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Parametric Certainty Equivalence Procedures in Decision-Making Under Uncertainty

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1. Introduction

Certainty equivalence procedures are popular in connection with decision-making under uncertainty because they simplify an otherwise complicated decision problem to something similar to decision-making under certainty. In management science and in the theory of economic policy and planning the best known case of certainty equivalence (known especially from several works by H. Theil) is the case of a quadratic objective function combined with a structural model with additive error terms with a constant variance-covariance matrix. The certainty equivalence results in this case lead to simple and practical computational procedures. They are strong in the sense that they do not require specific forms of the probability distributions involved apart from the existence and constancy of the first and second order moments. On the other hand, in many contexts the quadratic form is not very attractive, partly because of the attitude towards risk implied by this function, and partly because of the symmetry properties of the quadratic function.

The purpose of this paper is to introduce a somewhat more general type of certainty equivalence procedure than the usual one, to be called "parametric certainty equivalence". The idea is to formulate a procedure for deriving the optimal decisions under uncertainty which is similar to the one which would be valid under certainty by permitting some adjustments of the values of parameters involved. A quite attractive case of such parametric certainty equivalence can

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be established on the basis of an objective function expressed in terms of combinations of exponential functions. In several respects this formulation is more attractive than the quadratic objective function; on the other hand, it makes it necessary to assume the error terms to be normally distributed.

For computational procedures it is useful to compare the exponential objective function with the quadratic function. Computational methods which are valid for the quadratic case may be used as parts of an iterative scheme which solves the problem for the case of exponential objective functions.

The discussion of the objective function to be proposed in this paper gives rise to some considerations concerning the concept of risk aversion, which will be taken up in an appendix.

2. Concepts of Certainty Equivalence

I shall not try to review all definitions of certainty equivalence found in the literature on decision-making under uncertainty, but only give a representative definition in general terms. Next I shall introduce the concept of "parametric certainty equivalence".

We consider a decision problem involving the following variables:

x = a variable or a vector of variables to which we attach preferences;

a = a decision or action which may in a purely quantitative case be represented by a vector of instrument variables;

A = a set of possible actions, i. e. we have $a \in A$;

z = a vector of random variables.

The values obtained for x are determined by the action taken and the random variables by a reduced form system which we write as

$$x = f(a, z). \tag{2.1}$$

The objective function is written as

$$U = U(x; \alpha) \tag{2.2}$$

where α is a vector of parameters entering the form of the objective function.

The decision problem is to make a decision a so as to maximize the expected value of U. In making the decision a we do not know the actual values of the random variables z, but only the probability distribution of z. We let the optimal decision be a^*. This is determined by

$$\underset{a \in A}{\text{Max}}\ E\left[U(x;\alpha)\right] = \underset{a \in A}{\text{Max}}\ E\left[U(f(a,z);\alpha)\right]. \tag{2.3}$$

A certainty equivalence procedure for solving the problem (2.3) will consist in replacing the stochastic variables by some representative non-stochastic values $\tilde{z}$ and then solving the problem

$$\underset{a \in A}{\text{Max}}\ U(f(a,\tilde{z});\alpha) \tag{2.4}$$

where $\tilde{z}$ is derived from the probability distribution of z in a prescribed manner. In most cases the expected values will serve, i. e. one will set $\tilde{z} = E(z)$.

Sometimes a procedure like (2.4) is called a certainty equivalence procedure with no further reservation. However, the solution of (2.4) will of course, in general, not be the same as a solution a^* of the original problem (2.3). The term "equivalence" then seems somewhat misleading. At least for the present purpose I shall reserve the term certainty equivalence for the case in which the solution of (2.4) leads to the same solution a^* as the solution of (2.3), and the solution procedure indicated by (2.4) will then be called a certainty equivalence procedure.

In order to distinguish this case from the case to be described below we shall call it *simple* certainty equivalence. The simplicity refers to the fact that the same vector of parameter values α is used in the certainty equivalence procedure (2.4) as in the formulation of the original problem (2.3).

The most well-known case of simple certainty equivalence is the case of a quadratic objective function and a structure (2.1) where random elements enter in an additive manner. The expected value of the objective function can then be written as

$$E\left[U(x;\alpha)\right] = U(E(x);\alpha) + C \tag{2.5}$$

where C is a term depending on the variance-covariance matrix of x. If the random elements z enter additively in the structure (2.1) and have zero expectations, then we can write

$$x = F(a) + Hz,\ E(x) = F(a) \tag{2.6}$$

where H is a constant matrix. The equivalence between the outcomes of (2.3) and (2.4) then follows. (It should be observed that this equivalence is valid regardless of the form of the possibility set A and the form of the function F. The essential thing, in addition to the quadratic form of the objective function, is the fact that the variance-covariance matrix of x is independent of the action a chosen, which is secured if z has a constant variance-covariance matrix and z enters additively as indicated in (2.6).)

Simple certainty equivalence is elegant and useful when it is valid, but one might feel that uncertainty then enters in an almost trivial way. The existence of uncertainty does not make us change our decision as compared with what we would do in the absence of uncertainty. In this sense simple certainty equivalence is perhaps too elegant and too strong.

There are essentially two different ways in which the presence of uncertainty might have more interesting and realistic consequences. One way is to assume that the random elements enter in a different way, so that the variance-covariance matrix of x will depend upon the decision a. [See for instance W. Brainard (1967) and L. Johansen (1973) and (1978).] The other direction is to introduce another form of the objective function, especially permitting forms implying less symmetry than the quadratic function. [The relevance of asymmetric objective functions has been emphasized especially by R. N. Waud (1976); see also the discussion in L. Johansen (1978).]

For non-quadratic, asymmetric objective functions it is not easy to obtain simple certainty equivalence results. However, one may obtain "parametric certainty equivalence". This means that there will be equivalence between the results of a procedure like (2.4) and the problem as formulated in (2.3), with the modification that the parameter vector α in (2.3) is replaced by a modified vector in the procedure as given by (2.4).

