by the second, which

Property	Shiko	Son	Soukous	Rumba	Bossa	Gahu
Maximal evenness	—	—	—	—	Yes	—
Swap-distance evenness	4	5	6	4	6	7
Rhythmic oddity	—	Yes	—	Yes	Yes	—
Rhythmic oddity amount	6	7	5	5	6	4
Off-beatness	0	1	2	2	2	1
Weighted off-beatness	2	4	6	5	6	5
Metrical complexity	2	4	6	5	6	5
Closure	Yes	Yes	—	Yes	—	—
Main-beat onsets	3	2	1	2	1	1
Distinct durations	4	5	7	6	4	7
Distinct adjacent durations	2	3	4	3	2	3
Onset distinct distances	16	16	17	19	16	17
Deep	Yes	—	—	—	Yes	—
Shallowness	0	4	6	4	0	6
Tallness	4	3	2	3	4	2
Swap-distance centrality	10	6	10	11	8	12
Mirror symmetry	Yes	Yes	—	—	Yes	—
Diagonal mirror symmetry	—	Yes	—	—	—	—
Shadow contour isomorphism	—	Yes	—	—	—	—

FIGURE 37.11 Table of properties possessed by the six distinguished timelines.

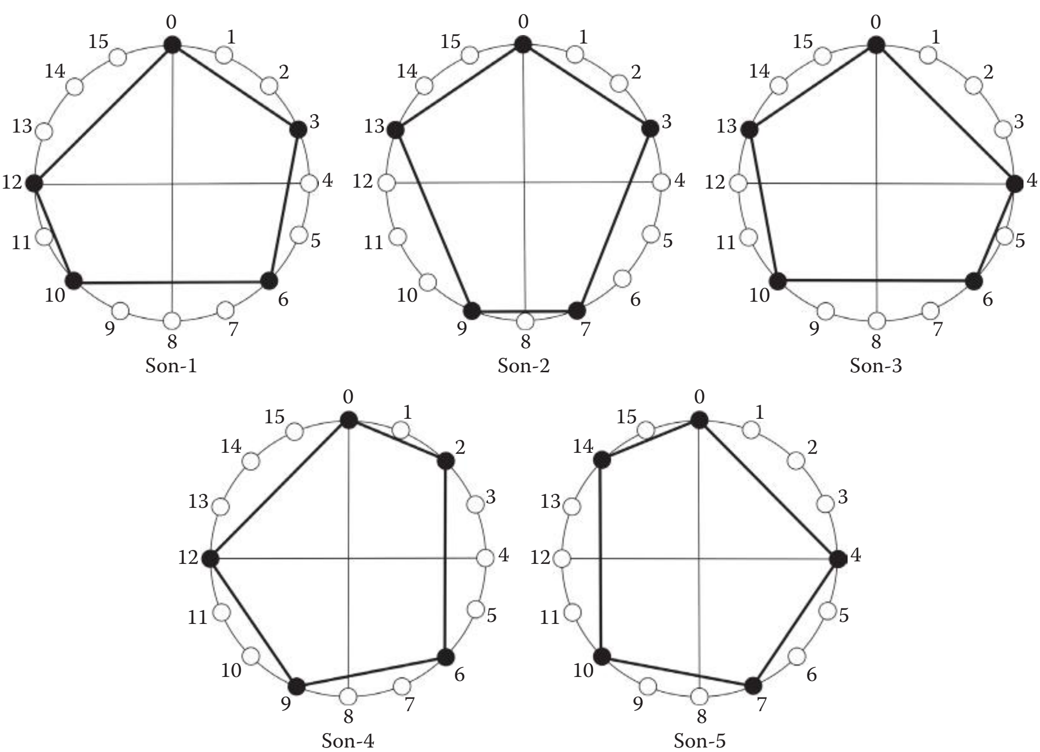


FIGURE 37.12 All five onset rotations of the clave son timeline. (From Toussaint, G. T., *Percussive Notes*, 2011, November Issue, pp. 52–59. With permission.)

Property	Son-1	Son-2	Son-3	Son-4	Son-5
Call–response	Strong	—	—	Weak	—
Metric ambiguity	Yes	—	—	—	—

FIGURE 37.13 Table of properties of the five rotations of the clave son timeline.

"gives an answer." This property of music is inherent in much of sub-Saharan African music, and beyond. It is somewhat related to the property of *closure*. Call–response provides closure, but closure does not necessarily convey call–response. Of the five rotations of the clave son shown in Figure 37.12, only son-1 (the true son) and son-4 have closure. Of these, son-1 has *strong* closure, provided by the fourth onset, which lies at pulse 10, at the midpoint of the third quarter cycle. By contrast, the call–response property of son-4 may be called *weak* because the fourth onset is situated at pulse nine, which is close to the midpoint of the cycle and far from the last onset.

It is well known that an effective method by which to add spice and surprise to a rhythm is through syncopation. Syncopation introduces a touch of cognitive insecurity, or metrical ambiguity, or what Neil McLachlan calls a *gestalt despatialization*.* In the case of the clave son timeline, which has four strong beats at pulses 0, 4, 8, and 12, metrical ambiguity can be introduced by first misguiding the listener's brain into predicting a sequence of equal-duration intervals of three pulses each. For this to happen, there must be at least two such intervals at the start of the sequence. This means that the first three onsets must occur at pulses zero, three, and six. However, this is not sufficient to cause the listener to experience metrical ambiguity. To do so, the last interval must have duration equal to four units, and hence be determined by onsets at pulses 12 and 0. In this way, the rhythm ends with a clear and contradictory four-pulse interval. Only son-1 has this property.

* McLachlan, N. (2000). In the words of Thaut, M. H. (2008), p. 5, effects such a gestalt despatialization cause a temporary violation of musical predictions which in turn leads to a heightened state of arousal to search for a meaningful resolution of the musical tension.

