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CORES, AGGRESSIVENESS AND THE BREAKDOWN OF COOPERATION IN ECONOMIC GAMES

Leif JOHANSEN*

University of Oslo, Oslo 3, Norway

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The paper develops the use of the core as a solution concept in game theory in two interrelated directions. In the first place, an indicator of aggressiveness of claims is introduced in a modified definition of the core. The modified core may be smaller than the usual core, and may fail to exist if aggressiveness increases beyond some critical level. In the second place the article gives a formulation of a mixed cooperative/non-cooperative game, in which the game will be played cooperatively within coalitions, but non-cooperatively as between coalitions. A mixed cooperative/non-cooperative solution obtains if the grand coalition of all players fail to materialize because the various claims are incompatible. The two directions referred to are interrelated because the level of aggressiveness may be decisive for whether or not the grand coalition, and possibly other coalitions, will break down. The final section of the paper draws some general conclusions and relates the approach to other ideas in the literature.

1. Introduction

The core of a game is a useful solution concept in cooperative game theory. This is especially so when the core contains only one imputation. This core-imputation would then seem to be a reasonable prediction of the outcome of the game. In other cases the knowledge about the core of a game provides useful information, but is insufficient as a basis for prediction. If the core contains more than one imputation, there is a sort of surplus to be divided among the players, and when the core is empty there is a sort of deficit to be shared. In both cases there must be some kind of bargaining process which determines what is left undetermined by the ordinary core consideration.

My purpose is to present some considerations concerning the determination of the outcome of the game in such cases. I shall mainly have in mind the case where the core contains more than one imputation. A common-sense observation is that the final outcome in such a case would be determined by how aggressive the various players and coalitions of players are. I shall take this as a starting point and introduce aggressiveness into the game model in the simplest possible way. With some degree of aggressiveness

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on the part of the various players, the set of possible outcomes narrows down as compared with the ordinary core. If there is too much of aggressiveness, then a cooperative solution may fail to exist. One may say that the game breaks down as a cooperative game, and the situation may disintegrate into non-cooperative behaviour or a mixture of cooperative behaviour within some coalitions and non-cooperative behaviour as between coalitions. Thus the introduction of aggressiveness into the theory leads naturally to a sort of generalized game theory in which it is not necessarily determined a priori whether the game is a cooperative or a non-cooperative game. In such a theory aggressiveness may lead to 'collective irrationality' in the sense that the set of all players fail to realize a potential gain to all players which is implicit in the situation. Such breakdowns or failures are facts of life. I think, therefore, that a realistic game theory must somehow include such possibilities. [Some arguments pertaining to the fact that a bargaining situation does not necessarily lead to a collectively rational outcome are given in Johansen (1979).]

If we introduce aggressiveness into the game, then doubt will also be cast on the ability of the ordinary core theory to predict the outcome when the core contains only one imputation. This point will, however, not be pursued in detail in the following.

For the bargaining situation which occurs when the core is empty, the question is not about aggressiveness in attempts to conquer parts of the 'surplus', but rather a question about willingness to yield. I shall not consider this situation further, but only suggest that the considerations which I am going to present might be applicable, with some modifications, also to this situation.

To characterize a game I shall, as a starting point, use the characteristic function. If there are n players, $1, 2, \ldots, n$, I shall use N as a symbol for the set of all players, and S as a notation for any coalition from the set of all players. The notation $N \setminus S$ will be used for the complementary coalition of S, i.e., the coalition consisting of those players in N who do not belong to S. I shall assume transferable utility, and as usual $v(S)$ is the characteristic function. It will be interpreted in the usual way as the maximum value that coalition S can secure for itself without cooperating with other players.

It would be possible to develop the ideas to be presented in the following also for the case of non-transferable utility. Furthermore, one could distinguish between different types of cores according to the more specific definition of what is meant by 'the maximum value that a coalition can secure for itself without cooperating with other players'. I shall, however, proceed on assumptions which make the analysis as simple as possible with regard to aspects which are not specific to the present approach.

Section 2 introduces the aggressiveness concept and considers the corresponding modification of the core concept. Section 3 considers the question of what happens when cooperation breaks down, and introduces a mixed cooperative/non-cooperative solution. One might say that we treat the degree of cooperation as an endogenous element in the model, instead of specifying it in advance. Section 4 summarizes and discusses the interpretation of the results. A more general formulation of a mixed cooperative/non-cooperative game and some brief technical considerations are given in an appendix.

2. The introduction of aggressive claims

Let the outcome for the n players be represented by a 'payoff vector' $x = (x_1, \ldots, x_n)$. The ordinary core is a set of payoff vectors defined by the following requirements:

$$\Sigma_N x_i = v(N),$$
$$\Sigma_S x_i \geq v(S) \quad \text{for every } S \subset N. \tag{2.1}$$

The first requirement is Pareto optimality, and the second requirement says that each coalition shall obtain at least as much as indicated by the value of the characteristic function for that coalition. This requirement is imposed for all possible coalitions contained in N, including the degenerate coalitions consisting of single members. The set of all outcomes $x = (x_1, \ldots, x_n)$ which satisfy requirements (2.1) is the ordinary core.

In the definition of the core in this sense each coalition is assumed to acquiesce in an outcome if it receives the value $v(S)$, which is defined as the value the coalition will be able to secure for itself independently of any cooperation with players not in the coalition. If there is a surplus in the game above what is needed to satisfy these levels for the various coalitions smaller than the grand coalition of all players, then nothing is assumed about claims in this surplus to be put forward by the various coalitions. Accordingly we may call the core in this sense the acquiescent core, a slightly illogical abbreviation for 'the core of a game between acquiescent players'.

Let us now consider the possibility that the players are concerned with the surpluses mentioned (which will exist when the game is an essential game). If we have two coalitions S_1 and S_2 which contain no common member, and

$$v(S_1 \cup S_2) - v(S_1) - v(S_2) > 0, \tag{2.2}$$

then (2.2) represents a surplus reachable for coalitions S_1 and S_2. If S_1 and S_2 do not make up the whole set N then there may be surpluses above this again in which they may also try to get their share.

Let us now consider a coalition S and the aspiration it may have to acquire a share in the surplus available in the game. Coalition S knows that it can obtain $v(S)$ without cooperation with other players. Furthermore, it knows that the total value of the game is $v(N)$. Out of this the complementary coalition can secure for itself without the consent of S the value $v(N\backslash S)$. Accordingly there is a surplus available which amounts to

$$v(N)-v(S)-v(N\backslash S). \tag{2.3}$$

Knowing the availability of this surplus coalition S may aspire to get more than $v(S)$. Let the aspiration of coalition S be to achieve $R(S)$, which we shall call the claim of coalition S. The most aggressive claim the coalition can make is to acquire the whole surplus, i.e., to claim the value (2.3) in addition to $v(S)$. Thus the most aggressive claim for any coalition would be

$$R(S)=v(N)-v(N\backslash S). \tag{2.4}$$

If a core should exist when all coalitions claim $R(S)$ instead of only the value $v(S)$, then we could call this the maximally aggressive core. However, for essential games such a core will obviously not exist. If we add the claims of coalition S and the complementary coalition $N\backslash S$, then we have

$$R(S)+R(N\backslash S)=v(N)-v(N\backslash S)+v(N)-v(S)$$

which exceeds $v(N)$ if $v(N)>v(S)+v(N\backslash S)$.

Let us therefore introduce the degree of aggressiveness of coalition S as the share λ_S which the coalition will aspire to get in the surplus. Then the claim of coalition S will be

$$R(S)=v(S)+\lambda_S[v(N)-v(S)-v(N\backslash S)]. \tag{2.5}$$

With $\lambda_S=0$ for all coalitions we have the ordinary or acquiescent core; with $\lambda_S=1$ for all coalitions we have the maximally aggressive core, which fails to exist for essential games; with coefficients λ_S in the intermediate range at least for some coalitions we may have a core which is more narrow than the ordinary core. According to the level of the λ-coefficients we may speak about acquiescent and more or less aggressive cores (or perhaps acquiescent and more or less acquisitive cores).

It should be observed that according to the definitions given above each coalition adopts a realistic attitude in the sense that it assumes that the players not in the coalition are able to form the complementary coalition so as to secure at least its value $v(N\backslash S)$. A coalition which is overly optimistic might claim

$$R(S)=v(S)+\lambda_S[v(N)-\Sigma_{N\backslash S}v(\{i\})],$$

where $v(\{i\})$ is the value of the characteristic function for the single player i. This would correspond to a sort of successful divide and conquer strategy where the cooperation of the members not belonging to S is secured so as to realize the total value $v(N)$ without the complementary coalition $N \backslash S$ being able to form and put forward a claim on the basis of the value $v(N \backslash S)$. Our aggressive players corresponding to (2.5) limit their claims to a share in the surplus which is available if the non-members of the coalition are able to form the complementary coalition and act on that basis.

We now proceed on the basis of claims as indicated by (2.5). Corresponding to (2.1) the core is now defined by

$$\begin{aligned} &\Sigma_N x_i = v(N), \\ &\Sigma_S x_i \geqq R(S) = v(S) + \lambda_S[v(N) - v(S) - v(N \backslash S)] \qquad (S \subset N). \end{aligned} \tag{2.6}$$

For a core in this sense to exist it is clearly *necessary* that

$$R(S) + R(N \backslash S) \leqq v(N) \quad \text{for all} \quad S \subset N. \tag{2.7}$$

If there is a surplus corresponding to the partition into S and $N \backslash S$, i.e., if $v(S) + v(N \backslash S) < v(N)$, then condition (2.7) is equivalent to the obvious requirement

$$\lambda_S + \lambda_{N \backslash S} \leqq 1. \tag{2.8}$$

If we have the special case of equal degree of aggressiveness for all coalitions, then it is necessary that this common degree λ is $\lambda \leqq \frac{1}{2}$. (In some cases it might be more natural to define equal degree of aggressiveness in other ways, for instance by letting the values of the λ_S's be proportional to the size of the coalitions. However, the players may be of different types — firms, persons, organizations etc. — so that a per-head consideration is irrelevant. In any case, the idea of equal degree of aggressiveness is used here and occasionally in the sequel only for illustrative purposes.)

However, the conditions just mentioned are not sufficient for the existence of the core. In other words, even if the condition (2.8) should be fulfilled for all coalitions the core may still fail to exist. In particular, for the case of a uniform degree of aggressiveness, the core may still fail to exist even if this degree is $\lambda = \frac{1}{2}$ or less. *The last statement means that a principle of equal split for all coalitions versus the complementary coalitions may be infeasible.*

Considering the definition (2.6) of the aggressive core we may think of a process by which we increase the λ_S's from zero in some way. The core will then shrink from the ordinary core and become narrower as we increase the λ_S's. For some constellation of the aggressiveness coefficients the core will consist of one uniquely determined outcome. We may also reverse the question and ask if every outcome in the ordinary core can be generated as a unique outcome by some constellation of aggressiveness coefficients for various coalitions. The answer is affirmative. Consider an outcome satisfying (2.1). We may then put

$$\Sigma_S x_i = R(S) \geqq v(S),$$

$$\Sigma_{N\setminus S} x_i = R(N\setminus S) \geqq v(N\setminus S).$$

This means that both coalition S and coalition $N\setminus S$ see their claims precisely met. Using definition (2.5) this gives the following aggressiveness coefficients:

$$\lambda_S = \frac{\Sigma_S x_i - v(S)}{v(N)-v(S)-v(N\setminus S)}, \qquad \lambda_{N\setminus S} = \frac{\Sigma_{N\setminus S} x_i - v(N\setminus S)}{v(N)-v(S)-v(N\setminus S)} \tag{2.9}$$

which add up to unity for λ_S and $\lambda_{N\setminus S}$.