Let G be the distribution function of the random elements in x, i. e. of $x-E(x)=Hz$ in the case of (2.6). This distribution will depend on the distribution of z and on the matrix H, but not on the decision a. We then introduce a modified parameter vector $\tilde{\alpha}$ as a function of the original parameter vector α and the form of the distribution function, i. e.

$$\tilde{\alpha} = \psi\,(\alpha, G), \tag{2.7}$$

where the vector $\tilde{\alpha}$ has the same number of elements as α and corresponds to α element by element. When the form of G is known,

G may be represented by a parameter vector characterizing the distribution.

We can now formulate the optimization problem

$$\operatorname*{Max}_{a \in A} U(f(a, \tilde{z}); \tilde{\alpha}) \tag{2.8}$$

where again $\tilde{z}$ is a non-stochastic representative value of z, for instance $E(z)$. We then say that (2.7, 2.8) represents a certainty equivalence procedure if the solution of this problem leads to the same decision a^* as the solution of the problem in the original formulation (2.3).

The "non-simple" aspect of this procedure is that we have to establish a function ψ which indicates how the original parameter vector should be transformed into a new parameter vector $\tilde{\alpha}$, depending on properties of the distribution function G.

It is seen that the problem as formulated by (2.8) is of precisely the same form as the form into which (2.3) would collapse if the distribution of z, and accordingly of x, degenerated into the case of full certainty so that the expectations operator could be removed. We should expect the transformation (2.7) to be such that it gives $\tilde{\alpha}=\alpha$ for the case in which G degenerates in this manner.

The problem as formulated by (2.8) involves no more complexity than what originates from the form of the objective function U, the structural form f and the form of the possibility set A, i. e. those elements which are present also in the case of full certainty. The usefulness of the formulation (2.8) as a substitute for the original formulation (2.3) then depends on whether or not we can establish a sufficiently simple transformation ψ to give the vector $\tilde{\alpha}$ to be used in (2.8).

In the following section we shall derive such a procedure for the case of an objective function constructed by combining exponential functions. We shall also restrict the probability distribution of the random elements to belonging to the class of normal distributions. Instead of entering the form G as an argument in the transformation (2.7), we can then of course enter the parameters of the normal distribution.

Although the *procedure* in the case of parametric certainty equivalence is the same as under certainty, once the modified parameter vector $\tilde{\alpha}$ has been established, *the decision actually taken* will now in general be different under uncertainty than under certainty because the parameter values $\tilde{\alpha}$ used in calculating the decision a from (2.8) will depend on the probability distribution of the random elements and influence the decision.

3. Exponential Objective Functions

For a scalar x the exponential objective function is

$$U(x) = -Be^{-\beta x} \quad (B>0, \quad \beta>0). \tag{3.1}$$

This is a nicely increasing function with decreasing slope:

$$u(x) = \frac{dU(x)}{dx} = \beta Be^{-\beta x}, \; u'(x) = \frac{du(x)}{dx} = -\beta^2 Be^{-\beta x}. \tag{3.2}$$

In the context of decisions under uncertainty this form implies a constant degree of absolute risk aversion:

$$-\frac{u'(x)}{u(x)} = \beta = \text{coefficient of absolute risk aversion.} \tag{3.3}$$

For a vector of variables $x = (x_1, \ldots, x_n)$ a sum of exponential functions will yield the objective function:

$$U(x) = -\Sigma B_i e^{-\beta_i x_i} \; (B_i>0, \; \beta_i>0). \tag{3.4}$$

In this form the function implies strong separability, which is of course a restrictive assumption in many contexts. Otherwise the function has attractive properties. It has nicely shaped indifference curves (which cut the axes). The corresponding expansion curves are linear and parallel.

A sum-of-exponentials utility function was used for illustrative purposes by J. Chipman (1965). For applications in consumer demand theory such utility functions were used by R. A. Pollak (1971); see also L. Johansen (1979).

In some applications of objective functions it may be necessary to have functions which permit saturation levels for some variables. This is especially important if we want a function which can serve as an alternative to quadratic objective functions which are usually constructed as quadratic forms in the deviations from some most desired values of the various variables. For the scalar case a peaked function can be obtained by adding another exponential term in (3.1) so as to write

$$U(x) = -Be^{-\beta x} - Ce^{\gamma x} \quad (B, C, \beta, \gamma > 0) \tag{3.5}$$

with the signs of the coefficients as stipulated to the right in (3.5).

The marginal utility corresponding to this function, and its derivative, will be

$$u(x) = \beta Be^{-\beta x} - \gamma Ce^{\gamma x}, \; u'(x) = -\beta^2 Be^{-\beta x} - \gamma^2 Ce^{\gamma x}. \tag{3.6}$$

It is seen that we always have decreasing marginal utility.

This utility function will have a maximum for x determined by

$$\beta Be^{-\beta x} - \gamma Ce^{\gamma x} = 0, \tag{3.7}$$

or

$$x = \frac{\ln(\beta B) - \ln(\gamma C)}{\beta + \gamma} = x^*. \tag{3.8}$$

If we want to write the objective function in terms of deviations from the most desired value x^* as given by (3.8), it is convenient, in addition to x^* as defined by (3.8), to introduce the constant K defined as follows:

$$K = [(\beta B)^{\gamma} (\gamma C)^{\beta}]^{\frac{1}{\beta+\gamma}} \tag{3.9}$$

Then (3.5) can be written as

$$U(x) = -K\left[\frac{1}{\beta} e^{-\beta(x-x^*)} + \frac{1}{\gamma} e^{\gamma(x-x^*)}\right] \tag{3.10}$$

From this form the marginal utility can be written as

$$u(x) = K[e^{-\beta(x-x^*)} - e^{\gamma(x-x^*)}] \tag{3.11}$$

from which it is seen that we have positive marginal utility for $x < x^*$ and negative marginal utility for $x > x^*$, and of course $u(x) = 0$ for $x = x^*$. For $\beta = \gamma$ the function is evidently symmetric around $x = x^*$, but for $\beta \neq \gamma$ the function is not symmetric. For $\beta > \gamma$ the function will be steeper to the left of x^* than to the right of x^*, and for $\beta < \gamma$ the function will be steeper to the right of x^*. By suitable choices of the parameters we can make $U(x)$ fall off as steeply as we like to the left of x^* and be as close to flat as we like to the right of x^* (or vice versa), which may be relevant for cases in which we are concerned to express something like a critical value for a variable.