CHAPTER 38

The Origin, Evolution, and Migration of the Clave Son

Let us return to the Spanish sailor that we encountered in Chapter 12. Recall that in the sixteenth century, he travels from Sevilla to Havana on a Spanish fleet of galleons on its way to pick up gold from Mexico and silver from Bolivia. During one of his wanderings in Havana, he comes upon a group of black former slaves drumming and dancing passionately in the street, and is captivated by one of the crisp invigorating rhythms that one of the musicians is playing with a pair of wooden sticks. The sailor's brain interprets a signal relayed by his ear, which in turn is stimulated by a sound wave traveling from the sticks to the ear, causing his eardrum to vibrate. This sound wave is the *acoustic signal* that embodies the rhythm that the musician is playing. The acoustic signal is a complex waveform that contains much information about the sound, as well as the environment in which the sound is produced, in addition to the points in time at which the sticks are struck together. However, a greatly simplified and idealized representation of this acoustic signal, after being recorded by electronic equipment, and graphically displayed on a computer screen, might look something like the waveform in Figure 38.1 (left). Of course, the sailor is not privy to this graphical information.

It is well known that human perception does not result from a mere *bottom-up* processing of the objective scientific stimulus presented to the perceiving mechanism. It is rather a partly subjective and constructive interactive process that also involves *top-down* processing, in which the perceiver projects a medley of competing hypotheses about what is perceived. These hypotheses emerge from a variety of biological as well as cultural expectations possessed by the listener.[*] In the words of David Huron, Rhythms are perceived categorically.[†] The brain has, as it were, a collection of rhythmic templates, and when it

[*] Morrison, S. J. and Demorest, S. M. (2009).

[†] Huron, D. (2007), p. 191. See also Schulze, H.-H. (1989), Honing, H. (2002), and Patel, A. D. (2008), p. 112.

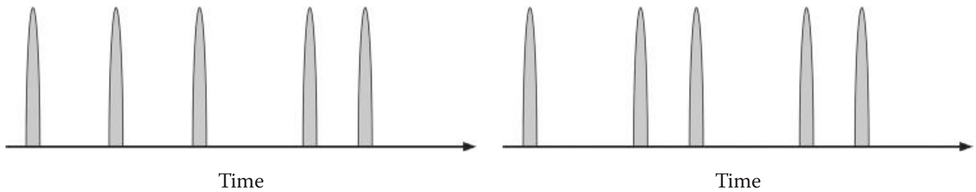


FIGURE 38.1 Two idealized acoustic signals or waveforms in real time.

encounters an unknown rhythm it perceives it as one of its close rhythmic templates.* Therefore, the rhythm that our sailor perceives in the situation described may correspond to an idealized waveform different from that shown in Figure 38.1 (left), and closer to a rhythm that is much more familiar to the sailor.

Back on board the ship on his return to Sevilla, the sailor, while trying to entertain his shipmates one evening, tries to remember and reconstruct the rhythm with a pair of wooden spoons. However, had there been a recording of it, the rhythm performed that night might have looked more like the rhythm on the right in Figure 38.1. Note that in this rhythm the second onset is closer to the third onset than the first. In the rhythm on the left, the corresponding three onsets are equally spaced.

The scenario just described illustrates a *mutation* step in the type of evolution known as *Lamarckian* evolution.† The original rhythm heard by the sailor has been changed slightly for whatever reason (maybe misperception in the first place due to aural illusions or to cultural expectations, perhaps bad memory during reconstruction, possibly poor performance capability, or perchance purposeful creativity). Another sailor on board may later try to play this mutated rhythm for his friends back home upon arrival in Sevilla, only to make a further slight change to it. Thus, in Lamarckian evolution, the progressive changes or errors introduced into the original rhythm may accumulate rapidly, resulting in its speedy extinction. In the process of transmitting the rhythm in this way from one person to another a dozen times, the final rhythm obtained may bear little resemblance to its progenitor. In the context of biology, Lamarckian evolution espouses the idea that an organism may transfer to its offspring, characteristics that are acquired by the organism during its lifetime.

Consider, on the other hand, a culture in which the village master drummer teaches a young pupil to play a rhythm on a drum using the following instructions. First, strike the drum with both hands 16 times starting with the right hand, to make a steady pulse alternating between right and left hands, much like walking. When this steady pulse is well established, play five of these pulses loudly and the remaining 11 softly. In particular, play the first stroke loud, the next two strokes soft, another loud, two more soft, another loud followed by three soft, a loud, a soft, a loud, and finally three soft. When the pattern of loud

* The mental templates may be partly culturally acquired in the form of the "perception norms" discussed by Anku, W. (1997), p. 212. They may also be determined by the perceptual figural grouping mechanisms in the brain, as suggested by experiments performed by Handel, S. (1992, 1998).

† Blackmore, S. (2000), p. 59. The term "Lamarckian" comes from the evolutionary theories of Jean-Baptiste de Lamarck and today refers exclusively to the principle of the inheritance of acquired features. See also Diettrich, O. (1992).

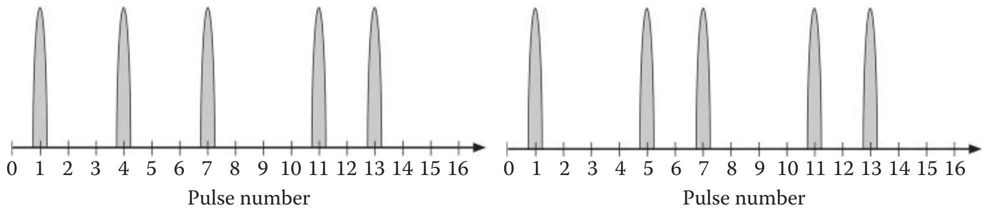


FIGURE 38.2 Two rhythms notated in pulse-measured time.

beats is clear, skip all the soft beats altogether by stopping the hands just before they touch the drum. The resulting rhythm will have the second onset played with the left hand, and the other four with the right hand. Were we to represent this scenario graphically, the acoustic signal would look like the waveform in Figure 38.2 (left), with the added information of pulse numbers implicitly encoded either in the oral instructions set forth by the master teacher or in the alternating right–left hand motions that mark out the pulses. The onsets would be played on pulses 1, 4, 7, 11, and 13. These instructions would make it virtually impossible for a student, once having attained a steady motion, to produce the rhythm on the right in which all the onsets are played with the right hand on pulses 1, 5, 7, 11, and 13.