The determination of all coefficients λ_S according to (2.9) is in fact more than needed to produce the vector $x_1,\ldots,x_n$ as a unique outcome. A given vector $x_1,\ldots,x_n$ can be generated as a unique outcome by conditions referring to a subset of possible coalitions, while the claims of other coalitions are satisfied with some slack. Accordingly, *there will be several constellations of aggressiveness coefficients which will produce a given vector* $x_1,\ldots,x_n$ *as a uniquely determined outcome.*

The denominators in the formulas in (2.9) are positive if there is a surplus for the partition $S, N\setminus S$. If there is no such surplus, then the values of λ_S and $\lambda_{N\setminus S}$ are arbitrary.

Let us illustrate the concepts above briefly by means of a small example. Let there be a game of three players, and let the characteristic function be [where we, for simplicity, write $v(1)$ instead of $v(\{1\})$ etc.]

$$\begin{aligned} &v(1)=1, \quad v(2)=2, \quad v(3)=4,\\ &v(1,2)=4, \quad v(1,3)=6, \quad v(2,3)=6,\\ &v(1,2,3)=10. \end{aligned} \tag{2.10}$$

The corresponding ordinary or acquiescent core consists of outcomes x_1, x_2, x_3 which satisfy

$$\begin{aligned} &x_1 \geqq 1, \quad x_2 \geqq 2, \quad x_3 \geqq 4,\\ &x_1+x_2 \geqq 4, \quad x_1+x_3 \geqq 6, \quad x_2+x_3 \geqq 6,\\ &x_1+x_2+x_3=10. \end{aligned} \tag{2.11}$$

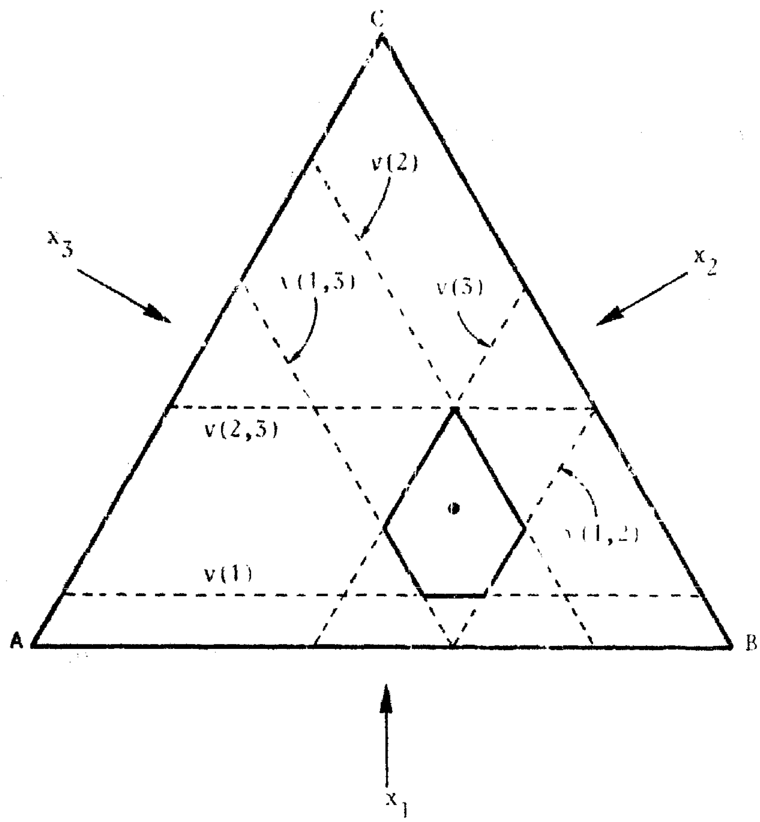

Fig. 1. The core of the game described by (2.10)–(2.11). The core is the pentagon inside the triangle. The point marked off inside the core is the point corresponding to $\lambda=\frac{3}{4}$.

The core is illustrated in fig. 1 where x_1 is measured vertically from AB, x_2 is measured vertically from BC, and x_3 is measured vertically from CA. The core is the pentagon inside the triangle. The extreme imputations (the corners of the pentagon) are the following:

$$(1,4,5),\quad (1,3,6),\quad (2,2,6),\quad (4,2,4),\quad (2,4,4).$$

If we take an arbitrary point in the core, for instance $x=(2,3,5)$ then we can easily calculate the corresponding aggressiveness coefficients from (2.9). For λ_1 we have for instance

$$\lambda_1=\frac{x_1-v(1)}{v(1,2,3)-v(1)-v(2,3)}=\frac{2-1}{10-1-6}=\frac{1}{3}.$$

Altogether we obtain

$$\lambda_1=\tfrac{1}{3},\quad \lambda_2=\tfrac{1}{2},\quad \lambda_3=\tfrac{1}{2},\quad \lambda_{1,2}=\tfrac{1}{2},\quad \lambda_{1,3}=\tfrac{1}{2},\quad \lambda_{2,3}=\tfrac{2}{3}.$$

As suggested above, this is, however, more than necessary to produce $x=(2,3,5)$ as a unique outcome. For instance, the values for λ_1, λ_2 and λ_3 would suffice (while $\lambda_{1,2}$, $\lambda_{1,3}$ and $\lambda_{2,3}$ were smaller than given above), and several other selections would also do.

For an extreme outcome some coalitions are pressed down to getting no more than their values. For generating such points the corresponding aggressiveness coefficients must, of course, be zero. For instance, for the outcome $x=(1,4,5)$ we have

$$\lambda_1=0,\quad \lambda_2=1,\quad \lambda_3=\tfrac{1}{2},\quad \lambda_{1,2}=\tfrac{1}{2},\quad \lambda_{1,3}=0,\quad \lambda_{2,3}=1.$$

It was pointed out above that not all constellations of aggressiveness coefficients which satisfy (2.8) will generate a feasible point. In other words, it is not sufficient for the existence of a solution that the aggressiveness of one coalition can always be accommodated by the complementary coalition. For instance, assume that players 1 and 2 are rather acquiescent players, having $\lambda_1=0.1$ and $\lambda_2=0.1$, while coalitions $\{1,3\}$ and $\{2,3\}$ are aggressive, having $\lambda_{1,3}=0.9$ and $\lambda_{2,3}=0.9$. (Players 1 and 2 are acquiescent individually, but player 3 infuses every coalition in which he participates with aggressiveness.) Then we have, according to (2.5),

$$R(1)=1+0.1(10-1-6)=1.3,$$

$$R(2)=2+0.1(10-2-6)=2.2,$$

$$R(1,3)=6+0.9(10-6-2)=7.8,$$

$$R(2,3)=6+0.9(10-6-1)=8.7.$$

The two last lines imply $x_1+x_3\geqq 7.8$ and $x_2+x_3\geqq 8.7$. Regardless of how acquiescent coalition $\{1,2\}$ is, it must at least have $x_1+x_2\geqq 4$. In conjunction these conditions imply $2(x_1+x_2+x_3)\geqq 20.5$, which is impossible, in spite of the fact that requirement (2.8) is fulfilled.

Let us next assume that there is a uniform degree of aggressiveness λ in the game, and let us calculate the maximum degree of aggressiveness for which the core still exists. With all $\lambda_S=\lambda$ we easily get from (2.5) when we use the characteristic function of the example:

$$R(1)=1+3\lambda,\qquad R(2)=2+2\lambda,\qquad R(3)=4+2\lambda,$$

$$R(1,2)=4+2\lambda,\quad R(1,3)=6+2\lambda,\quad R(2,3)=6+2\lambda.$$

We should now find the maximal value of λ for which these claims are compatible with the condition $x_1+x_2+x_3=10$. One finds that this is determined by

$$R(1)+R(2)+R(3)=7+7\lambda=10,$$

i.e., $\lambda=3/7$. The corresponding outcome is

$$x_1=2\tfrac{2}{7},\quad x_2=2\tfrac{6}{7},\quad x_3=4\tfrac{6}{7}.$$

This outcome satisfies the coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ with some slack. The corresponding point is marked off in fig. 1.

If there had been a common maximal level of aggressiveness allowing a solution for all games, then one might have expected a sort of convention to develop, and we might have had a theory with some predictive power. However, this is clearly not so. If we consider the game as a bargaining game, then the aggressiveness coefficients of the various coalitions are characteristics of the bargaining strategies of the various players and coalitions. One might imagine that the coalitions determined their λ_S-values as strategies in a non-cooperative game against the other players. Now for constellations of aggressiveness coefficients λ_S which exhaust the total value of the game, i.e., $v(N)$, the payoff of a coalition increases with its aggressiveness λ_S. (For instance, in the example given above x_1 would increase from 1 to 4 when λ_1 increases from 0 to 1 if the other players accommodate so that a solution exists.) In the terminology of non-cooperative game theory, there would then exist many *equilibrium* constellations of aggressiveness coefficients, but there would not exist a non-cooperative *solution*. We should then be prepared for the possibility that the aggressiveness in the game may turn out to be excessive so that no core in the sense defined here would exist. Then the game would break down in the present form, and we would need a more general framework. This is the subject of the next section. [For a more detailed discussion of bargaining and possible reasons for failure to reach agreements, see Johansen (1979).]

Before proceeding a few more words on the aggressiveness coefficients may be in order. They have been introduced above as given coefficients. However, I do not consider them to be autonomous or constant coefficients like coefficients in many other behavioural equations in economic theory and econometric models. They are meant to represent in a simple form the bargaining strategy of the various players and coalitions, as levels below which they are not prepared to reduce their claims, hoping that the other side (or sides) will acquiesce if necessary for reaching a settlement. They may be determined by many subjective factors characteristic of the specific situation, perhaps by reference to considerations which go beyond the formalized game (perhaps representable as a meta-game), and so on. In the formulations above we have introduced as many aggressiveness coefficients as there are possible coalitions (less than the grand coalition N), as if they are independent coefficients. There may be reasons for some relations between them, for instance between the coefficient λ_S of a coalition S and the coefficients corresponding to subsets of S. However, it would be premature to discuss such finer points at the present exploratory stage.