With several variables we can write

$$U(x) = -\Sigma (B_i e^{-\beta_i x_i} + C_i e^{\gamma_i x_i}). \tag{3.12}$$

In general we could have a sum of two exponentials for some variables and only one for others.

4. Parametric Certainty Equivalence with Exponentional Utility Functions

The basic fact which creates the possibility of a convenient parametric certainty equivalence in the case of exponential utility functions is the following: If x is a stochastic variable, normally distributed with expectation $E(x)$ and variance σ_x^2, then the expected value of the objective function (3.1) is

$$E[U(x)] = -Be^{-\beta[E(x)-(1/2)\beta\sigma_x^2]}. \qquad (4.1)$$

This result is well known and easily established by direct integration. Formula (4.1) is very useful in many contexts in which a decision problem has only one target variable, for instance "income". Then $E(x)$ is the expected value of income and σ_x^2 is the variance of income. If we have a decision problem in which the decision $a \in A$ is to be determined so as to maximize $E[U(x)]$, and the economic mechanism is such that the decision a will influence $E(x)$ or σ_x^2 or both, then the optimal decision can be determined by maximizing $E(x) - 1/2\, \beta\sigma_x^2$, and the role of β as a measure of risk aversion is brought out in a very clear way. This fact was noticed quite early and utilized in a very interesting study of the production planning of American farmers by R. J. Freund (1956), a study which has later been followed up by others; see also the discussion in L. Johansen (1978). The same fact has been utilized in portfolio analysis, see for instance J. M. Parkin et al. (1970). In a related context it has been shown by M. Saito (1977) that convenient results, though not of a certainty equivalence type, can also be obtained when the exponential utility function is combined with a distribution function of the gamma type. Combinations with the Cauchy distribution and with the Poisson distribution give rise to parametric certainty equivalence results equally convenient as in the case of the normal distribution, but the normal distribution is in most cases more attractive. Relevant formulas from which these assertions are easily seen are given for instance in R. L. Keeney and H. Raiffa (1976).

The formula given by (4.1) is closely related to properties of the lognormal distribution. In fact, since $e^{-\beta x} = (e^x)^{-\beta}$ and e^x is lognormally distributed when x is normal the expected value of $e^{-\beta x}$ follows from the formulas for moments of the lognormal distribution.

In the following we are mainly interested in objective functions formed as sums of exponentials, either as (3.4) or as in the more general case (3.12) where all $B_i > 0$ and $C_i > 0$ at least for some i. Then the reduction to maximizing something like $E(x) - 1/2\, \beta\sigma_x^2$ which is valid for the case of one variable no longer works. But we can write (4.1) as

$$E[U(x)] = -Be^{(1/2)\beta^2\sigma_x^2} e^{-\beta E(x)}, \qquad (4.2)$$

or as

$$E[U(x)] = -\tilde{B}e^{-\beta E(x)} \text{ where } \tilde{B} = Be^{(1/2)\beta^2\sigma_x^2}. \qquad (4.3)$$

By this formulation we can link up objective functions like (3.4) or (3.12) with the general idea of parametric certainty equivalence procedures from section 2. For the objective function (3.4) we can consider the parameter vector α as introduced in section 2 to be

$$\alpha = (B_1, \ldots, B_n, \beta_1, \ldots, \beta_n). \tag{4.4}$$

If now $x_1, \ldots, x_n$ are normally distributed with expected values $E(x_i)$ and variances σ_i^2, then from (3.4) we have

$$E[U(x; \alpha)] = -\Sigma \tilde{B}_i e^{-\beta_i E(x_i)} \tag{4.5}$$

which means that we can write

$$E[U(x; \alpha)] = U(E(x); \tilde{\alpha}) \tag{4.6}$$

where the modified parameter vector $\tilde{\alpha}$ is given by

$$\tilde{\alpha} = (\tilde{B}_1, \ldots, \tilde{B}_n, \tilde{\beta}_1, \ldots, \tilde{\beta}_n) =$$
$$(B_1 e^{(1/2)\beta_1^2 \sigma_1^2}, \ldots, B_n e^{(1/2)\beta_n^2 \sigma_n^2}, \beta_1, \ldots, \beta_n). \tag{4.7}$$

The function ψ introduced in section 2, which transforms the original parameter vector α into the modified parameter vector $\tilde{\alpha}$ now corresponds to the transformation from (4.4) to (4.7). In section 2 we indicated that the transformation will depend on the distribution of the random elements. We have now limited the class of distributions to the normal distribution. In (4.7) we see that the transformation is influenced by the variances of $x_1, \ldots, x_n$. If the variances go to zero, then we have $\tilde{\alpha} = \alpha$.

Similar considerations hold, of course, when we have an objective function of the type (3.12). We then have

$$E[U(x)] = -\Sigma (\tilde{B}_i e^{-\beta_i E(x_i)} + \tilde{C}_i e^{\gamma_i E(x_i)}) \tag{4.8}$$

where

$$\tilde{B}_i = B_i e^{(1/2)\beta_i^2 \sigma_i^2}, \; \tilde{C}_i = C_i e^{(1/2)\gamma_i^2 \sigma_i^2}. \tag{4.9}$$

Again (4.6) is valid, now with the parameter vector α enlarged so as to include the γ_i and C_i, and correspondingly for $\tilde{\alpha}$.