This second scenario described above differs considerably from the first. In the first, a new instance of a rhythm is brought about by means of an attempt to *copy the rhythm*. The information transmitted from one generation to another consists of a *mutated* facsimile of the rhythm itself. This replica of the rhythm is a complicated continuous acoustic signal susceptible to the smallest of perturbations, and thus easily morphed into other rhythms. In the second scenario, a rhythm is brought about by means of *executing a set of instructions* for producing it. Here, the information transmitted across generations is not the rhythm itself, but rather the *instructions* for *creating* the rhythm. As long as the instructions are passed along intact, the rhythm produced with them will likely remain the same, barring other drastic outcomes such as a lack of performance skill. In contrast to the *concrete continuous* acoustic signal, these instructions are *abstract discrete* logical entities. As such they are more robust and stable than the malleable continuous objects that make up acoustic signals. Hence, rather than a small quantitative error, a large qualitative error would have to be made in order to mutate the rhythm. Alternately, the rules themselves would have to be changed in order to mutate the rhythm. This type of evolution is called *Weismannian* evolution,[*] after the German evolutionary biologist Friedrich Leopold August Weismann, who was strongly opposed to Lamarckism. It is analogous to the modern view in biology that holds that an organism is produced by executing the discrete "instructions" encoded in DNA molecules.

Both types of evolution described above may be at work in a culture based on oral tradition, such as sub-Saharan Africa.[†] In the case of Lamarckian evolution, the acoustic signal itself is passed on aurally. In Weismannian evolution, the instructions may be passed on

[*] *Ibid*., see also Weismann, A. (1889).
[†] Kaufman Shelemay, K. (2000), p. 24.

orally and memorized for subsequent oral transmission. A third oral approach that a master may use to teach rhythms to a pupil is by means of language mnemonics. As poets know very well, words, phrases, and sentences possess rhythm. In recent years, much research has been done to uncover the fascinating relationships between music and language.* It is therefore not surprising that many cultures have passed on rhythms from one generation to another by means of mnemonics. Certain syllables in a language sound stronger than others, thus automatically introducing accents and strong beats. Such is the case, for example, with *ta* and *na*, *ta* being stronger than *na*. Also, some syllables are naturally longer than others when sounded, such as *tan* and *ta*, the former being longer than the latter. By stringing a group of such syllables together, rhythmic patterns are automatically created.†

One of the earliest mnemonic syllable systems for teaching and transmitting musical rhythms was developed in the eighth century by the well-known prosodist al-Khalīl, who wrote one of the first books on the theory of rhythm, *Kitāb al-iqā*. According to this system, the shortest beat, corresponding to one pulse, was *ta*, represented here in box notation in Figure 38.3 (row 1). The syllable *tan* was used for a sounded pulse that is followed by an unsounded pulse, as in row two. The combinations *tana*, *tanan*, and *tananan* correspond to the rhythms in rows three, four, and five, respectively, where a pulse containing a black circle represents the strong beat, the pulse with a gray circle denotes a secondary (optional, or ornamental) beat, and the empty box is a silent pulse.

In the early thirteenth century, in the year 1258, at a time when Baghdad was one of the most brilliant intellectual centers of the world, a leading musician and music theorist (also a calligrapher and physicist) living there, by the name of Safi al-Din al-Urmawi

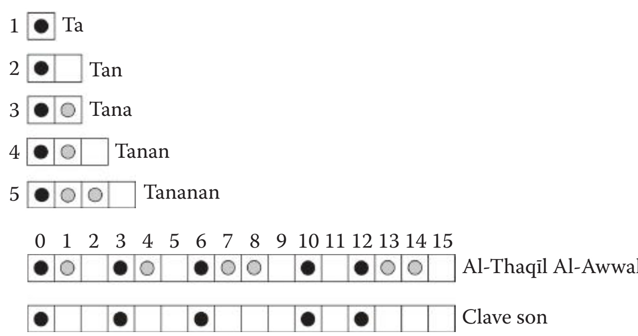


FIGURE 38.3 A mnemonic system for the rhythm *al-thaqīl al-awwal*. (1) Box notation for the sound “Ta,” (2) box notation for the sound “Tan,” (3) box notation for the sound “Tana,” (4) box notation for the sound “Tanan,” (5) box notation for the sound “Tananan.”

* See Longuet-Higgins, H. C. and Lee, C. S. (1982) and Longuet-Higgins et al. (1994) for early analogies between music and natural languages, and Patel, A. D. (2006, 2008) and Cohen, D. and Katz, R. (2008), p. 18, for more recent work. There is ongoing debate about which came first, music or language. Brown, S. (2001) offers a possible resolution to this issue with his “musilanguage” model of music evolution, which hypothesizes that music and language both evolved simultaneously from a communication method he calls “musilanguage.” For an introduction to evolutionary musicology, see Brown et al. (2001). See also Fitch, W. T. (2005) and McGowan, R. W. and Levitt, A. G. (2011).

† Ester et al. (2006) describe a syllabic system called *Takadimi* that has found successful application to the teaching of rhythm, and Colley, B. (1987) compares several syllabic methods for improving rhythm literacy.

(AD 1216–AD 1294), managed to survive the destruction of Baghdad at the hands of the Mongol invaders. Safi al-Din produced a seminal book, by some accounts estimated to be written in the year 1252, which included a theory of rhythm, titled *Kitāb al-adwār* (The Book of Musical Modes).* One of the most important rhythms in this book identified as *al-thaquīl al-awwal* is described using the mnemonic syllable system as *tanan tanan tananan tan tananan*. Figure 38.3 shows this rhythm in box notation. If we disregard the secondary onsets (labeled as optional) and retain only the fundamental *ta* beats, we obtain the clave son shown directly underneath the *al-thaquīl al-awwal* rhythm. A mnemonic system such as this provides better copying-fidelity than merely replicating an acoustic signal. However, in practice there may still be some inexactness present in the pronunciation of the syllables. Hence, this system of cultural inheritance could be said to lie somewhere in between Lamarckism and strict Weismannian evolution.