3. Possible breakdown of cooperation. Mixed cooperative/non-cooperative solutions

A game specified in terms of a characteristic function, perhaps modified by aggressiveness so as to be described in terms of a claim function $R(S)$, may fail to have a core. In our context a crucial question is what will then happen. A natural idea is that the grand coalition may fail to behave as a coalition and accordingly fail to realize the total value $v(N)$, and/or that one or more of the coalitions may break down so that they fail to make their claims effective according to the first specification. A solution may then exist corresponding to the reduced set of claims. However, the description of the game in the original form does not contain sufficient information for saying which coalition(s) will break down. Another approach might be to introduce a system of modification of claims or aggressiveness when the core fails to exist, perhaps along lines used in some types of bargaining theory for two parties. Such approaches may be realistic for certain situations, but they do not cover cases in which a failure to reach agreement may entail a breakdown of cooperation, so that the total value of the game as a cooperative game will not be realized. For a more general theory intended to cover such cases, we must give a more informative description of the game than that contained in the usual characteristic function and its extension by means of claims influenced by aggressiveness. A possible formalization is suggested in the following.

If we have as before a game of n players, then this game can now be played in many different ways. Let us designate by P a *partition* or a *coalition structure* of the set N of players into coalitions $S_1, S_2, \ldots, S_K$. Then the game can be played cooperatively as between players within each coalition, while the coalitions play non-cooperatively against each other. To be a little bit more specific we may think of each player as having at his disposal a set of possible actions (or strategies) A_i from which he has to choose an action (or a strategy) a_i. A coalition S will have at its disposal all actions $a_i \subset A_i$ for all players i belonging to coalition S. A coalition may also have action possibilities which are not simply combinations of individual actions, so that we could, in general, speak of the action a_S of coalition S, belonging to a set A_S of possible actions for the coalition. Now the pay-off to a coalition will depend on the actions taken by all coalitions so that we can write, for any coalition S_k,

$$\Sigma_{S_k} x_i = \phi_{S_k}(a_{S_1}, \ldots, a_{S_k}, \ldots, a_{S_K}), \qquad k = 1, \ldots, K, \tag{3.1}$$

or more briefly

$$\Sigma_{S_k} x_i = \phi_{S_k}(a \mid P), \qquad k = 1, \ldots, K, \tag{3.2}$$

where $a \mid P$ designates the actions by the coalitions under the partition P.[1]

We now assume that there exists a non-cooperative solution to this game, each coalition trying to maximize its total payoff. Let the corresponding actions be $\hat{a}$. Coalition S_k will then obtain $\phi_{S_k}(\hat{a} \mid P)$. We may use this for defining a sort of characteristic function by

$$v(S \mid P) = \phi_S(\hat{a} \mid P). \tag{3.3}$$

This is the total value reached by coalition S under a partition of the game corresponding to P, where S is one of the coalitions constituting the partition P.

For a more complete specification we must also say something about what finer sub-coalitions of the coalitions $S_1, \ldots, S_K$ in the partition P can get. If S is one of the coalitions in the partition P, then let S^* be a sub-coalition of S. Coalition S^* will claim a part of the value given by (3.3). The source of strength of coalition S^*, which can form the basis for the claim, will be the value that S^* can obtain by breaking away from the rest of the coalition S. This would be the value

$$v(S^* \mid P^*), \tag{3.4}$$

where P^* is the partition obtained from P by splitting S into S^* and $S - S^*$. Thus, for every sub-coalition S^* of a coalition S in the partition P the payoff to members of sub-coalition S^* should be not less than given by (3.4). [An alternative definition at this point might be to define the value v of coalition S^*, where $S^* \subset S$, as the value that S^* would obtain if the players in S, but not in S^*, did not necessarily form the complementary coalition $S \backslash S^*$. Instead we could define the value of $v(S^* \mid P^*)$ by considering what S^* obtains in the non-cooperative game where the worst possible sub-partition of $S \backslash S^*$ takes place. The worst possible case for S^* is not necessarily that

[1]The concept of a *partition* of the set of players so as to form a *coalition structure* is common in cooperative game theory, see, e.g., Luce and Raiffa (1958). However, in the present context it plays a somewhat different role in that the coalitions play non-cooperatively against each other. In fact, for some purposes it might be relevant, as in usual cooperative game theory, to consider coalition structures inside each of our coalitions $S_1, \ldots, S_k$ forming P.

$S \setminus S^*$ plays as one coalition — see the numerical example later on in this section. Some further comments on this point are given in the concluding section. An analogous problem exists in ordinary cooperative game theory about the definition or interpretation of the core.]

A solution for the mixed cooperative/non-cooperative game which we have described above will be a pair consisting of an outcome vector combined with a partition of the set of players, written as

$$(x \mid P) = (x_1, x_2, \ldots, x_n \mid S_1, \ldots, S_K). \tag{3.5}$$

The meaning of this is that the solution is obtained by cooperation within each coalition S_k contained in partition P, and non-cooperative play as between coalitions.

An outcome vector/partition which represents a solution should now satisfy several requirements. In the first place, each coalition in P should receive a total amount which corresponds to what it gets in the non-cooperative game, as defined by (3.3). Furthermore, each sub-coalition of a coalition in P should get at least as much as it can claim according to the considerations referring to (3.4). We may say that a core should exist for the internal cooperative game inside each coalition in the partition P. This can be interpreted as a condition for the coalitions in P to be viable, i.e., not to break down because of internal conflicts. The partition should also be such that no pair of coalitions can gain by merger, when the new coalition structure thereby established is also viable, since the two coalitions would then be likely to try to realize this potential gain — or we may introduce a similar condition referring to mergers of an arbitrary number of coalitions.

There are many problems involved in the definition of a solution, some of which will be discussed below. For clarity we will, however, put down the definition of a solution a little bit more precisely for the case based on the conditions already referred to:

Definition of a solution (special case). Let the function $v(S \mid P)$ be defined as explained by (3.1)–(3.3). An outcome vector $x = (x_1, \ldots, x_n)$ and a partition P of the players into coalitions $S_1, \ldots, S_K$ then form a solution if the following conditions hold:

Condition (a). Each coalition in P receives a total value given by

$$\Sigma_{i \in S_k} x_i = v(S_k \mid P), \qquad k = 1, 2, \ldots, K. \tag{3.6}$$

This condition gives to each coalition in P the value it achieves in the non-cooperative game between the coalitions.

Condition (b). For each coalition S_k which contains more than one player each sub-coalition S_k^* in S_k receives at least as much as it would obtain by breaking out of S_k, the partition otherwise remaining the same, i.e., we have for each $k=1,2,\ldots,K$, where S_k is not a single player,

$$\Sigma_{S_k^*} x_i \geqq v(S_k^* \mid P_k^*) \quad \text{for each} \quad S_k^* \subset S_k, \tag{3.7}$$

where P_k^* is the partition obtained when S_k is split into two, i.e.,

$$P_k^* = (S_1, \ldots, S_k^*, S_k \backslash S_k^*, \ldots, S_k).$$

For each S_k (with more than one player) these conditions say that the outcome for players in S_k should be in the core of the cooperative game that takes place within coalition S_k. (When this core exists coalition S_k is said to be viable.)

Condition (c). In the partition P there is no pair of coalitions, say S_j and S_k, such that the new partition obtained by merging S_j and S_k into the joint coalition $S_j \cup S_k$ (leaving the other coalitions unchanged) contains only viable coalitions and gives a gain to S_j and S_k. In other words, for each pair of coalitions S_j and S_k in P, either

$$\Sigma_{S_j} x_i + \Sigma_{S_k} x_i \geqq v(S_j \cup S_k \mid P_{jk}^U), \tag{3.8}$$

where P_{jk}^U is the partition obtained when S_j and S_k merge (and the other coalitions remain unchanged), or the coalition structure P_{jk}^U is not viable, i.e., it is not possible to satisfy conditions (a) and (b) for this new partition. This condition says that it is either not tempting for any pair of coalitions to merge, or the new coalition structure that would be established by such a merger would not be viable.

In what sense is it reasonable to call an output vector and a partition satisfying the conditions of this definition a solution? I think the most natural consideration is to start out from a proposed $(x \mid P)$ as a sort of initial proposal and test if there are tendencies to change it. With no change in the coalition structure the proposed solution has the same claim to being a solution as a non-cooperative solution in usual non-cooperative games, which need not to be discussed here. Condition (b) guards against changes through the most obvious action by a sub-coalition: unilateral withdrawal from the coalition to which it belongs according to the initial proposal. Condition (c) guards against the most obvious changes by joint actions by two coalitions. The considerations alluded to in connection with conditions (b) and (c) are rather simple-minded or myopic on the part of the coalitions or sub-coalitions involved, i.e., the players settle for $(x \mid P)$ if there are no rather easily detected opportunities of doing better for some of the players.

In connection with condition (c), coalitions S_j and S_k retreat to the initial coalition structure P if the contemplated merger proves to produce a structure which is not viable.

The definition given above could obviously be modified in various ways, especially according to the degree of sophistication or optimism/pessimism of the various players regarding action possibilities and analyses of consequences of possible actions. For instance, the value used to the right in condition (3.7) could be defined in other ways. The crucial issue is what kind of alternative arrangements of the coalitions the sub-coalition under consideration will take into account when it contemplates breaking out. In connection with condition (c) one could obviously, as already mentioned, stipulate similar conditions for mergers of more than two original coalitions. One could also construct more complex conditions by combining considerations from (b) and (c), referring to possibilities of members defecting from different original coalitions to form a new coalition. I shall not go through all possible combinations of assumptions here. Different combinations may be relevant or realistic in different situations. In the appendix a rather general solution is suggested (together with some comments on the relation to the concept of ψ-stability). The formulation given above is perhaps the simplest one possible for this kind of a mixed cooperative/non-cooperative game, and may serve as a basis for the following discussion.

In discussions of solution concepts in game theory the existence issue is often important. In the present case a solution will always exist, provided that the non-cooperative games involved have solutions. A 'constructive' argument will show this. Start by considering the partition of the set of players into the degenerate coalitions containing only single players, i.e., a situation in which the whole game is played non-cooperatively. For this situation condition (a) of the definition is fulfilled, while (b) is irrelevant. We should then explore whether it is possible to form a coalition of two players so that they gain by merging, thus violating condition (c) of the definition. (The other 'coalitions' of single players do, of course, remain viable.) If this fails, then the fully non-cooperative solution is a solution of our more general game. If it is possible to join two players into a viable coalition in the way indicated, then we have a new candidate for solution. We could then proceed by considering the possibilities of forming further coalitions, now checking not only whether there is a gain to the participants of a merger, but also exploring whether the coalitions in the new structure which contain more than one player are viable. We continue until this is no longer possible. We may end up with a fully cooperative solution, a solution consisting of a larger or smaller number of coalitions playing non-cooperatively against each other while playing cooperatively within coalitions, or a fully non-cooperative solution.

This procedure, starting from below with regard to cooperation, is bound to reveal at least one solution. But there may exist more solutions, some of which are not detectable by this procedure. A safe (but probably inefficient) procedure would be to explore all possible coalition structures, impose (3.6), and check which ones of the structures that can be made to satisfy conditions (b) and (c) of the definition. If no other structures satisfy the requirements, then the fully non-cooperative solution will do so. This solution has a sort of robustness which the other ones do not have, since the viability condition for coalitions — condition (b) — is not relevant for this case.