These formulas now give rise to a parametric certainty equivalence procedure as indicated in general by (2.8). Suppose that the model relating outcomes to actions is as indicated by (2.6). Then from the distribution of the random elements z we must establish the variances of $x_1, \ldots, x_n$. (In many cases these will be known directly, as the random elements in the reduced form equations.)

Then we modify the parameters of the objective function as indicated by (4.7) or (4.9). Then the problem of maximizing $E[U(x;\alpha)]$ is the same as maximizing $U(E(x);\tilde{\alpha})$ because of (4.6). $E(x)$ is again equal to $F(a)$ corresponding to (2.6). It is then clear that maximizing $U(F(a);\tilde{\alpha})$ with respect to $a \in A$ gives the same result as the problem in the original form. The mathematical form of the modified problem is precisely the same as the form of the original problem if uncertainty were absent, but the parameter vector has been modified. In other words, we have a parametric certainty equivalence procedure.

The modification of the parameters involved in the procedure outlined above affects only $B_1, \ldots, B_n$ (and $C_1, \ldots, C_n$ when these are relevant), not $\beta_1, \ldots, \beta_n$ (and $\gamma_1, \ldots, \gamma_n$). This would be true also for the procedure emerging if $x_1, \ldots, x_n$ were distributed according to the Cauchy distribution. On the other hand, in the procedure which would emerge if $x_1, \ldots, x_n$ were distributed according to the Poisson distribution the parameters $\beta_1, \ldots, \beta_n$ (and $\gamma_1, \ldots, \gamma_n$) would be affected by the transformation.

An objective function like (3.5) consisting of a positive and a negative exponential function, combined with an assumption about normality of the error terms, has been used by A. Kunstman for representing asymmetric preferences in the context of a dynamic control problem (in an unpublished paper which came to my attention during the preparation of this paper). Also in decision theory objective functions consisting of two exponentials, combined with the normal probability distribution, have been used, see especially R. L. Keeney and H. Raiffa (1976). However, Keeney and Raiffa use exponentials with exponents of the same sign, the purpose being mainly to use a functional form which permits the degree of risk aversion to change with the value of the argument instead of being constant as indicated by (3.3) for the case of one exponential function. Neither Kunstman, nor Keeney and Raiffa, explore the possibility of certainty equivalence procedures which is the main concern of this paper.

5. Consequences of Uncertainty

This section gives some illustrations of the consequences of uncertainty when the objective function belongs to the class considered above. We restrict considerations to the case in which the variances of $x_1, \ldots, x_n$ are not influenced by the decision a so that the certainty equivalence procedure is valid. In contrast to the case of certainty equivalence based on quadratic objective functions the existence of uncertainty will in the present case influence the decision through the transformation of the parameters of the objective function.

Consider first the objective function (3.4) for $n=2$, i. e.

$$U(x;\alpha) = -B_1 e^{-\beta_1 x_1} - B_2 e^{-\beta_2 x_2} \tag{5.1}$$

with the corresponding expected value according to (4.5—4.7)

$$U(E(x);\tilde{\alpha}) = -\tilde{B}_1 e^{-\beta_1 E(x_1)} - \tilde{B}_2 e^{-\beta_2 E(x_2)}$$

$$\text{where } \tilde{B}_1 = B_1 e^{(1/2)\beta_1^2 \sigma_1^2} \text{ and } \tilde{B}_2 = B_2 e^{(1/2)\beta_2^2 \sigma_2^2}. \tag{5.2}$$

If there were no uncertainty involved, then variation of the decision a within the set A would generate a feasible set in the x_1, x_2-plane, and we should choose from among these feasible points so as to maximize (5.1). The indifference curves of (5.1) in the x_1, x_2-plane would then be relevant. The marginal rate of substitution between x_1 and x_2 according to this objective function is

$$\frac{dx_2}{dx_1} = -\frac{u_1(x_1)}{u_2(x_2)} = -\frac{\beta_1 B_1}{\beta_2 B_2} e^{\beta_2 x_2 - \beta_1 x_1} \tag{5.3}$$

for constant level of $U(x;\alpha)$.

When there is uncertainty about x_1 and x_2 of the form assumed, then variations of a will generate a feasible set for $E(x_1)$ and $E(x_2)$. The certainty equivalence procedure consists in choosing from this feasible set so as to maximize the value of (5.2). The contour curves of (5.2) will then be relevant. Corresponding to (5.3) we now have the following expression for the marginal rate of substitution

$$\begin{aligned} \frac{dE(x_2)}{dE(x_1)} &= -\frac{\beta_1 \tilde{B}_1}{\beta_2 \tilde{B}_2} e^{\beta_2 E(x_2) - \beta_1 E(x_1)} \\ &= -\frac{\beta_1 B_1}{\beta_2 B_2} e^{(1/2)(\beta_1^2 \sigma_1^2 - \beta_2^2 \sigma_2^2)} e^{\beta_2 E(x_2) - \beta_1 E(x_1)} \end{aligned} \tag{5.4}$$

for constant level of $U(E(x);\tilde{\alpha})$.

Now compare corresponding points in the x_1, x_2-plane and in the $E(x_1), E(x_2)$-plane for increasing uncertainty. We then see that the marginal rate of substitution, and in fact the whole set of contour curves, remain the same if

$$\beta_1 \sigma_1 = \beta_2 \sigma_2. \tag{5.5}$$

This means that if the product of the coefficient of partial risk aversion and the standard deviation remains the same for both variables, then the marginal rate of substitution does not change. Furthermore, an increase in uncertainties which is such that the uncertainty as measured by the standard deviation increases more for the variable with the smaller partial risk aversion than for the variable with the larger risk aversion may leave the decision unaffected. Otherwise the existence and the degree of uncertainty will usually influence the decision. We see from (5.4) that a partial increase in σ_1^2 will make the indifference curve steeper while a partial increase in σ_2^2 will make the indifference curve flatter. With an unaltered feasible set this will in general mean that a larger σ_1^2 tends to induce a change in the decision in the direction of a larger value of $E(x_1)$ while a larger value of σ_2^2 tends to induce a change in the decision in the direction of a larger value of $E(x_2)$. Thus, an increase in the uncertainty referring to the value of a variable tends to make the decision maker safe-guard against this by taking a decision which implies a higher value of the expected value of the variable in question. If the variance is the same for both variables, i. e. $\sigma_1^2=\sigma_2^2$, then we see from (5.4) that the indifference curve will become steeper because of the uncertainty if $\beta_1>\beta_2$, i. e. if the partial degree of risk aversion is larger for x_1 than for x_2. This will again tend to push the decision in a direction which produces a larger expected value of x_1, and vice versa if $\beta_2>\beta_1$. In other words, if the uncertainty is the same concerning both variables, then one will safeguard against the uncertainty by making larger the expected value of the variable to which one attaches the highest degree of risk aversion.