In addition to the mnemonic syllable system, Safi al-Din's book includes an exact written notation system for describing the rhythms within lines of text, as well as a geometric version that uses circles to represent the rhythmic cycles. In both notations, the pulses are denoted by dots, and circles denote the partition of the pulses into groups that start with a main beat. Thus, the *al-thaqīl al-awwal* rhythm is written by Safi al-Din in running text as O . . . O . . . O . . . . O . . O . . . . using this notation. He also includes a circular representation illustrated in Figure 38.4. The circle is divided into 16 pulses indicated by the black dots. A smaller circle inside the large circle contains the name of the rhythm, and the lines connecting the inner circle to the small white-filled circles on the large outer circle indicate the partition of the cycle into the groups. An arrow indicates where the rhythmic cycle starts, and in which direction time flows (counterclockwise). Hence, this diagram denotes the rhythm [3-3-4-2-4].

Safi al-Din's book appears to contain the earliest historical records of the *clave son* rhythm. Written notations such as these are examples of sets of instructions that provide an even higher copying-fidelity than mnemonic syllable systems, and therefore they greatly facilitate Weismannian evolution. Therefore, we can be pretty certain that the clave

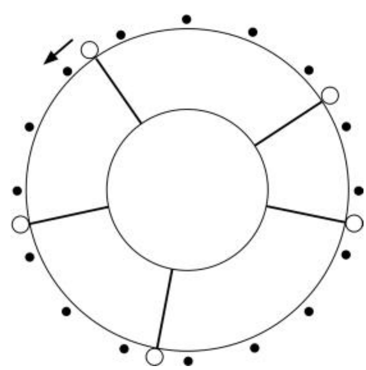


FIGURE 38.4 Safi al-Din's circular representation of the rhythm *al-thaqīl al-awwal*.

* Wright, O. (1995).

son in its binary form [3-3-4-2-4] existed in Baghdad as early as the thirteenth century. However, it was not until the twentieth century that this rhythm took center stage on the world scene. During the 1960s in the West African country of Ghana, there appeared a dance and music called *kpanlogo* that uses this rhythmic pattern played on an iron bell. Furthermore, in Central Africa, there is a music named *gome*, popular during the 1920s, which also uses this bell pattern as a timeline. Some musicologists believe that the migration of gome from Central to West Africa is responsible for the adoption of this bell pattern in the kpanlogo dance. Others believe that this rhythm made its way more than once back and forth between South America and West Africa via the slaves, and that it returned to Africa in the 1880s in a mutated form with West Indian regimental bands stationed at the Portuguese slave trading post of Elmina on the coast of Ghana. It is known that 1000 years earlier while the Arabs lived in North Africa, they traveled extensively southward to Ghana and beyond.* Whether this rhythmic pattern originated in Baghdad and made its way to Ghana, or vice versa, or whether it emerged independently in both regions, or whether it originated in yet a third region is an open question.

There is debate among ethnomusicologists concerning the types of mutations that may transform one rhythm to another as it migrates from one culture to another. On the one hand, there are those such as Rolando Pérez Fernández who believe that African ternary rhythms were transformed into binary rhythms in America when they were exposed to Spanish music.† In an oral tradition that is governed by Lamarckian transmission of acoustic signals, it is plausible that a ternary rhythm such as the fume-fume of Figure 38.5 (left) could be transformed to its binary version of Figure 38.2 (left) by virtue of cultural expectation. After all, from the purely acoustic point of view, the binary and ternary versions are almost the same, especially when played at fast tempos. Comparing their ternary and binary polygon representations of Figure 10.2, it is clear that both rhythms have the same perceptual grouping: one group of three onsets followed by a space, followed by one group of two onsets followed by another space. Experiments by Stephen Handel demonstrated that "Two rhythms that had the same perceptual grouping were judged as being identical, even if the timing between the groups was different."‡ The two rhythms also have identical *rhythmic contours*, a feature that has been shown by psychologists to be more easily

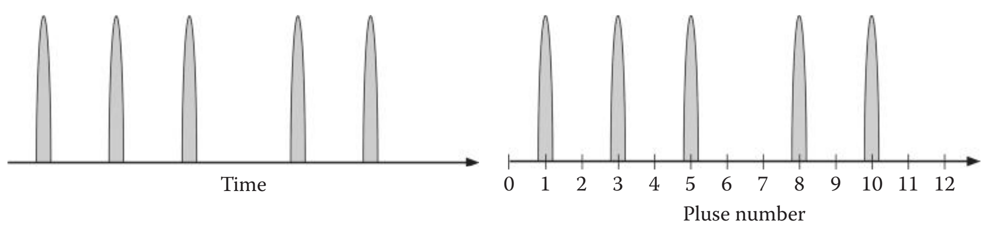


FIGURE 38.5 The fume-fume rhythm notated in real time and pulse-measured time.

* Stone, R. M. (2000), p. 14.
† Pérez-Fernández, R. A. (1986, 2007). See Chapter 12.
‡ Handel, S. (1992), p. 497.

perceptible than precise relations. It is easier for subjects to judge whether an inter-onset interval duration is equal to, greater than, or less than, the preceding or the following duration, than to judge quantitative relations such as twice as long or half as long.* Furthermore, there is plenty of evidence that rhythms are perceived *categorically*, meaning that rhythms that are slightly different from each other may be perceived as being identical.†

A contrary view is held by the anthropologist Gerhard Kubik who holds that social factors are not sufficient to explain the presence or absence of asymmetric timeline patterns in the music of the New World. Kubik believes that rhythmic timeline patterns possess discrete mathematical structures that are *cultural invariables*, and as a consequence, that they either do not migrate at all or migrate intact. In Kubik's words: "Accentuation, mnemonic syllables, and instrumentation of the time-line patterns are all highly variable, but no one can change their intrinsic mathematical structure. Any such attempt simply destroys the pattern."‡ If the instructions for teaching rhythmic timelines in a culture make use of mnemonics, then a rhythm such as the fume-fume, implicitly encoded as in Figure 38.5 (right), would enjoy stronger copying-fidelity, and be governed by Weismannian evolution. It would then likely migrate intact to another culture if the people carried the code with them and continued to use it in their teaching. However, the fact that the clave son existed in Baghdad hundreds of years before the slave migrations from West Africa to the New World, and that in the West African oral musical tradition rhythms were coded mnemonically, suggests that both the binary and ternary versions of the clave son probably existed simultaneously in West Africa before the advent of the slave trade, and that they migrated independently to the New World. One must be careful of ascribing evolutionary influences on the basis if similarity alone, since such similarity may arise independently as a result of functional constraints, especially where universals are involved.§ Of course, the Europeans living in the New World would probably have been unaware of the mnemonics in a foreign language used by the African slaves, and thus the process of binarization in a Lamarckian manner could have easily occurred among the Europeans. Mathematical structures per se are not immune to change; it depends on the type of mathematics employed. Durational patterns expressed in terms of continuous mathematics are quite malleable (Lamarckian), and can easily change, as has been repeatedly demonstrated with listening experiments in the laboratory. In general, discrete mathematics is more resistant to continuous change, it is more Weismannian, but asymmetric timeline patterns need not be coded mathematically in terms of the number of pulses, as suggested by Kubik. They could be coded in the