The formulation of condition (c) in the definition is important for the existence of a solution. According to the formulation given above a pair $(x|P)$ may be a solution even if (3.8) holds the other way around, i.e., even if

$$v(S_j \cup S_k \mid P_{jk}^U) > \Sigma_{S_j} x_i + \Sigma_{S_k} x_i, \tag{3.9}$$

provided that the new coalition structure P_{jk}^U is not viable. This is perfectly acceptable if it is the new coalition $S_j \cup S_k$ which is not viable. This case means that it is tempting to form the joint coalition because it can achieve an increased total value, but the new coalition does not manage to hold together. However, $(x|P)$ may be declared to be a solution even when (3.9) holds and the new coalition $S_j \cup S_k$ is viable, if the joining of S_j and S_k makes some other coalition, say S_m, non-viable. This may be all right, but will certainly also in some cases be a dubious consequence of the definition. This dubious consequence would be avoided if we replace condition (c) by

Condition (c′). In the partition P there is no pair of coalitions, say S_j and S_k, such that the joint coalition $S_j \cup S_k$ could be established as a viable coalition and give a gain while the other coalitions remain unchanged. In other words, for each pair of coalitions S_j and S_k, either (3.8) is fulfilled, or $S_j \cup S_k$ is not a viable coalition in the coalition structure P_{jk}^U [defined as in connection with (3.8)]

But then the existence of a solution would not be guaranteed. However, we could add the following condition to the definition:

Condition (*d*). If no $(x|P)$ satisfies conditions (a), (b) and (c′), then the solution is the fully non-cooperative case,

so that the full definition consists of (a), (b), (c′) and (d). This would seem to conform with the general tenor of the present approach: All attempts to establish coalitions with some cooperation would be frustrated, and one ends up playing the game in the fully non-cooperative manner.

Besides the existence problem we have the problem about uniqueness. As already suggested there is no reason to expect that a solution according to the definition given above (either in the original or the modified form) will be unique. There are two aspects of non-uniqueness, referring to the payoff vector for one and the same coalition structure, and to the coalition structure itself. It would be possible to add further conditions based on a comparison between the various $(x|P)$ which satisfy the conditions as stipulated above, but I shall not pursue this line here. (Some considerations are given in connection with the numerical example below.)

As pointed out the solution hinges on the existence of solutions to the non-cooperative games involved. As is well known non-cooperative *equilibria* will exist under quite general conditions (if mixed strategies are allowed), but the situation may be problematic if the non-cooperative equilibrium is not unique (unless the various equilibria are interchangeable in the terminology of the theory of non-cooperative games). This may be a serious issue, but it will not be discussed further here since it is not a problem created by the approach proposed here.

We have not yet introduced aggressiveness into the mixed cooperative/non-cooperative game. The place to do this is in connection with condition (b), i.e., (3.7) in the definition above. This condition refers to the internal viability of coalition S_k (consisting of more than one player). The sub-coalition S_k^* will, in analogy with formulation (2.3), perceive a surplus equal to

$$v(S_k \mid P) - v(S_k^* \mid P_k^*) - v(S_k \backslash S_k^* \mid P_k^*). \tag{3.10}$$

This is the gain to S_k when the sub-coalitions S_k^* and $S_k \backslash S_k^*$ play cooperatively as compared with non-cooperatively against the other coalitions in the partition P. [P_k^* is as defined after (3.7).] On this basis we can, corresponding to (2.5), formulate the claim of sub-coalition S_k^* under the partition P as

$$R(S_k^* \mid P) = v(S_k^* \mid P_k^*) + \lambda_{S_k^*}[v(S_k \mid P) - v(S_k^* \mid P_k^*) - v(S_k \backslash S_k^* \mid P_k^*)], \tag{3.11}$$

where S_k belongs to P and $S_k^* \subset S_k$. Then this claim will replace $v(S_k^* \mid P_k^*)$ in condition (3.7). Otherwise the definition may stand as it is rendered before. Players can raise claims only against a coalition of which they are members; against other players they have to accept the outcome of the non-cooperative game as far as this affects the total value of the coalition of which they are members.

Let us illustrate the concepts and definitions given above by a small example, similar to (2.10) but now extended as necessary in the present context:

$$\begin{aligned} &v(1 \mid 1,2,3)=0, \quad v(2 \mid 1,2,3)=2, \quad v(3 \mid 1,2,3)=3, \\ &v(1 \mid 1,\{2,3\})=1, \quad v(\{2,3\} \mid 1,\{2,3\})=6, \\ &v(2 \mid \{1,3\},2)=2, \quad v(\{1,3\} \mid \{1,3\},2)=6, \\ &v(3 \mid \{1,2\},3)=4, \quad v(\{1,2\} \mid \{1,2\},3)=4, \\ &v(\{1,2,3\} \mid \{1,2,3\})=10. \end{aligned} \tag{3.12}$$

In this arrangement we have used $\{\ \}$ to indicate coalitions, so that for instance $\{1,2\}$ is the coalition of players 1 and 2. For simplicity we omit the braces when a single player is a 'coalition' by himself.

In the set-up in (3.12) the first line then gives the values achieved by the individual players when the game is played as fully non-cooperative. On the second line are given the values obtained by player 1 and the coalition $\{2,3\}$ when the partition is such that player 1 plays by himself and players 2 and 3 join in a coalition. In the same way the third line gives the results when the partition is $\{1,3\},2$ and the next line the values when the partition is $\{1,2\},3$. On the fourth line is indicated that the total value achieved when the game is played cooperatively for all players is 10. The example is such that the total value achieved for all players is larger the more there is of cooperation. By comparing $v(1 \mid 1,2,3)$ and $v(1 \mid 1,\{2,3\})$ we see that player 1 gains something if players 2 and 3 join in a coalition. In the same way we see that player 3 gains if players 1 and 2 join, while player 2 is left unaffected if players 1 and 3 join. It could also easily be the case that a player loses if other players join in a coalition.

Let us now check that a solution exists according to the definition given by (3.6)–(3.8). In this case it is easily seen that an outcome vector and a partition given by

$$(x_1, x_2, x_3 \mid \{1,2,3\}), \tag{3.13}$$

where x_1, x_2, x_3 satisfy the same conditions as (2.11) is a solution according to the definition. Here the structure is the simple one in which all players join in one coalition. The conditions for viability of this coalition consist in applying (3.7) to single defectors and defecting sub-coalitions consisting of two players. These considerations give the inequalities listed in (2.11). Since we have already the largest coalition possible condition (3.8) is not relevant in this case. [If a defector has to be prepared for the worst possible coalition formation of the other players if he defects from the original coalition, then the result would be modified. If player 1 defects, he would then be prepared for players 2 and 3 acting as single players rather than in a coalition $\{2,3\}$, and he could claim only 0 instead of 1. In the same way player 3 could claim only 3 instead of 4. This would give a somewhat larger core than corresponding to (2.11) and illustrated in fig. 1.]

If we test other partitions than the one involved in (3.13), then we will see by means of condition (3.8) that these will not constitute solutions.

Let us now introduce aggression into the game as described by the characteristic function in (3.12). We then establish the claims for all sub-coalitions in coalitions contained in each partition by applying formula (3.11). For instance, the claim of player 1 within the coalition $\{1,2,3\}$ will be

$$R(1\,|\,\{1,2,3\})=v(1\,|\,1,\{2,3\})+\lambda_1[v(\{1,2,3\}\,|\,\{1,2,3\}) -v(1\,|\,1,\{2,3\})-v(\{2,3\}\,|\,1,\{2,3\})]. \tag{3.14}$$

Altogether we get the following claims:

$$\begin{aligned}
&R(1\,|\,\{1,2,3\})=1+3\lambda_1,\\
&R(2\,|\,\{1,2,3\})=2+2\lambda_2,\\
&R(3\,|\,\{1,2,3\})=4+2\lambda_3,\\
&R(\{1,2\}\,|\,\{1,2,3\})=4+2\lambda_{1,2},\\
&R(\{1,3\}\,|\,\{1,2,3\})=6+2\lambda_{1,3},\\
&R(\{2,3\}\,|\,\{1,2,3\})=6+3\lambda_{2,3},\\
&R(1\,|\,\{1,2\},3)=0+2\lambda_1,\\
&R(2\,|\,\{1,2\},3)=2+2\lambda_2,\\
&R(1\,|\,\{1,3\},2)=0+3\lambda_1,\\
&R(3\,|\,\{1,3\},2)=3+3\lambda_3,\\
&R(2\,|\,1,\{2,3\})=2+\lambda_2,\\
&R(3\,|\,1,\{2,3\})=3+\lambda_3.
\end{aligned} \tag{3.15}$$

It is assumed here that each player is equally aggressive within each coalition where he may be a member, and similarly that each coalition is equally aggressive within each larger coalition to which it may belong as a sub-coalition; for instance, player 1 has the same coefficient of aggressiveness λ_1 when it makes its claim within coalition $\{1,2,3\}$ as when it makes its claim within coalitions $\{1,2\}$ or $\{1,3\}$. There would be no difficulty in modifying this assumption.

For small values of the aggressiveness coefficients a core corresponding to full cooperation will exist. If all λ's are equal, we saw in section 2 that $\lambda=3/7$ is the largest value of λ for which such a solution exists.

If the common value of λ increases above 3/7, then the coalition $\{1,2,3\}$ is not viable, and cooperation breaks down. Let us see what happens if $\lambda=\frac{1}{2}$. We first try the partition $P=(\{1,2\},3)$. According to (3.6), using the relevant values from (3.12), we must then have

$$x_1+x_2=4, \qquad x_3=4.$$

For the coalition $\{1,2\}$ the claims of players 1 and 2 will, according to the relevant lines of (3.15), be $R(1\,|\,\{1,2\},3)=1$ and $R(2\,|\,\{1,2\},3)=3$. Requiring $x_1 \geqq 1$ and $x_2 \geqq 3$, corresponding to condition (3.7) in the definition of a solution, and at the same time having $x_1+x_2=4$, we get $x_1=1$ and $x_2=3$.

In a similar way we can check the other possible partitions. With $\lambda=\frac{1}{2}$ they will also satisfy the requirements. Thus we have the following solutions:

$$\begin{aligned}
&(x\,|\,P)=(1,3,4\,|\,\{1,2\},3) && [\Sigma_N x_i=8],\\
&(x\,|\,P)=(1\tfrac{1}{2},2,4\tfrac{1}{2}\,|\,\{1,3\},2) && [\Sigma_N x_i=8],\\
&(x\,|\,P)=(1,2\tfrac{1}{2},3\tfrac{1}{2}\,|\,1,\{2,3\}) && [\Sigma_N x_i=7].
\end{aligned} \tag{3.16}$$

In these solutions the total value achieved by all players are 8, 8 and 7 respectively, while the fully cooperative solution gave 10.

If we consider the fully non-cooperative case, which would give $x_1=0$, $x_2=2$ and $x_3=3$, then it follows from condition (c) in the definition that this will not be a solution of the game with the value $\lambda=\frac{1}{2}$ now specified. However, if $\lambda>\frac{1}{2}$, then also each coalition of two players will break down, and we get the fully non-cooperative result, with a total value $x_1+x_2+x_3=5$.