Fig. 1 illustrates how the indifference curves are twisted when the variances change. The indifference curves represent the function $E(U(x;\alpha))=U(E(x);\tilde{\alpha})$ as defined by (5.2) for $B_1=B_2=1$, $\beta_1=0.2$ and $\beta_2=0.4$. In the flattest curve we have $\sigma_1^2=4$ and $\sigma_2^2=24$ while the steepest curve represents $\sigma_1^2=50$ and $\sigma_2^2=1$. The variances for the intermediate curves are indicated in the figure. The indifference curves correspond to levels of the function which have been adjusted so that they all pass approximately through a common point, thus giving a clear impression of the rotation of the indifference curves with the changes in the variances of x_1 and x_2. Fig. 2 illustrates the families of indifference curves for the same values of B_1, B_2, β_1 and β_2. The flattest indifference curves represent full certainty, i. e. $\sigma_1^2=\sigma_2^2=0$, while the steeper indifference curves represent uncertainty with $\sigma_1^2=40$ and $\sigma_2^2=4$. (The same shift of the curves could have been obtained by other variances, smaller or larger, if only $\beta_1^2\sigma_1^2-\beta_2^2\sigma_2^2$ is kept constant.) If the restrictions defining the feasible set are such that for instance $E(x_1)+E(x_2)$ has to be less than or equal to some constant, then the corresponding expansion paths would be as indicated in the figure, with the expansion path

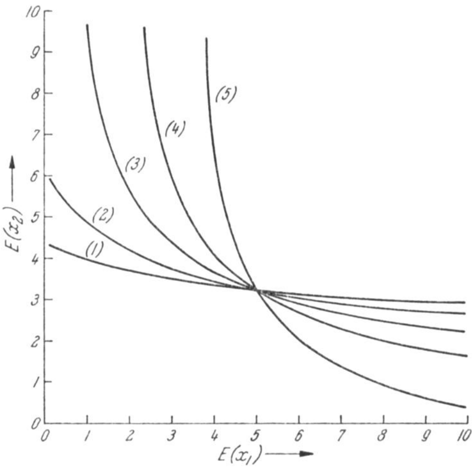

Fig. 1. Indifference curves corresponding to (5.2) for different values of σ_1^2 and σ_2^2. Curve (1): $\sigma_1^2=4$, $\sigma_2^2=24$; Curve (2): $\sigma_1^2=4$, $\sigma_2^2=16$; Curve (3): $\sigma_1^2=4$, $\sigma_2^2=8$; Curve (4): $\sigma_1^2=4$, $\sigma_2^2=1$; Curve (5): $\sigma_1^2=50$, $\sigma_2^2=1$

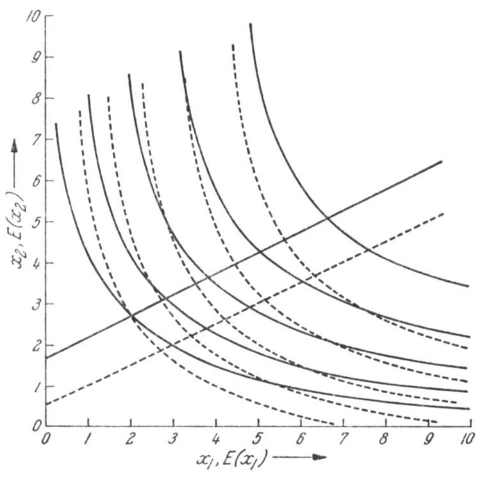

Fig. 2. Families of indifference curves and corresponding expansion paths for increasing values of x_1+x_2 and $E(x_1)+E(x_2)$
——— for (5.1), i. e. full certainty; – – – for (5.2), with $\sigma_1^2=40$, $\sigma_2^2=4$

under uncertainty below the expansion path under certainty. The expansion paths are linear and parallel, with a slope given $\beta_1/\beta_2 = 1/2$.

Next consider the case of a peaked curve formed by adding a negative and a positive exponential function. We consider the influence of uncertainty on this curve for one variable, i. e. we have

$$U(x; \alpha) = -Be^{-\beta x} - Ce^{\gamma x} \tag{5.6}$$

and

$$U(E(x); \tilde{\alpha}) = -\tilde{B}e^{-\beta E(x)} - \tilde{C}e^{\gamma E(x)} \tag{5.7}$$

$$\text{where } \tilde{B} = Be^{(1/2)\beta^2\sigma^2}, \quad \tilde{C} = Ce^{(1/2)\gamma^2\sigma^2}.$$

The effects of the uncertainty will in this case be most clearly shown by rewriting the form of (5.7) in accordance with the reformulation given by (3.8—3.10). This gives

$$U(E(x); \tilde{\alpha}) = -\tilde{K}\left[\frac{1}{\beta}e^{-\beta[E(x)-\tilde{x}^*]} + \frac{1}{\gamma}e^{\gamma[E(x)-\tilde{x}^*]}\right] \tag{5.8}$$

where

$$\tilde{K} = [(\beta\tilde{B})^\gamma(\gamma\tilde{C})^\beta]^{\frac{1}{\beta+\gamma}} \tag{5.9}$$

and

$$\tilde{x}^* = \frac{\ln(\beta\tilde{B}) - \ln(\gamma\tilde{C})}{\beta+\gamma} = x^* + {}^1/_2(\beta-\gamma)\sigma^2. \tag{5.10}$$

In these formulas $\tilde{B}$ and $\tilde{C}$ are as given in connection with (5.7), and x^* is the value given by (3.8).