* Roederer, J. G. (2008), p. 10. Cohen, J. E. (2007), p. 142, writes: "In recognizing the pattern represented by the score, the brain compares relations between successive events rather than absolute values." Jay Rahn (1995) develops music theoretical analytical tools based on such nonnumerical predicates. A related approach to soften the hardness of absolute values employs *fuzzy* set theory, and Ian Quinn (1997) provides an example of their usefulness when applied to the theory and practice of musical contours.

† Desain, P. and Honing, H. (2003).

‡ Kubik, G. (1998), p. 218.

§ Blench, R. (1982) argues on the basis of the similarity of musical instruments that Indonesia had a cultural influence on Central Africa through colonization. Jeffreys, M. D. W. (1966/1967) on the other hand suggests that the influence traveled in the converse direction.

discrete mathematical terms of equal, greater than, or less than, as reflected in rhythmic contours. Mathematics alone does not necessarily impose cultural invariability.

The *clave son* rhythmic pattern acquired its name from the type of Cuban music knows as the *son*. The cradle of the son music is the eastern Cuban province of Oriente, where it may have taken preliminary forms in the seventeenth century. However, in several accounts, its verified existence comes later, in the nineteenth century in the cities of Guantánamo, Baracoa, Manzanillo, and Santiago de Cuba, where it took on its more present-day form in the early twentieth century. On the other hand, Peter Manuel dates the practice of the clave concept earlier to the 1850s.* From Oriente, it was taken to Havana in 1909 by the soldiers of the Permanent Army, where in the early part of the twentieth century, it slowly acquired a faster tempo, and musicians incorporated other instruments such as the trumpet, congas, and piano.† In the 1940s and 1950s, Havana was the playground of the United States, and Cuban music such as the *son*, *mambo*, and *cha-cha-cha* made its way to New Orleans and New York. In New York, with the influence of the Puerto Rican community, it morphed into *salsa*, which in the last half of the twentieth century conquered the world. The question of when the clave son rhythmic pattern was incorporated into son music is more difficult to ascertain. Some musicologists believe that the son borrowed the pattern from the rumba music when it migrated to Havana.

In the rock-and-roll period of the 1950s and 1960s in the United States, there was one rhythm that was used by so many musicians in so many songs that it stuck out from among all others. It was called the *Bo-Diddley Beat*, named after the singer and song-writer Bo Diddley who made it popular in several of his recordings. In 1955, on the nationally televised *Ed Sullivan Show*, Bo Diddley performed one of his own songs called *Bo Diddley*, violating the contract he had signed to perform another song made popular by Tennessee Ernie Ford, titled *Sixteen Tons*. Bo Diddley's song, sometimes referred to as a "freight-train" stomp, had a hard driving guitar and drum rhythm, also known by the mnemonic *shave and a hair cut, two bits*, often used as a door-knock pattern, and referred to by H. C. Longuet-Higgins as the "cliché rhythm."‡ It is shown in box notation in Figure 38.6 (line two). Much music inspired by the Bo Diddley Beat has a similar feel but uses variations of this basic rhythmic pattern. A small sample of these is listed in Figure 38.6 starting with the clave son itself (on line 1), which does not have the Bo Diddley Beat onsets at pulses two and four.

The third example in Figure 38.6 shows the floor-tom rhythm (actually played on a cardboard box) used in Buddy Holly's *Not Fade Away* recorded in 1957. It has dropped the second onset of the Bo Diddley Beat and inserted an onset at pulse nine. This version has the feel of the *shiko* timeline [4-2-4-2-4] with anticipatory onsets at pulses three and nine. In addition to this floor-tom rhythm, another variant is heard on backup vocals sung by *The Crickets*, shown on line four. Here, the gray circles indicate the hummed consonant sound *mmm* and the black circles the plosive syllable sound *pa*.

* Manuel, P. (2009), p. 189.
† Orovio, H. (1992), p. 456.
‡ Longuet-Higgins et al. (1994), p. 105.

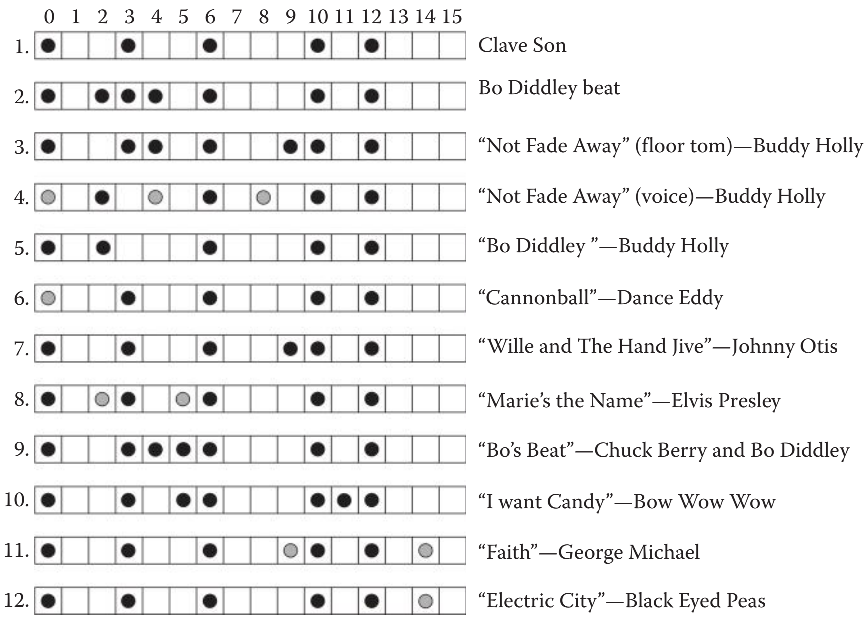


FIGURE 38.6 A small sample of the variants of the *clave son* rhythmic pattern.