We may summarize the discussion by considering what happens if we increase the common value of the aggressiveness coefficient λ from zero. We then first have a wide core with total outcome $x_1+x_2+x_3=10$. With increasing λ the solution remains fully cooperative, but the set of outcome vectors satisfying the requirements shrinks and becomes unique, namely $(2\frac{3}{7},2\frac{6}{7},4\frac{5}{7})$, when λ reaches $\frac{3}{7}$ (see the example in section 2). When λ increases above $\frac{3}{7}$, then there will be three different kinds of solutions, corresponding to the partitions ({1,2},3), ({1,3},2), (1,{2,3}), with total values 8, 8 and 7. With $\frac{3}{7}<\lambda<\frac{1}{2}$ the division within each coalition is not unique. When $\lambda=\frac{1}{2}$ the three types of solutions still exist, but they are now unique as given above. For $\lambda>\frac{1}{2}$ also the coalitions of two players break down, and we have the fully non-cooperative solution giving the outcome vector (0,2,3) with the total value 5.

The example demonstrates how the total value achieved by all players declines with increased aggressiveness. There is of course a similar tendency for the individual players, but this tendency is not quite uniform. If $\lambda=\frac{3}{7}$ player 2 will get $2\frac{4}{7}$. If λ increases to $\lambda=\frac{1}{2}$, then player 2 may receive $x_2=3$ if the solution realized corresponds to the partition $P=(\{1,2\},3)$, i.e., a larger value in spite of the fact that the total sum has gone down.

Fig. 2 illustrates the consequences of increasing aggressiveness when there is a common coefficient of aggressiveness λ as just discussed. At the top of the figure are indicated the various partitions or coalition structures valid for the different ranges of λ. The next line shows the stepwise decline of the total outcome $\Sigma_N x_i$ from 10 for $0\leq\lambda\leq\frac{3}{7}$, to 8 for $\frac{3}{7}<\lambda\leq\frac{1}{2}$, and down to 5 for $\lambda>\frac{1}{2}$. [For the intermediate range of λ we have chosen to show the partition ({1,3},2) which seems to be the most interesting one of the three alternatives in this range, see the discussion below.] Below in the figure are shown the values for x_1, x_2 and x_3. The value for x_1 will be between 1 and 4 for $\lambda=0$, the range narrowing down to the point $x_1=2\frac{2}{7}$ for $\lambda=\frac{3}{7}$. When λ increases

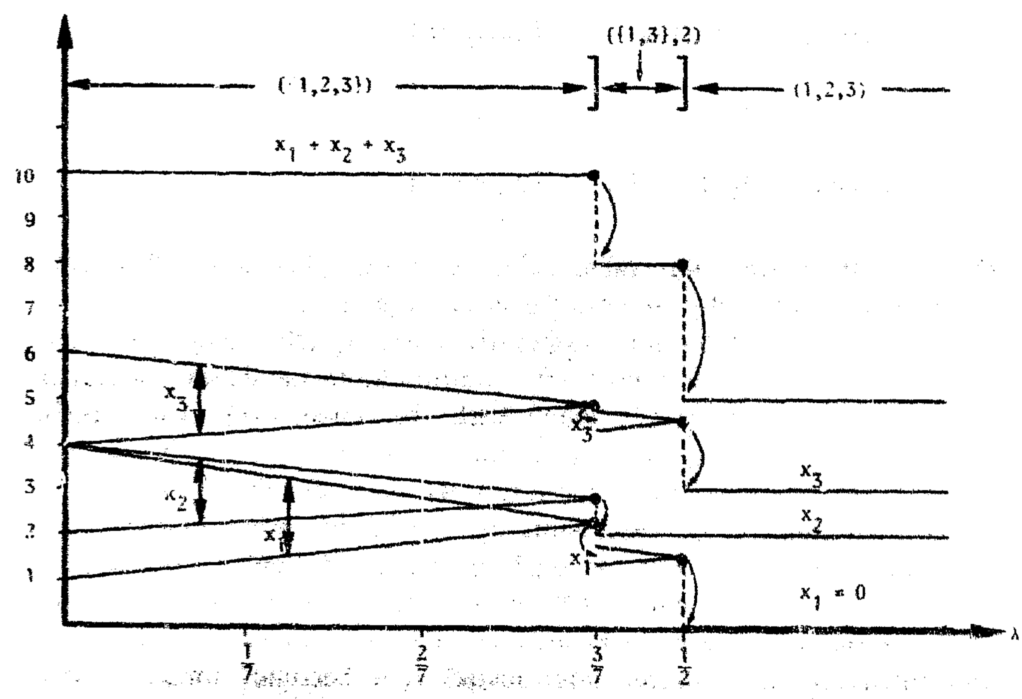

Fig. 2. Illustration of the solution of the game based on the claim function (3.15) for different values of a common coefficient of aggressiveness λ.

beyond $\frac{3}{7}$ we must have $x_1+x_3=6$ with $x_1 \geqq 3\lambda$ and $x_3 \geqq 3+3\lambda$. The range for x_1 narrows down from the interval $1\frac{2}{7}$, $1\frac{5}{7}$ to the point $1\frac{1}{2}$ for $\lambda=\frac{1}{2}$. For $\lambda>\frac{1}{2}$, $x_1=0$. In a similar way we see the possible developments for x_2 and x_3 when λ increases from zero to values exceeding $\frac{1}{2}$.

We conclude with a brief discussion of the uniqueness or non-uniqueness of the coalition structure in a solution. In the example given above there were three different types of solutions for intermediate levels of aggressiveness, i.e., for $\frac{3}{7}<\lambda\leqq\frac{1}{2}$. If we modify the definition of a solution a little bit the conclusion will be sharper. In the conditions in the definition as given above the various sub-coalitions considered only the possibility of breaking out and playing as new coalitions by themselves, and new, larger coalitions were contemplated only by merging existing coalitions. Let us require also that no sub-coalition should be able to break out of its present coalition and form a new, viable coalition by joining others, and furthermore make a total gain for the new coalition by this. Limiting for simplicity the following discussion to the case $\lambda=\frac{1}{2}$, where we originally had three alternative solutions as given by (3.16), we then see that the solution which builds on the coalition structure $(1,\{2,3\})$ will no longer qualify as a solution. In this case player 3 could defect from the coalition $\{2,3\}$ and join player 1 so that the new structure $(\{1,3\},2)$ is formed. Player 1 would then increase his payoff from 1 to $1\frac{1}{2}$ and player 3 would increase his payoff from $3\frac{1}{2}$ to $4\frac{1}{2}$. On the other hand, player 2 will see his payoff reduced from $2\frac{1}{2}$ to 2. However, he can do nothing about this if players 1 and 3 decide to join in a coalition. Considering the partition $(\{1,2\},3)$ we can reason in the same way. In this case we see that if player 1 defects from the coalition $\{1,2\}$ and joins 3 in the coalition $\{1,3\}$, then both player 1 and player 3 will increase their payoffs. It appears, then, that with this more demanding requirement for a solution, we end up with

$$(x\,|\,P)=(1\tfrac{1}{2},2,4\tfrac{1}{2}\,|\,\{1,3\},2)$$

as the unique solution when $\lambda=\frac{1}{2}$. This is the solution indicated in fig. 2.

Also a more unequal degree of aggressiveness could create uniqueness in the solution (retaining the unmodified definition of a solution). For instance, let player 1 be particularly aggressive corresponding to for instance the following stipulations:

$$\lambda_1=0.8,\quad \lambda_2=0.4,\quad \lambda_3=0.4,\quad \lambda_{1,2}=0.6\quad \lambda_{1,3}=0.6,\quad \lambda_{2,3}=0.4.$$

Using the figures in (3.12) and the claim functions as set out in (3.15) it is then easy to see that the grand coalition $\{1,2,3\}$ breaks down. Furthermore coalitions $\{1,2\}$ and $\{1,3\}$ break down, while $\{2,3\}$ remains viable. The solution in this case will be

$$(x_1, x_2, x_3 \mid 1, \{2,3\}) \quad \text{where} \quad x_1, x_2, x_3 \quad \text{satisfy}$$

$$x_1 = 1, \quad x_2 + x_3 = 6, \quad x_2 \geqq 2.4, \quad x_3 \geqq 3.4.$$

By being too aggressive player 1 has here made himself impossible in coalitions with others, and ends up with the poor result 1. The result is also not particularly good for the other players, the total outcome being only $\Sigma_N x_i = 7$.

In the discussion above we saw that aggressiveness may lead to a lower degree of cooperation, because the claims within a coalition may become incompatibly large. A related observation is that the model also makes it possible for cooperation to break down because the available amount to satisfy the claims of sub-coalitions within a coalition is too small. This may happen even if there is no aggressiveness in excess of what is already implied by the ordinary core-definition. This is easily illustrated by the game described by (3.12), with a solution as given by (3.13). Let in this game $v(\{1,2,3\} \mid \{1,2,3\})$ decline from 10 to 8. Then the claims from coalitions $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ can be jointly satisfied with no slack. If it declines below 8 the claims from these coalitions will become incompatible, and the solution must be found at a lower level of cooperation.[2] This effect conforms with many common-sense arguments and casual empirical observations. However, it is of interest to note that aggressiveness and a small total value for a coalition do not add up their effects as independent elements in explaining possible breakdown of cooperation. The reason is that aggressiveness applies to claims in a potential surplus for a coalition, and when this surplus declines with a decline in the total value of the coalition, the claims also decline. Consider again the example used above, now with a common level of aggressiveness λ. According to (3.15) the claims of the three two-coalitions will then be

$$R(\{1,2\} \mid \{1,2,3\}) = 4 + 2\lambda,$$

$$R(\{1,3\} \mid \{1,2,3\}) = 6 + 2\lambda,$$

$$R(\{2,3\} \mid \{1,2,3\}) = 6 + 3\lambda,$$

implying a total claim of $8 + \frac{7}{2}\lambda$. If $v(\{1,2,3\} \mid \{1,2,3\})$ declines to 8, the claims

[2] As the figures in the example are constructed, the game will not be superadditive if $v(\{1,2,3\} \mid \{1,2,3\})$ declines below 8. However, the effect described above would appear also if, for instance, $v(2 \mid \{1,3\},2)$ and $v(3 \mid \{1,2\},3)$ were lower so that the game was still superadditive even if $v(\{1,2,3\} \mid \{1,2,3\})$ declined a little below 8.

of $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ will decline to 4, 6 and $6+\lambda$ respectively, implying a total claim of only $8+\frac{1}{2}\lambda$ instead of $8+\frac{7}{2}\lambda$. With $\lambda>0$ this is still too much, but more moderately excessive than if the claim had been kept unchanged at $8+\frac{7}{2}\lambda$. We may say that even if our players are aggressive, they are informed or realistic in the sense that they make claims in an available surplus, and modify their claims when the surplus declines.

4. Summary and concluding comments

The motivations of the tentative explorations presented in the preceding sections are found partly in observations of how actual economic systems work, and partly in the structure of game theory as presently applied in economics.

On the factual side we observe mixtures of cooperative and non-cooperative behaviour, and we often see that the degree of aggressiveness of various groups and agents is decisive for the type of solution that emerges.