Formula (5.10) gives the most desired value of $E(x)$ when there is uncertainty. It is seen from the expression to the right that this most desired value deviates from the most desired value under certainty, x^*, if we have $\beta \neq \gamma$. The deviation is greater the greater is the deviation from symmetry by $\beta-\gamma$ and the greater is uncertainty as measured by σ^2. If $\beta > \gamma$ the preference function (5.6) is more strongly curved to the left than to the right, and the most desired value of $E(x)$ under uncertainty is accordingly moved to the right by formula (5.10), so as to decrease the chance of getting a value of x to the left of x^* as compared with the probability of getting a value of x to the right of x^*, and vice versa for $\beta < \gamma$.

The maximal value of the function now studied will be less than zero. From (5.8) it is seen that the maximal value will be

$$-\left(\frac{1}{\beta} + \frac{1}{\gamma}\right)\tilde{K} \tag{5.11}$$

where $\tilde{K}$ is given by (5.9). It is seen that this value will be lower (more below zero) the larger is the variance σ^2 since a larger σ^2

increases both $\tilde{B}$ and $\tilde{C}$. This is a natural consequence of the concavity of the function. The increase in K with the increase in the variance σ^2 also means that the function will be more strongly peaked when the uncertainty is greater. This is of no consequence if we have only one argument in the preference function. However, if we have more than one variable so that a trade-off between this

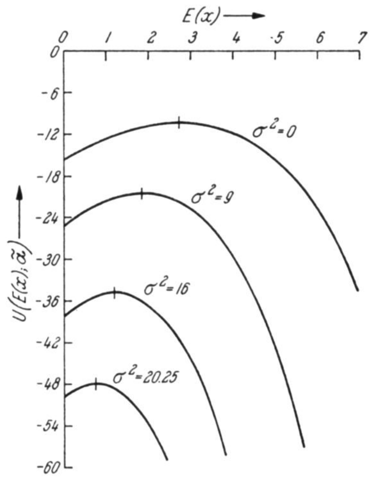

Fig. 3. The form of (5.7) or (5.8) for different values of σ^2

variable and other variables is relevant, then this stronger peakedness with larger uncertainty is of interest. It then means that we will be less willing to let the expected value of the variable in question deviate from its most desired value $\tilde{x}^*$.

Fig. 3 shows the form of the function (5.7) or (5.8) for different levels of uncertainty. The values of the parameters are $B = 15$, $C = 1$, $\beta = 0.3$ and $\gamma = 0.5$. The upper curve represents the case of no uncertainty, i. e. $\sigma^2 = 0$. In this case we have $x^* = 2.75$. The next curve has $\sigma^2 = 9$, then $\sigma^2 = 16$, and finally $\sigma^2 = 20.25$ ($\sigma = 4.5$). From its value 2.75 under no uncertainty the most desired value of $E(x)$, $\tilde{x}^*$, shifts to 1.85 for the second curve, 1.15 for the third curve, and 0.72 for the last curve in the figure. The direction of the shift is determined by the fact that in this case we have $\beta < \gamma$ so that the function falls off more steeply to the right than to the left. The figure also shows how the maximal value declines with increasing uncertainty, and suggests the more sharply peaked form of the curve for larger uncertainty.

The effects elucidated above would be relevant for decision-making under uncertainty in general, and more specifically for instance for the theory of consumer choice under uncertainty along the lines suggested in Y. Amihud (1977).

6. Some Practical Suggestions

Preference functions involving exponentials should be rather convenient from a practical point of view. There are two practical aspects involved: The question of how to establish a preference function, and the methods of solving the optimization problem to which the preference function is applied.

Most of the interview methods which have been suggested for other forms of preference functions, especially quadratic functions, could be taken over and used in connection with exponential preference functions with minor modifications. [For a survey and discussion of some such methods, see for instance L. Johansen (1974).] Careful discussion and practical examples of how to establish preference functions for use in decisions under uncertainty, including preference functions of the exponential type, are given in R. L. Keeney and H. Raiffa (1976).

Of special interest perhaps is the asymmetric version of the function given by (3.5) for one variable and used as components in preference functions for several variables as in (3.12). This form offers some obvious possibilities of simple questions which help to determine the parameters. If the most desired value x^* is within the range of reasonable imagination, then this is an obvious object for questions. If x^* has been located, then it gives the following constraint on the values of the parameters of (3.5):

$$\frac{\ln\beta-\ln\gamma+\ln(B/C)}{\beta+\gamma}=x^*. \tag{6.1}$$

Another simple type of question would refer to the possibility of locating indifferent values of the variable, on each side of x^*. One might for instance choose a value $x'<x^*$, at a suitable distance from x^*, and try to locate a value $x''>x^*$ which is indifferent with x'. This would give the following type of constraint on the parameters:

$$e^{\gamma x''}-e^{\gamma x'}=(\mathrm{B/C})\,(e^{-\beta x'}-e^{-\beta x''}). \tag{6.2}$$

Conditions (6.1) and (6.2) are not sufficient to determine the parameters. They give only two equations referring to three variables,

β, γ, and B/C. [Eq. (6.2) can be rewritten in terms of deviations from x^* corresponding to the formulation (3.10). Then we get an equation involving only β and γ. But B/C cannot be eliminated from (6.1) by any rearrangement.]

For a utility function in one variable it is natural that we cannot come further on the basis of deterministic questions, since under certainty any monotonically increasing function would represent the same preferences and indifferences. For a further determination questions involving "lotteries" would be necessary. In R. L. Keeney and H. Raiffa (1976) methods of assessing a preference function consisting of two negative exponentials are referred to (see especially section 4.10.3). The type of question used is to locate certainty equivalents of simple 50-50 lotteries. One should only observe that there will in the present case be two certainty equivalents of each such lottery, one inside the interval between the two possible outcomes of the lottery, and one outside of it.