Also in 1957, Buddy Holly released the song *Bo Diddley* that makes use of the strong drum timeline shown on line five. Rather surprisingly, in this version of the clave son, Buddy Holly moved the second onset from pulse three to pulse two, converting it in effect to a rotation of the *shiko* timeline. In another Buddy Holly song, the chorus sings another variant with the mnemonic: “DUM DE DUM DUM,” “OH BOY.”*

In the 1958 release of Duane Eddy’s *Cannon Ball*, the song starts with a strong high-pitched sound marking out the last four onsets of the clave son (the black circles in line six). As the song progresses, these synthesized sounds are replaced by hand claps. The first onset is sounded throughout the piece with a drum (gray circle).

Line seven shows one of the top 10 hits on the U.S. Pop Chart in 1958: “Willie and The Hand Jive” by Johnny Otis. It may be obtained by removing the third onset at pulse four in Buddy Holly’s drum pattern in *Not Fade Away*.

The eighth line shows the guitar rhythm in Elvis Presley’s 1961 rendition of *Marie’s the Name* (*of his latest flame*). The guitar riff has five strong downward strums that coincide with the clave son onsets, and two (anticipatory) upward strums on pulses two and five, in effect converting the first three onsets of the clave son to the *cinquillo* rhythm.

In 1964, Bo Diddley and Chuck Berry released a song called *Bo’s Rhythm* on the album *Two Great Guitars*. The guitar rhythm at times plays the pattern on line nine, which consists of the clave son with two additional filling onsets at pulses four and five. When the rhythm is strummed on a guitar at a fast pace, it is convenient for a performer to add these two onsets so that the hand strums the pattern *up-down-up-down*.

* Brady, B. (2002), p. 82.

In 1982, the group *Bow Wow Wow* released the song *I Want Candy* in which the rhythm shown on line 10 is played on the *timbales*. This rhythm consists of the clave son with two fill onsets added at pulses 5 and 11.

George Michael's *Faith* released in 1987 has a bass line that follows the first six onsets of the pattern in line 11. It embellishes the clave son with two onsets, one anticipatory bass note at pulse nine, and a high-pitched snapping sound at pulse 14 that helps to propel the cycle forward by breaking the two-onset closing response section of the clave son.

Much more recently in 2009, the Los Angeles group *The Black Eyed Peas* released the album *The E.N.D.* (short for *The Energy Never Dies*). The song *Electric City* in this album uses a strong bass in the form of the clave son with a snapping high-pitched onset at pulse 14, as in George Michael's *Faith* shown on line 12.

Figure 38.7 shows the histogram of the frequencies with which each onset is used in the variants of the Bo Diddley rhythm listed in Figure 38.6. It is clear from this histogram that the indispensable beats are the five onsets that make up the clave son, whereas the embellishment onsets occur mainly between the second and third onsets of the clave son, and just before the fourth onset.

In the *Dictionary of Cuban Music* Helio Orovio writes that the 16-pulse pattern shown in Figure 38.8 forms the rhythmic cell that characterizes Cuban music. It consists of the syncopated cinquillo pattern [2-1-2-1-2] followed by the regular pattern [2-2-2-2]. The former he calls *compás fuerte* (strong timeline), and the latter *compás débil* (weak timeline), and he emphasizes that the strong and weak parts always alternate. This rhythmic cell contains the clave son as a subset. Comparing this rhythmic cell to the clave son variants listed in Figure 38.6 shows that it contains several others as well. This pattern is also used as a bell timeline rhythm in West African music. The further slight variant shown in Figure 38.9 is the timeline of the *kassa* rhythm of Guinea.

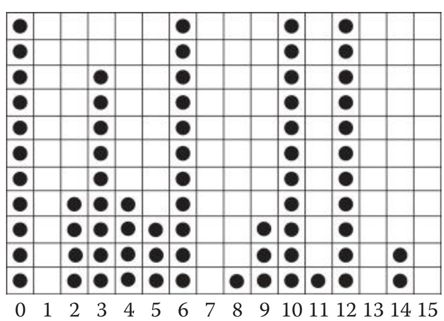


FIGURE 38.7 The histogram of the onsets in the Bo Diddley variants of Figure 38.6.

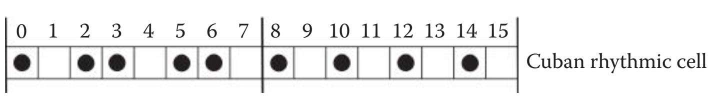


FIGURE 38.8 The 16-pulse rhythmic cell that characterizes Cuban music.

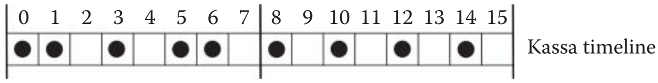


FIGURE 38.9 The *kassa* timeline from Guinea.

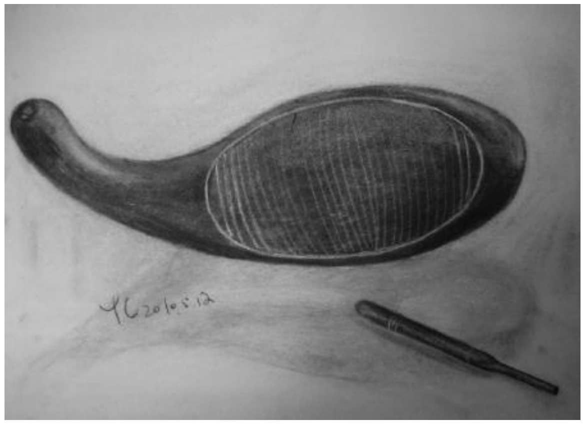


FIGURE 38.10 The *guiro*. (Courtesy of Yang Liu.)