On the theoretical side, game theory (as well as most economic theory also when it is not presented in terms of game concepts) usually assumes in advance of the analysis whether a game is to be considered as cooperative or non-cooperative. If it is assumed to be cooperative, then the most common solution concept, the core, assumes a very defensive behaviour on the part of the various players and coalitions; there is no representation of aggressiveness in the model, and there is no such thing as break-down of cooperation. It may be found that the core does not exist; however, further interpretations of this situation are usually not given (but other types of solutions than the core may be developed).

On this background I think there is a need for a more general theoretical framework in which games can be played as a mixture of cooperation and non-cooperation, and where the mixture or degree of cooperation/non-cooperation is determined endogenously. Furthermore, there ought to be some sort of representation of aggressiveness in the model since there is hardly any doubt that such a thing influences outcomes in practice.

In the framework as described in the previous sections aggressiveness is introduced in a very simple manner, by means of 'coefficients of aggressiveness' which reflect the claim a player or a sub-coalition will raise referring to the surplus which is within reach because sub-coalitions cooperate in a larger coalition instead of playing non-cooperatively. These levels of aggressiveness assert themselves in the bargaining process. There is nothing mysterious about them. They may represent internal decisions in coalitions about minimum levels below which they will under no circumstance agree to move in the bargaining process, or they may represent levels to which negotiators have openly committed themselves as strategic moves in the bargaining process. In other cases they may be more obscure or flexible, representing psychological resistance rather than decisions or open commitments. I do not, of course, expect the 'coefficients of aggressiveness' to be amenable to successful econometric estimation. There are inherent difficulties about predictability in bargaining behaviour, which may carry over to the type of games considered in this paper. [Cf. my discussion of bargaining in Johansen (1979).] Nevertheless I think it helps to gain insight into important and realistic situations to have a framework in which there is a place for different degrees of aggressiveness.

If one hesitates to accept the introduction of aggressiveness into formal representation of games, then it should be pointed out that the type of mixed game in which the degree of cooperation/non-cooperation is endogenous as set out in section 3 is not logically dependent upon the introduction of aggressiveness. The reasoning is relevant also for the ordinary specification of the characteristic function of a cooperative game in the case where the core is empty and we want to pursue the analysis beyond a statement about the non-existence of the core. But a model of less than full cooperation is, of course, more likely to be relevant when there is some degree of aggressiveness in the system.

Some of the points and conclusions reached in the preceding sections can be summarized as follows.

(1) Introducing aggressiveness into a cooperative game we can establish a claim function on the basis of the usual characteristic function. The definition and further analysis of the core can be carried out on the basis of the claim function in a similar way as the analysis of the core on the basis of the characteristic function for the usual case.

(2) The core of the game when aggressiveness has been introduced will be smaller than the usual core with defensive players, and the core for the case of aggressive players may fail to exist even if the ordinary core does exist.

(3) There is no simple rule which says in advance what degree of aggressiveness a game may accommodate before the core disappears. This is illustrated by the example in section 2 where the maximal uniform degree of aggressiveness which the game could accommodate was $\frac{3}{7}$.

(4) A game in which it is not determined in advance whether or not the game shall be cooperative, or what degree of cooperation there will be, requires the specification of a sort of generalized characteristic function which indicates the values various coalitions will receive under various partitions or coalition structures, where players are cooperative within coalitions, and coalitions play non-cooperatively against each other.

(5) A solution of a game in this form must specify both the partition of players into coalitions and the outcome for the various players. The

solution must satisfy different kinds of conditions: Each coalition receives what is determined by the non-cooperative game between coalitions; the partition or coalition structure is such that no sub-coalition can break out and gain by the more non-cooperative game then established, and different coalitions cannot gain by joining into larger coalitions. The more precise specifications can be given in different ways as discussed in section 3, or more generally as in the appendix.

(6) For this generalized type of game a solution will always exist under the special definition given in section 3, provided that the non-cooperative games involved possess solutions. The solution can be anywhere between full cooperation and fully non-cooperative play. The solution is not necessarily unique.

(7) The degree of cooperation in the solution of the game will depend on the degree of aggressiveness, roughly speaking in such a way that the degree of cooperation is smaller the higher is the level of aggressiveness. Small surpluses in the various coalitions may also contribute to breakdown of cooperation.

(8) As a consequence of the preceding point, there is a tendency for the total payoff for all players taken together to decline in a stepwise manner if the degree of aggressiveness increases from zero up to the point where all individual players play non-cooperatively.

(9) The results for individual players can be more varied according to the degree of aggressiveness. As long as a certain coalition structure remains valid, a player may gain by being more aggressive, but if his aggressiveness increases beyond a certain point he may drop out of favourable coalitions and lose because of his aggressiveness.

(10) Although it has not been illustrated by the example in section 3, the analysis can easily be extended so as to say more about the pattern of coalition formations. For instance, in a four-player game with two aggressive and two more acquiescent players the model would predict, if the total degree of aggressiveness is too large for full cooperation, that there will appear two coalitions with one aggressive and one acquiescent player in each coalition.

These points are not cited because they are particularly striking or surprising. On the contrary, most of them sound quite obvious or plausible. The point I want to make is that it is desirable to have a sort of conceptual framework and a model structure in which such points can be made *within* the framework and on the basis of the model, and not only as common-sense observations outside the model.

The model of a mixed cooperative/non-cooperative equilibrium as explained in section 3 is a sort of generalized theory as compared with strictly cooperative or strictly non-cooperative theory. In other respects it is as simple as possible and less general than some other models found in the literature. For instance, we have assumed throughout that the result achieved by a player or a coalition is measured in terms of some homogeneous, divisible and transferable good, most directly interpretable as money. In a similar way as ordinary game theory has been generalized so as to cover the case of 'non-transferable utility', one may generalize the ideas of section 3. Furthermore, there are other solution concepts than the core which might be useful as a starting point for constructing a mixed cooperative/non-cooperative game theory.

It has been assumed above that all logically possible coalitions can potentially be formed and that they can coordinate the use of the action possibilities associated with the members of the coalition. For practical reasons this may not always be possible. For instance, it may be impossible for the coalition to make sure that all members actually behave as agreed upon in the coalition. For such reasons it may in some cases be relevant to impose restrictions on coalition structures which can actually be formed.

A crucial point is the definition of an equilibrium as discussed in section 3, especially the conditions which refer to the possibilities of gaining by defecting from existing coalitions or by merging existing coalitions into larger coalitions. For these aspects of the definition some alternatives were suggested in section 3, and a more general definition is proposed in the appendix. I do not think it is very important to settle for one very specific definition in this respect. I consider it more realistic to have in mind several alternative types of conditions. An outcome vector/coalition structure may possess some stability if it is not vulnerable to the most obvious and simple rearrangements of coalition structures, for instance as indicateed in the main definition given in section 3. It may still be vulnerable to reorganizations of coalitions which can be discovered and implemented only by more active and sophisticated players. Accordingly we may have 'solutions' of varying degrees of stability or vulnerability. This consideration could lead to interesting observations concerning the importance of inventiveness and sophistication on the part of the various players in considering alternatives to an existing coalition structure. There would seem to be some contradictory tendencies. On the one hand, lack of inventiveness and sophistication may lead to a situation with a wide set of solutions, some good and some bad. One may get stuck in a bad situation, or remain safe in a good situation. On the other hand, with more inventiveness and sophistication some bad situations may fail to be solutions of the game, because the players see favourable opportunities of reorganization. But it may also happen that otherwise robust, good situations fall apart as solutions because the players see tempting alternatives in various directions so that coalitions needed to realize the good situation will no longer be viable, i.e., disintegrate so that the game takes a more non-cooperative form. Along a similar line of thought one could also be led into speculations about the possible role of institutional rules and regulations for the protection of good cooperative solutions against tendencies towards disintegration.

The concepts and the model introduced in this paper are very tentative and meant only to help intuition and provide insight at a rather general level. The model is not meant to be ready for applications or specific interpretations for concrete cases. However, some problems seem to lend themselves in a rather natural way to formalizations along the lines of the mixed cooperative/non-cooperative game.

The most obvious example would perhaps be oligopoly situations where the players are the producers. A fully cooperative solution would in this case correspond to collusion, whereas the fully non-cooperative case would be the Cournot solution. Intermediate cases would be cases where some producers form coalitions, i.e., behave in a collusive manner within the coalition, but play in a Cournot manner against other coalitions. The characteristic function corresponding to (3.3) could in this case be established on the basis of the demand function in the market and the cost functions of the various producers.

Another example would be a set of producers who exploit some sort of common natural resource, for instance fishing countries around a sea. The payoffs could in this case be, for instance, the present value of the flow of income from fishing for the various countries. A coalition could limit its present catch in order to keep future catch at a higher level. A country which is not a member of the coalition could play non-cooperatively and thus to some extent take advantage of the situation created by the coalition.

In such cases as those mentioned here it will often be to the advantage of single players to break out of a larger coalition provided that the other players remain in the coalition. Whether a cooperative solution will establish itself in such a case depends on what formulation of the second condition [condition (b)] in the definition of a solution in section 3 is most relevant. If no player is satisfied with less than what he would obtain if the other ones remain in the coalition even if he himself breaks out, then coalitions will not hold together. However, if we, in defining the stability conditions for the coalitions, assume that each player, when he contemplates breaking out, considers the possibility that also the rest of the coalition breaks down if he himself defects, then the coalition may hold together. (In the general definition in the appendix these considerations refer to the form of the function π.)

A third example, but perhaps somewhat more indirect, would be wage claims in relation to inflation problems. A large coalition may limit its claim because it recognizes the effects of wages on prices. For smaller coalitions this connection appears in a different light. An important question is, at what level between the full-scale national union and the completely atomistic situation will wage bargains be settled? One might approach this problem along the lines of the theory outlined in this paper.

These are just brief suggestions. I think it is not difficult to find a number of examples where formulations along the lines indicated in section 3 would have some appeal.

I conclude with some brief observations on relevant literature.

As indicated before, games with coalition structures are well known in the literature. However, when a coalition structure is introduced into a game it is usually done in the form of a *given* coalition structure imposed on the game.

A particularly interesting study of cooperative games with coalition structures is given by Aumann and Drèze (1974). This paper clarifies the role of coalition structures in connection with several different solution concepts in game theory. The mixed cooperative/non-cooperative game theory suggested in section 3 above (and generalized in the appendix) could perhaps be linked up in a natural way with some other solution concepts than the ordinary core which we took as the starting point. Aumann's and Drèze's paper would provide a basis for such extensions or reformulations.

As discussed by Aumann and Drèze the interpretation of 'games with coalition structures' remains somewhat obscure. The authors suggest some possible interpretations of such games as representing actual situations, but the most convincing interpretation is to consider 'games with coalition structures' as an element of a broader analysis. In the words of Aumann and Drèze: 'If the reader wishes, he may view the analysis in this subsection as part of a broader analysis, which would consider simultaneously the process of coalition formation and the bargaining for the payoff.' Games with a *given* coalition structure take up one part of this broader study 'and should thus be understood as a contribution to partial equilibrium analysis'. The cooperative/non-cooperative game studied in the present paper is an approach to the broader analysis. However, the models of 'games with coalition structures' do not provide elements which could be [illegible] immediately into the broader analysis; the reason is that these models usually assume efficiency or Pareto-optimality, whereas a model like the one studied in section 3 permits coalitions to break down with the consequence that potential gains are not realized.[3] The concluding discussion in Aumann's and Drèze's paper also contains several other considerations which are of interest in connection with the present paper.