By combining information from such questions with information obtained by (6.1) and (6.2) one could determine the values of β, γ, and B/C. However, the absolute values of B and C separately could not be determined by questions involving only one variable; for scaling of the various components of the multivariable preference function in proportion to each other questions involving the trade-offs between variables would be necessary.

The computational problem for optimization with the type of objective function considered in this paper should not be too difficult. The function is smooth and concave, and if the feasible region is not too awkward various methods of non-linear programming will be convenient and efficient. In any case, if one is able to solve the corresponding problem with a quadratic preference function, then one will also be able to solve the problem with the present preference functions, at least approximately. The simplest procedure would be as follows:

1. Establish the preference function in terms of exponential components, either as (3.4) or as (3.12).
2. Specify the degree of uncertainty referring to each variable x_i, as represented by the variance σ_i^2. (For the certainty equivalence procedure discussed here it is necessary that this be independent of the decision a.)
3. Modify the preference function so as to express it in terms of expected values $E(x_i)$, with parameter values modified according to (4.7) or (4.9).

4. Work out a quadratic approximation to the modified preference function in terms of $E(x_1), \ldots, E(x_n)$ established under step 3.
5. Solve the optimization problem with the quadratic objective function established under step 4.

Computationally everything which precedes step 5 here is very simple, so whenever we can solve a problem involving a quadratic objective function, we can follow the above procedure. This does not give the exact solution of the original problem, but it does reflect the consequences of uncertainty in an approximate manner. By first specifying the preference function in the way indicated here, and then introducing uncertainty and modifying the objective function accordingly, it will reflect such shifts as a consequence of uncertainty as were illustrated in the previous figures. For instance, referring to fig. 3 we would, under uncertainty, use a quadratic approximation to a lower curve rather than to the upper curve which corresponds to no uncertainty. Even if this quadratic approximation is symmetric once it has been established, it will reflect the shift of the maximum point of the curve and the sharper curvature as a consequence of uncertainty. These effects will accordingly also be transferred to the outcome of the optimization.

The point here is that we introduce uncertainty before we carry out the quadratic approximation; if we had first introduced the quadratic approximation, then we would be back to the simple certainty equivalence case in which the uncertainty would have no consequences for the decision.

Now the approximation obtained by the procedure outlined above could clearly be improved. There is a question about the way in which one approximates the original preference function by a quadratic function. For the function $U(x)$ as written in (3.1) we would have the following simple approximation at any point x^0:

$$U(x) \approx -Be^{-\beta x^0}\left[1-\beta(x-x^0)+{}^1/_2\beta^2(x-x^0)^2\right]. \tag{6.3}$$

For the composite function (3.5) an approximation like (6.3) could be used for each term. When the approximation is taken at the most desired value x^*, then a particularly simple expression emerges form the form (3.10). We then have

$$U(x) \approx -K\left(\frac{1}{\beta}+\frac{1}{\gamma}\right) - {}^1/_2 K(\beta+\gamma)(x-x^*)^2. \tag{6.4}$$

The approximations (6.3, 6.4) have been expressed in terms of x. As used in the five-step procedure outlined above, one would establish the approximation for the functions considered as functions of $E(x)$, and the parameters used would be the modified values as given by (4.7), (4.9), (5.9) and (5.10).

In the procedure suggested above one would first have to approximate the original preference function at some more or less arbitrary point, perhaps having a preliminary guess about the range in which one will find the optimal solution. For variables with a saturation value one might perhaps first use the approximation as given by (6.4). Having solved the optimization problem one could go back and work out a new approximation to the original preference function at the new point found, and then repeat the procedure. For practical purposes it would hardly be necessary to go through many such rounds of iteration.

An objective function of the sum-of-exponentials type has been used as a local criterion in inter-active approaches to socalled "multi-attribute decision-making" in management science. In this context both of the practical problems touched upon above, i. e. the problem of establishing the function numerically and of solving the computational problem of optimization, have been dealt with; see especially K. R. Oppenheimer (1978).

Appendix:
Some Observations on Risk Aversion

We have observed that the coefficient of absolute risk aversion corresponding to the preference function (3.1) is simply β, see (3.3). When such functions are combined into a sum of exponentials such as (3.4) the parameters β_i retain their relevance as measures of risk aversion; in fact, they are for instance the diagonal elements of the "absolute risk aversion matrix" according to the matrix measure of multivariate local risk aversion proposed by G. T. Duncan (1977).

When we introduce the function (3.5), which is not monotonic but involves a saturation level for the variable, the measure of risk aversion is more problematic. For the range where the function is increasing, we have the following measure of absolute risk aversion:

$$-\frac{u'(x)}{u(x)} = \frac{\beta^2 Be^{-\beta x} + \gamma^2 Ce^{\gamma x}}{\beta Be^{-\beta x} - \gamma Ce^{\gamma x}} \quad (x < x^*). \tag{A.1}$$

On the basis of the formulation (3.10, 3.11) this can also be written as

$$-\frac{u'(x)}{u(x)} = \frac{\beta e^{-\beta(x-x^*)} + \gamma e^{\gamma(x-x^*)}}{e^{-\beta(x-x^*)} - e^{\gamma(x-x^*)}} \quad (x < x^*). \tag{A.2}$$

If we consider the range where $x > x^*$, then the objective function is decreasing. In this case it is usual to calculate the measure of absolute risk aversion simply by omitting the negative sign of the definition of the risk aversion coefficient. Corresponding to the formulas above we then have

$$\frac{u'(x)}{u(x)} = \frac{\beta^2 Be^{-\beta x} + \gamma^2 Ce^{\gamma x}}{\gamma Ce^{\gamma x} - \beta Be^{-\beta x}} \quad (x > x^*), \tag{A.3}$$

$$\frac{u'(x)}{u(x)} = \frac{\beta e^{-\beta(x-x^*)} + \gamma e^{\gamma(x-x^*)}}{e^{\gamma(x-x^*)} - e^{-\beta(x-x^*)}} \quad (x > x^*). \tag{A.4}$$

For $x = x^*$ the coefficient of absolute risk aversion does not exist, since at this point $u(x) = 0$. This difficulty concerning the definition of risk aversion for a non-monotonic utility function has been noted by R. L. Keeney and H. Raiffa (1976). They make the following comment (p. 188): "Perhaps an alternative definition of a local risk aversion exists for this case, but this seems to be academic. For operational problems, a reasonable approach would be to divide the range of the attribute into intervals so that preferences are monotonic in each interval and then treat each interval separately using the theory relevant to the respective cases."