Both rhythms shown in Figures 38.8 and 38.9 are rhythms played on the *guiro* in the Danzón style of Cuban music. The guiro, shown in Figure 38.10, is made from a hollowed-out gourd, on which parallel ridges are carved out at small intervals. It is played by means of scraping a thin stick made of wood or metal over the ridges.

Rhythm timelines are a type of *cultural object*. A rhythm that is considered successful from an evolutionary perspective will survive and multiply. The power of a rhythm timeline to disseminate itself among different cultural communities is the property of *fecundity*. Abundant evidence has been put forward that the clave son is such a rhythm, and thus it is only fitting that it should be investigated to determine the musicological and mathematical reasons for its fertility.* By analyzing the reasons for the success of the clave son, we should be able to gain insight into what makes a "good" rhythm good in general.

* Toussaint, G. T. (2011).

Epilogue

In this book, I have presented some of the computational and mathematical foundations for the characterization of good musical rhythms, as well as for their phylogenetic analysis, by focusing on the structure and the roots of one rhythm in particular, that has withstood the test of time, the *clave son*. The attempt to discover the origin of this rhythm raises a more general question. How old is musical rhythm? This question is difficult to answer in absolute terms, but I can venture a guess that musical rhythm is probably as old as the human mind. Even before humans were creating rhythm with musical instruments, the sound of rhythm was abundantly inherent in nature as the effects of wind, rain, rivers, waterfalls, thunderstorms, and galloping horses, not to mention the "music" produced by animals.[*] Although many of these sounds may lack a discernible cyclic pattern with precise small interval ratios determined by integers, once in a while nature will provide one. Furthermore, the perception of rhythm, as we have seen, is very much a constructive process in which the mind projects a categorical pattern on an input stimulus. Vijay Iyer considers music perception and cognition to be *embodied activities*.[†] Therefore, even if nature produces an arrhythmic (continuous) sound pattern, the human mind will mold it into a categorical (discrete) rhythm. Thus, in the human mind, rhythm is as old as the mind's ability to recognize perceptual categories.

The perception of rhythm in nature is one thing, but what about the creation of rhythm with musical instruments? Some archeologists have claimed that the first deliberately built musical instrument is a flute made out of bone in the late Stone Age, found in Germany.[‡] There is still debate about whether the holes found on this bone were made judiciously by the human hand or serendipitously by some animal's teeth. In any case, the question of designing and building sophisticated musical instruments by hand, such as the flute, violin, or harpsichord may be dispensed with outright, since human hands themselves become musical instruments when clapped, as exemplified so plainly in the *hambone* rhythms used by the slaves in African-American plantations, the *flamenco* music of Southern Spain, or the minimalist musical composition *Clapping Music* by Steve Reich. Elevating the flute to the status of the earliest musical instrument reflects a long-standing bias against rhythm as the fundamental property of music. Jeremy Montagu has suggested that during the early Stone Age when people were making stone tools, stones may have become the first musical

[*] Gray et al. (2001), p. 52, describe the structural similarities in the rhythms and scales used by animals and humans.
[†] Iyer, V. (2002).
[‡] Kunej, D. and Turk, I. (2000), p. 235.

instruments used for making concussive sounds.* The stones may have given way to bone or hard wooden sticks such as the *clappers* used in ancient China, or the *claves* so popular today in many parts of the world. Stone chimes have been used in ancient China for thousands of years, and have made their way to most parts of the world. In many places, stones are still used in music today. In the Basque country of the northeastern part of Spain for instance, *Xalaparta* music incorporates wooden sticks pounded on stone slabs (lithophones) of different thickness and composition that produce sounds with a wide variety of different pitches and timbres.

In this book, I have focused on more specific questions. How old is the rhythmic pattern with inter-onset interval structure [3-3-4-2-4], what is so special about this particular permutation of the numbers two, three, and four, and how did it manage to conquer the world, to become a household rhythm, a popular door-knock pattern? There are surely cultural, sociological, technological, economical, military, and political factors that may be partially responsible for the migrations of this rhythm from one culture to another, and from one continent to another. In addition to such diffusion or *ethnogenesis*, this rhythm may have been discovered in different places, and at different times, completely independently, a process referred to as *polygenesis*.† This is almost certainly true for the ubiquitous [3-3-2] rhythm timeline, and it is not unreasonable to suppose that the concatenation of two instances of this rhythm, given by [3-3-2-3-3-2] may have been independently transformed to [3-3-4-2-4] in geographically isolated cultures. However, the emphasis in this book has been the exploration of which mathematical, musicological, and psychological properties have made this pattern the salient universal cultural meme that it is today? To analyze this rhythm in a meaningful context, the related more general question that has been considered is: what makes a "good" rhythm good? This question led to the consideration of a wide variety of mathematical properties that good rhythms possess, thus narrowing down the infinitude of possible candidate rhythms to smaller groups of rhythms that could then be narrowed down further so as to characterize the pattern [3-3-4-2-4] as distinct from all others, in terms of these properties. In the end, the combination of the mathematical property of *shadow contour isomorphism* with the psychological property of *gestalt despatialization* accomplished the task of reducing the smaller groups to the unique rhythm: the *clave son*.

An implicit question repeatedly considered in this book has been: What can *mathematics* tell us about musical rhythm, and is what it tells us useful? From the examination of all the mathematical, musicological, and psychological properties discussed and applied to rhythm timelines, the general pattern that emerges in attempting to characterize the *clave son* rhythm may be described by the philosophy of the *golden mean*. This philosophy has surfaced in several different parts of the world at different times in history. The Chinese philosopher Confucius espoused it, the Greek philosopher Aristotle promoted it, and Buddhist philosophy embraced it, calling it the *middle path*. This philosophy proposes that for achieving success in life no single property or attribute of humanity should be

* Montagu, J. (2004), p. 171.

† List, G. (1978), p. 46.

present in excess; the properties should exist in moderation between that of excess and deficiency. From the 21 rhythmic properties listed in Figures 37.11 and 37.13, it may be observed that for almost all these properties, the clave son takes on a value between the possible extremes, and indeed close to the middle of the range of values afforded by each of these properties.