[3]In their concluding section Aumann and Drèze briefly consider a small three-person game in which they find it reasonable that two players form a coalition, excluding the third, even if the total outcome is then inefficient. It seems to me that this example and the intuitive arguments given in connection with it fit in nicely with the formulation of mixed cooperative/non-cooperative games as formulated in the present paper.

A special branch of economics which has developed tools and ideas which could be related to the mixed cooperative/non-cooperative game of this paper is the 'economic theory of clubs'. A recent survey is provided by Sandler and Tschirhart (1980). This theory is concerned with the formation of coalitions for the provision of public goods which are not public in an absolute sense, but which beyond a certain point provide the members with less of utility when the number of members increases. There is therefore an economic reason for establishing coalitions which are smaller than the grand coalition of all players. In this theory the formation of coalitions is an endogenous element in the theory. The mechanism is, however, different from the present model. When the theory of clubs is formulated in terms of game concepts, one usually introduces a characteristic function which is not super-additive. The property of not being super-additive reflects the fact that smaller coalitions than the grand coalition may be most efficient in providing the public goods. The use of non-super-additive characteristic functions requires special interpretations and may seem somewhat artificial. It is possible that a reformulation along the lines of a mixed cooperative/non-cooperative game might be worthwhile.

Recently Guesnerie and Oddou (1979) have provided a study of cooperative games with characteristic functions which are not necessarily super-additive, inspired by the theory of clubs. The formation of a coalition structure is endogenous in the model. However, in contrast to the present theory, they make it clear that the reason for the partition into a coalition structure is that 'efficiency may require that the grand coalition N breaks down'. In our case the 'grand coalition' may break down because of incompatible claims, and the partition which then occurs may result in an inefficient solution.

Referring to the possibility of non-super-additive functions, the claim function $R(S)$ or $R(S^* \mid P)$ introduced previously in this paper may fail to be super-additive even if the underlying characteristic function $v(S)$ or $v(S \mid P)$ is super-additive. But, in our case, if the grand coalition breaks down because the claim function R is not super-additive, then this will prevent rather than promote efficiency as measured by the actual total payoff.

Finally it should be observed that the use we have made of the outcome of a non-cooperative game as a reference for the definition of the characteristic function in section 3 is not entirely alien to the von Neumann–Morgenstern theory of games. Von Neumann and Morgenstern defined the characteristic function on the basis of a description of the game where the strategies were explicit, in contrast to many later expositions in which the characteristic function is stipulated directly, and they referred to the game between a coalition and the complementary coalition in the definition of the characteristic function. Furthermore, von Neumann and Morgenstern have several ideas on 'composition and decomposition of games' which may contain ideas of relevance to the present study. However, coming to the general non-zero-sum games, the authors state that 'in our theory all solutions correspond to attainment of the maximum collective profit by the totality of all players'. [See von Neumann and Morgenstern (1953, p. 541).] This basic consideration would seem to make it difficult to reinterpret von Neumann and Morgenstern so as to cover the type of game we have studied.

Some remarks on the relationship to the concept of ψ-stability introduced by Luce and Raiffa (1958) are given in the appendix in connection with the discussion of a more general definition of a solution than the one given in section 3.

Appendix A: Suggestion of a more general definition of a solution

In section 3 we used a rather simple definition of a solution of the mixed cooperative/non-cooperative game, consisting of conditions (a), (b) and (c) given in connection with (3.6)–(3.8). An alternative definition was suggested by replacing condition (c) with (c') and (d). In the discussion of the definition some other points were also taken up.

I shall here suggest the logical structure of a more general definition which will encompass the main definition in section 3 and the variations discussed there as special cases. This general definition is in some respects similar to the definition of ψ-stability proposed by Luce and Raiffa (1958), but is nevertheless different since we consider a mixed cooperative/non-cooperative game.

As before a solution is a pair $(x|P)$, where P is a coalition structure $(S_1, \ldots, S_k, \ldots, S_K)$.

The first thing which must be specified is which coalitions that can be formed with a given coalition structure P as an 'initial structure'. These possibilities are represented by a function $\psi(P)$ defined as follows:

> For each initial coalition structure P, $\psi(P)$ is a set of coalitions. The interpretation of $\psi(P)$ is that from a situation described by P, players contained in some set S^* are able to form a coalition (by joining, splitting, leaving or reorganization of coalitions in P) if and only if S^* is an element of $\psi(P)$, $S^* \in \psi(P)$. (A.1)

Now suppose players in S^* contemplate forming a coalition. What they would expect to gain by this depends on what they think the coalition structure of players outside S^* will become when S^* is established. Let this be defined by the function $\pi(S^*, P)$, defined for every P and every $S^* \in \psi(P)$, with the following interpretation:

> Consider an initial coalition structure P and suppose that members of S^*, where $S^* \in \psi(P)$, contemplate forming a new coalition. Then

members of S^* believe that a new coalition structure P^* will be formed (where S^* is one of the coalitions in P^*), and we write this as

$$P^* = \pi(S^*, P). \tag{A.2}$$

Now a pair $(x|P)$ will not be 'stable' if there is a feasible coalition S^* which would be viable under the structure P^* which members of S^* believe will emerge if they form S^*, and which gives a gain to the members of S^*.

On this basis we may suggest the following more general definition of a solution than the one given in section 3:

Definition of a solution (general case). A pair $(x|P)$ is a solution if the following conditions hold:

Condition (α). Each coalition S_k in P receives a total payoff given by

$$\Sigma_{S_k} x_i = v(S_k | P), \qquad k = 1, 2, \ldots K. \tag{A.3}$$

Condition (β). Let $\psi(P)$ and $\pi(S^*, P)$ be defined as given by (A.1) and (A.2). Then for each S_k containing more than one player we have for all feasible sub-coalitions of S_k, i.e., for all S_k^* which are such that $S_k^* \subset S_k$ and $S_k^* \in \psi(P)$,

$$\Sigma_{S_k^*} x_i \geqq v(S_k^* | \pi(S_k^*, P)). \tag{A.4}$$

Condition (γ). For every coalition S^* which can be formed from P, i.e., $S^* \in \psi(P)$, and which contains members from at least two of the coalitions $S_1, \ldots, S_K$ constituting P, we have

$$\Sigma_{S^*} x_i \geqq v(S^* | \pi(S^*, P)). \tag{A.5}$$

In this definition we have, as in the case of the simpler and more special definition in the text in section 3, distinguished between an internal viability condition, (β), and a sort of external stability condition, (γ). It appears from the formulations above that we could subsume conditions (β) and (γ) under one joint formulation. However, since they refer to somewhat different considerations, considerations which are internal versus external with respect to an initial coalition, it is convenient to distinguish between them. Furthermore, if we should introduce aggressiveness into the situation, as we did in section 3, then we have to keep the two kinds of conditions apart. Aggressiveness is relevant in connection with condition (β) since we assume that it is only a pure sub-coalition within a coalition which can raise claims in the total payoff of the coalition. Thus aggressiveness should be appended to condition (A.4) just as we have done in the earlier sections, while condition (A.5) should be left unaffected.

The various conditions specified in the definition in section 3 and the various modifications suggested there can now be seen as specializations of the definition now given. Various specifications of the functions ψ and π give different special definitions.

In the main definition — called a 'special case' — in section 3 condition (b) corresponds to condition (β) in the general definition, and condition (c) corresponds to condition (γ). In the special case the ψ-function is very simple: To every partition P only two types of new coalitions are able to constitute themselves: (1) coalitions which are sub-coalitions of some coalition S_k in the initial coalition structure P, and (2) coalitions which can be formed by joining two coalitions in the original coalition structure P. This defines the ψ-function for this special case.

Next there is the role of the function $\pi(S^*, P)$, which specifies what coalition S^*, which belongs to $\psi(P)$, believes will be the new coalition structure if S^* constitutes itself as a coalition. In the special definition we have

$$\begin{aligned}\pi(S^*,P) &= (S_1,\ldots,S^*,S_k\backslash S^*,\ldots,S_K) \quad \text{for } S^*\subset S_k,\\ &= (S_1,\ldots,S_j\cup S_k,\ldots,S_K) \quad \text{for } S^*=S_j\cup S_k,\end{aligned} \tag{A.6}$$

where the first line is meant to suggest the partition where no other changes have taken place than the splitting of an initial coalition S_k into S^* and $S_k\backslash S^*$, and the second line is meant to suggest the coalition structure where no other changes have taken place than the joining of S_j and S_k. Since there are no other types of S^* in $\psi(P)$, this specifies the function $\pi(S^*, P)$ completely in the present case.

A slight complication will be taken up in a moment.

Various modifications discussed after the special definition in section 3 can be seen as other specifications of ψ and π. The possibility to form new coalitions consisting of more than two initial coalitions is an obvious extension of the ψ-function. In connection with the numerical example towards the end of section 3 we introduce the possibility that new coalitions could be formed by a player defecting from one coalition and joining another player originally not in the same coalition. This would be another type of change in the ψ-function: In $\psi(P)$ there would now be coalitions consisting of players from different original coalitions.

We have also on some occasions touched upon the question as to whether members of a potential new coalition S^* would believe the coalition structure for the players not involved in S^* to remain unaffected by the formation of S^* (as they do in the definition in the special case). For instance, when a sub-coalition of some S_k breaks out of S_k, then it might believe that also the remaining $S_k\backslash S^*$ disintegrates in some way, or the players in S^* might behave as if they assume that the worst possible coalition structure for the remaining set of players would be established. Such modifications mean different formulations of the function $\pi(S^*, P)$.

In concluding this part of the appendix, we must return to a 'complication' hinted at above. It refers to the supplementary statement made in condition (c) in the definition in section 3 about the viability or non-viability of the coalition structure P_{jk}^{U} which emerges when S_j and S_k merge. This statement means that condition (A.5) does not apply for potential coalitions S^* when the coalition structure on the second line of (A.6) is not viable. This element of the definition can be taken care of through the specification of the function $\psi(P)$: We can simply say that if some $S^* = S_j \cup S_k$ implies that the structure $(S_1, \ldots, S_j \cup S_k, \ldots, S_K)$ is not viable according to the internal conditions in each coalition (i.e., cannot satisfy α and β), then S^* is excluded from $\psi(P)$.

As this condition (c) in the first formulation in the definition in section 3 is perhaps somewhat dubious, we suggested an alternative by condition (c'). This formulation of the condition could also be taken care of by the specification of $\psi(P)$.

The formulation of condition (c) was crucial for the question about the existence of a solution. In connection with the general definition given in this appendix not all conceivable specifications of the functions ψ and π would ensure the existence of a solution. One could then ensure the existence of a solution by adding a condition similar to condition (d) in section 3, to the effect that if no other coalition structure can produce a solution satisfying requirements (α), (β) and (γ), then the fully non-cooperative case is declared to be the solution. Whether this is a reasonable thing to do, is of course not a matter of formalities, but a question which must be settled by reference to the realities which the model is meant to represent.