In view of the properties of the function discussed here, it seems that the difficulties refer not only to the non-existence of the coefficient of risk aversion at the saturation point. The variation of the coefficient for the ranges where it exists is also of interest. This variation can most clearly be seen from the forms (A.2) and (A.4) above. From (A.2) it is clear that the coefficient of absolute risk aversion is close to β when the value of x is much smaller than x^*. When x increases towards x^*, the numerator will remain positive and finite, while the denominator, which is also positive, tends to zero as x tends to x^*. Accordingly the coefficient of risk aversion increases and tends to infinity as x approaches x^*. From (A.4) we see that the behaviour in the range $x > x^*$ is similar. For x much larger than x^* the coefficient is close to γ in value, while it again increases and tends to infinity when x approaches x^* from above. Since the utility function is nice and smooth throughout, it seems somewhat disturbing or misleading that the coefficient of risk aversion shows this dramatic behaviour not only at the point $x = x^*$, but also in the ranges on both sides of x^*. The decision-maker would hardly be so dramatically more averse to a gamble or a risk in the vicinity of x^* than at values of x somewhat more distant from x^*. The reason for the difficulty lies in the type of experiment used to define the risk aversion coefficient. This experiment is defined in terms of a lottery involving different possible outcomes specified in terms of x, and a compensation which the decision-maker requires for being willing to undertake the risk as compared with having the mathematical expectation of x in the lottery as a sure outcome. The point now is that this compensation is also given in terms of x. However, at x^* a marginal addition to or deduction from x has no value, and in the vicinity of x^* the marginal value of x is very small. Accordingly, in the experiment defining the coefficient of risk aversion we offer the decision-maker a compensation in terms of a good which is of no or only very small marginal value to him. There is then no wonder that he requires a very large compensation in terms of this good in proportion to the size of the risk, but this is due not to his aversion towards risk, but to the form of the compensation we offer.

In uncertain prospects involving only money, which is assumed to have a positive marginal utility throughout the relevant range, this difficulty will not appear in this strong form. Still it seems to be useful to be aware of the fact that the behaviour of the risk aversion coefficient does not reflect how "unhappy" the decision-maker is about a risk; if for instance the risk aversion coefficient increase, then this could be said to reflect the fact that the decision-maker is more well-off so that he requires a larger sum as a sure compensation although he is in fact not very averse to the risk.

These considerations suggest that the variation of the risk aversion coefficient with the value of the argument is due to two different factors: In the first place, the aversion to risk concerning the value of a variable varies with the initial value of the variable around which the possible outcomes of the risk situation — the "lottery" — are located. In the second place, the risk aversion coefficient varies with the marginal value which the decision-maker attaches to the good in terms of which the compensation for risk is offered. If we want to separate these effects and have a purer measure of risk aversion, it might for some purposes be useful to stipulate the compensation for risk-taking in terms of another good than the one which the lottery refers to. If the lottery refers to money, then one might for instance ask how much time or effort a decision-maker would be willing to expend in order to avoid a specific risk concerning money, for different levels of initial money holdings. This way of looking at it may, however, be more relevant when we have preference functions involving several variables, and when some of them may enter the overall preference function through components involving a saturation level. Consider a variable x_i which enters the overall utility function through an additive component $U_i(x_i)$, and suppose there is a risk attached to the value of x_i. We could then ask what compensation the decision-maker requires in terms of some other good x_j for undertaking the risk referring to x_i, and then see how this compensation varies with the initial value of x_i while the value of x_j to which the compensation is added is kept constant. Then apart from a scale factor the compensation required, and accordingly the measure of risk aversion, will vary as the absolute value of the second order derivative of the utility function $U_i(x_i)$. For a component with a utility function given by (3.5) or equivalently (3.10), this value is

$$\begin{aligned} |u_i'(x_i)| &= \beta_i^2 B_i e^{-\beta_i x_i} + \gamma_i^2 C_i e^{\gamma_i x_i} \\ &= K_i \left[\beta_i e^{-\beta_i (x_i - x_i^*)} + \gamma_i e^{\gamma_i (x_i - x_i^*)}\right]. \end{aligned} \tag{A.5}$$

This shows no dramatic variations around the value $x_i = x_i^*$. For this special value of x_i we simply have $|u_i'(x_i)| = K_i[\beta_i + \gamma_i]$. This, however, is not the minimum value of $|u_i'(x_i)|$, unless $\gamma_i = \beta_i$. The minimum value of $|u_i'(x_i)|$ occurs for

$$x_i = x_i^* + 2\,\frac{\ln \beta_i - \ln \gamma_i}{\beta_i + \gamma_i} \tag{A.6}$$

i. e. to the left or to the right of $x = x^*$ according as $\beta_i < \gamma_i$ or $\beta_i > \gamma_i$. Furthermore, it is seen that the value of $|u_i'(x_i)|$ increases strongly both when x_i becomes very small and when x_i becomes very large, indicating that risk aversion is strong both for very small and very large values of x_i. This all gives a much more reasonable picture than saying that risk aversion goes to infinity as x_i approaches x_i^* from either side, and is infinite at $x_j = x_i^*$.

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