One of the main goals of this book has been to promote the application of phylogenetics tools, first successfully applied to DNA analysis by biologists, and later to the evolution of cultural objects such as language, to the analysis of musical rhythms. Although in the past few decades phylogenetic techniques have been used by archaeologists and anthropologists to study the evolution of a wide variety of cultural objects besides language, such as stone projectile points, helmets, swords, pottery compositions, pottery designs, baskets, puberty rituals, marriage patterns, written text, textile designs, and even musical instruments, music itself has been conspicuously left out of these studies. The phylogenetic analysis of musical rhythm is just beginning. A challenging problem for future research is whether we can obtain a phylogeny of the world's rhythms? The phylogenetic analysis of musical rhythm, especially in the form of timelines and ostinatos, has much to contribute not only to musicology but also to the study of human migrations as well, as has already been done successfully with language and DNA molecules.

There is an ongoing debate about which came first, music or language.[*] A new model of the evolution of music put forward by Steven Brown suggests that *both* music and language evolved from a common ancestral cultural object that he calls *musilanguage*. Musilanguage has the properties of lexical tone, combinatorial phrase formation, and expressive phrasing mechanisms. Just as rhythm is the most important and fundamental aspect of music, it is probably also the case that rhythm is the fountain of musilanguage. Indeed, Bruce Richman has suggested that groups of sounds evolve into definite patterns with meaning by means of the creation of expectancies based on the repetition and regularity of rhythms.[†]

The nature of the studies on the relationship between music and culture has been mostly qualitative, descriptive, and ethnographic rather than quantitative. The folklorist and ethnomusicologist Alan Lomax was one of the pioneers in steering ethnomusicological research away from such an approach, and into a more quantitative direction. Over many years, he carried out an extensive cross-cultural project to determine to what extent song-style reflects other cultural patterns. To carry out this project, Lomax developed a system he called *Cantometrics* for measuring song styles and correlating the resulting features with cultural data. To label his data consisting of thousands of songs from all over the world, he trained human subjects to detect a variety of features of song style. Among these many features there were five rather coarse measures of rhythm type: (1) perfectly regular rhythms he called *one-beat rhythms*, (2) *simple* measures consisting of 2, 4, 6, 8, 9, or 12 pulses per cycle, in which the accent falls on the first pulse, (3) *complex* meters in which the cycle

[*] Gray et al. (2001), p. 54, describe evidence that "seems to signal . . . that music has a more ancient origin even than human language."

[†] Richman, B. (2001), p. 304.

time span is not divisible by two or three, (4) *irregular* rhythms in which the accents occur at irregular intervals, and (5) rhythms in which no accent pattern may be distinguished, which he called *free rhythms* or *parlando rubato*. With the resulting data, the song styles were submitted to classical cluster analyses to obtain a categorization of songs that could then be correlated with both culture and geographical location. The approach proposed to study the evolution of music in this book differs from *Cantometrics* on several fronts. For one it concerns rhythm only, on the grounds that rhythm is the most fundamental aspect of music, and should be explored and understood in detail first, before analyzing the other parameters of music. The features of rhythm proposed here are more numerous and much more detailed than the coarse rhythmic features employed in *Cantometrics*. The rhythmic features used here are calculated objectively by quantitative mathematical formulas rather than subjectively by trained human subjects. The measures of similarity between pairs of rhythms employed here are more sophisticated than those used in the classical cluster analysis programs. Finally, and perhaps most importantly, the new phylogenetic analysis tools developed in the field of computational evolutionary biology are being applied in the new approach proposed here. These phylogenetic techniques have been illustrated with two examples of their application to musical rhythm: flamenco meters and Afro-Cuban timelines. In both cases, interesting and promising results are evident. In the case of flamenco meters, the reconstruction of the ancestral rhythm generated from the phylogenetic tree coincided with the *fandango de Huelva*, confirming the musicologists' consensus concerning the origin and geographical location of flamenco music. In the case of the six distinguished Afro-Cuban timelines, the phylogenetic analysis located the clave son at the center of the phylogenetic tree, indicating that this rhythm is most like all the others, and suggesting that it is also the oldest of the six. Being "most like all the others" is an independent confirmation of its "middle-path" or "golden-mean" property prevalent in the mathematical properties listed in Figures 37.11 and 37.13, and provides a mathematical explanation for how the clave son became such a distinguished rhythm. Also, in terms of historical records, it has the oldest entry, dating back to a thirteenth century manuscript written in Baghdad by music scholar Safi al Din.

The systematic computational methods presented here for generating and classifying good rhythm timelines within the broad context of designing tools for composition, and fitness functions for genetic algorithms, fall under the general umbrella that includes structural and generative methods for the analysis of the rhythm timelines of West and Central Africa in a cultural context. It is hoped that the structural properties of the rhythm timelines explored here, their mathematical formulations, and the algorithms used to generate these rhythms will help not only in the quest to determine a characterization of what makes a "good" rhythm good, but will also provide useful ideas for the ethnomusicologist and the composer.

It is both surprising and encouraging that much useful information may be garnered from a purely mathematical analysis of the durational patterns of musical rhythm, an analysis that ignores all the other less mathematical parameters associated with musical rhythm. Nevertheless, to characterize the uniqueness of the clave son rhythm, it was necessary to enlist the psychological property of *gestalt despatialization*. Perhaps the moral of this story, at least at this point in time, is that since the perception of rhythm is in part intrinsically subjective and constructed in our minds, a purely mathematical solution to this characterization problem is not attainable. On the other hand, it may be that the right mathematical model has not yet been found, and that to be successful such a model must take into account the neurobiology of rhythm. Thus, the geometrical foundations laid out in this book beckon the infusion of higher-level quantitative musicological knowledge into the mathematical structures considered thus far, in order to determine the "correlation between external stimuli and internal structures," and thus to more fully and accurately understand musical rhythm.[*] Such a venture opens up a minefield for systematic, comparative, and computational musicology.[†]

[*] Goldenberg, J., Mazursky, D., and Solomon, S. (2001), p. 2433.
[†] Parncutt, R. (2007).

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ISBN: 978-1-4665-1202-3
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