Appendix B: Some technical notes

This part of the appendix offers some more technical notes on the structure of solutions and problems of computing solutions.

The basic element of computations for finding solutions of the types outlined in the main text will be computations of cores. The finding of an ordinary core is a linear programming problem of a special form. In the notations used in this paper an ordinary core is defined by the following requirements:

$$\begin{aligned} &\Sigma_N x_i = v(N), \\ &\Sigma_S x_i \geqq v(S) \quad \text{for every } S \subset N. \end{aligned} \tag{B.1}$$

One way of finding out whether a core exists is to solve the linear programming problem

$$\min y = \Sigma_N x_i, \quad \text{subject to}$$
$$\Sigma_S x_i \geqq v(S) \quad \text{for every } S \subset N. \tag{B.2}$$

Let the minimum value of y in this problem be y^*. Then the core exists if $y^* \leqq v(N)$, while it does not exist if $y^* > v(N)$. If we have $y^* < v(N)$, then this solution does not immediately give us the core. Formally we could add the condition $\Sigma_N x_i \geqq v(N)$ to the set of inequalities for coalitions S which are strict subsets of N in the minimization problem (B.2) and accordingly obtain only solutions which satisfy the first line of (B.1) (if such solutions exist).[4]

The structure of the problem is illustrated in fig. 3, where we represent the case of 3 players. The plane with corners α, β and γ represents the condition $\Sigma_N x_i = v(N)$. The condition $x_3 \geqq v(3)$ is represented by the requirement that the point (x_1, x_2, x_3) must be above the plane ABC in the diagram, while the condition $x_1 + x_2 \geqq v(1,2)$ says that the point must be to the right of the plane $DEFG$. The constraints for other players and coalitions would yield similar conditions, only rotated in relation to the axes. With the conditions introduced in the figure, the core must be in the set $P_1P_2P_3P_4$, but further constrained when we introduce all constraints.

If we introduce aggressiveness in the form used in section 2 of the paper, then the computational problem will remain exactly as for the ordinary core. We only have to replace the right-hand sides of the various constraints by the values of the claim function $R(S)$. In fig. 3 $v(3)$ should be replaced by $R(3)$, $v(1,2)$ by $R(1,2)$, etc. These changes will mean moving the various planes representing the various constraints outwards in the diagram, while the plane $\alpha\beta\gamma$ remains where it is. It is clear that the core will narrow down, and that the various claims may become incompatible so that the core based on the claim function ceases to exist even if the ordinary core exists.

If the grand coalition disintegrates because there is no solution in the plane $\alpha\beta\gamma$, then the solution of the mixed cooperative/non-cooperative game will switch to another configuration in fig. 3. Suppose that the grand coalition is not viable, while the coalition structure $P = (\{1,2\}, 3)$ is viable. Then the solution will give $x_3 = v(3)$, and $x_1 + x_2 = v(1,2)$, where x_1 and x_2 are limited by $x_1 \geqq v(1)$ and $x_2 \geqq v(2)$. In the diagram this solution is represented by the line segment MN where the total payoff $x_1 + x_2 + x_3$ is smaller than $v(1,2,3)$. The core in the internal cooperative game in coalition $\{1,2\}$ is represented by the projection of this segment onto the line EF in the x_1, x_2-plane.

[4]Computational aspects of finding the core of an n-person game, covering the case of non-transferable utility which is the more intricate case, are treated by Scarf (1967).

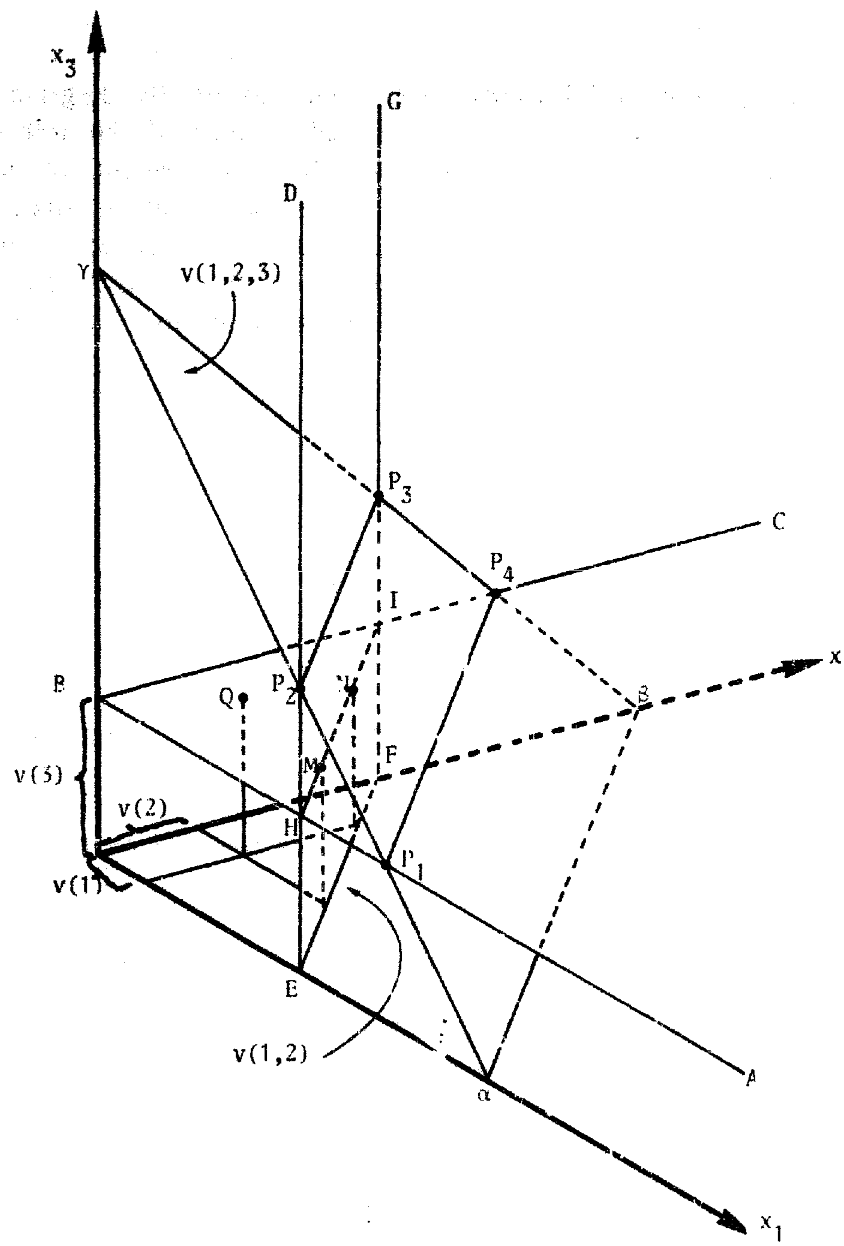

Fig. 3. Illustration of the constraints defining the core and mixed solutions in a three-person game.

If cooperation in the coalition $\{1,2\}$ should also break down, then the fully non-cooperative solution will be realized. This is represented by point Q in the figure.

As an illustration of a mixed cooperative/non-cooperative game fig. 3 is somewhat simplified. For instance, the value $v(3)$ which is relevant for the limitation of the core in the case of a fully cooperative solution would (according to the special case definition in section 3) be $v(3\,|\,\{1,2\},3)$, while the value which would be relevant for determining x_3 in the fully non-cooperative outcome would be $v(3|1,2,3)$, and similarly for $v(1)$ and $v(2)$. These distinctions and modifications could be entered into the diagram.

We conclude with some suggestions for calculations of the solution of mixed cooperative/non-cooperative games. The key element of such a computation would be to check whether a proposed coalition structure $P=(S_1,\ldots,S_k,\ldots,S_K)$ can yield a solution. Let us use the general definition given in part A of this appendix. The functions $\psi(P)$ and $\pi(P^*,P)$ are then given. Corresponding to the conditions (B.1) for the ordinary core we now have the following conditions:

$$\Sigma_{S_k} x_i = v(S_k | P), \qquad k=1,2,\ldots,K,$$

$$\Sigma_{S_k^*} x_i \geq v(S_k^* | \pi(S_k^*, P)) \quad \text{for each } S^* \text{ such that} \tag{B.3}$$

$$S_k^* \subset S_k \quad \text{and} \quad S_k^* \in \psi(P), \qquad k=1,\ldots,K,$$

$$\Sigma_{S^*} x_i \geq v(S^* | \pi(S^*, P)) \quad \text{for each } S^* \text{ such that}$$

$S^* \in \psi(P)$ and S^* contains members from more than one of the coalitions $S_1,\ldots,S_K$.

For a given P the right-hand side in all conditions in (B.3) are given magnitudes. The problem is therefore again a problem of finding out whether there exists a solution to a system of linear equations and inequalities, but a somewhat more complicated system than the one for the ordinary core.

One way to approach the problem of finding a solution to (B.3) by means of linear programming would be to replace the equalities on the first line of (B.3) by

$$\Sigma_{S_k} x_i \geq v(S_k | P), \qquad k=1,2,\ldots,K \tag{B.4}$$

and then minimizing the sum $y=\Sigma_N x_i$. Let the minimal value of y be y^*. Then a solution to (B.3) exists if

$$y^* = \sum_{k=1}^{K} v(S_k | P) \tag{B.5}$$

while no solution exists if

$$y^* > \sum_{k=1}^{K} v(S_k | P). \tag{B.6}$$

This linear programming problem has a special structure. The variables $x_1, \ldots, x_n$ are divided into groups corresponding to the partition P. For each S_k the conditions on the second line of (B.3) for the given k and the corresponding condition in (B.4) are internal constraints in the group defined by S_k, while the constraints on the last line of (B.3) are 'global constraints', involving variables from different groups. This means that the system should lend itself naturally to solutions by means of decomposition methods of linear programming.

If we have aggressiveness with corresponding claim functions in the problem, then this would only modify the right-hand side on the second line of (B.3). The form of the programming problem would be unaffected by this.

Having a method of finding whether or not a solution exists for a given partition P, one could approach the problem of finding a solution to the full game, including the unknown partition P, in various ways. One possibility is to test first the possibility of a solution involving full cooperation. If such a solution does not exist, one could test all possible ways of partitioning the players into two coalitions, and so on. The other obvious possibility is to start from below, checking first the fully non-cooperative case, and next move on to the case where two players form coalitions, and so on. If one is not satisfied with finding one solution, one would have to trace through very many coalition structures, since the existence of a solution at one level of cooperation does not preclude the existence of solutions also with other coalition structures. Even with a moderate number of players, the task of finding all solutions may therefore be a very big task. It would perhaps be possible to find iterative methods by which one would not necessarily have to check all coalition structures, for instance by observing at each step which constraints that are the effective constraints, and then choosing the next coalition structure to be tested in the light of this information. It might also be possible to establish rules such that if one coalition structure is found not to qualify as a solution, then immediately several other coalition structures are also ruled out. However, it is beyond the scope of this paper to go further into the computational problems.

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