A Calculus Approach to the Theory of the Core of an Exchange Economy

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The theory of the shrinking of the core of an exchange economy to the competitive equilibrium (or set of equilibria) when the number of participants increases is one of the most important and interesting contributions to general equilibrium theory in recent decades, and ought to become part of standard courses in economic theory. It is important to have an exposition of this idea which appears as a simple and natural extension of the tools of analysis familiar to most students of economics. The purpose of the present paper is to make an attempt at such an exposition along traditional calculus lines. The paper does not contain results which are new to specialists in the field. In the literature there are, of course, some expositions which point in the direction taken here, but I have not seen the approach spelled out in the way it is done in the sequel. (Some relevant references are given at the end of the paper.)

I. Background and Perspectives

Let me first state very briefly why I consider the result mentioned to be interesting and important. It is then necessary to emphasize the difference between the meaning of the concepts of competitive equilibrium and core allocations.

A competitive equilibrium presupposes the existence of a price system. Under this system individual agents act in isolation in the sense that each of them decides how much to supply and how much to demand of the various commodities on the basis of his own preferences, without making conscious and explicit arrangements with other agents. Each agent considers prices as given in an impersonal way, not subject to bargaining or manipulation through his own supply and demand. We have equilibrium if prices are such that supply and demand for all agents taken together are equal for each commodity. Provided that we have somehow established equilibrium prices in this sense, they solve a complicated multiagent problem by transforming it into a set of rather simple individual decision problems. (It is not necessary for our purpose to go into the problem of how the prices are established and the associated dynamic stability problems.)

A core allocation is defined by an entirely different approach. In this case we consider only a set of agents with initial holdings of commodities who may improve their positions by reallocation, but we do not presuppose the existence of a price system. We start at a more basic level, assuming only that there are possibilities for the agents to communicate and make agreements to reallocate commodities between them—by unilateral gifts, by bilateral exchange, or by some more complicated multilateral exchange arrangement. The individuals are free to form "coalitions" for the purpose of improving the situation for members of the coalition. In our context a coalition is simply a group of agents who agree on a certain reallocation of the initial quantities of goods held by its members. It should be observed that the initial quantities are individually owned, and ownership respected in the sense that nothing can be taken away from an agent without his consent, as part of a voluntary exchange or reallocation. We may now ask whether it is possible to predict the outcome of the exchange or reallocation process in such a system. The "core" gives an answer to this question. It is based on the following observation: Consider an outcome which is feasible in the

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sense that the total amounts of commodities held after the exchange or reallocation are equal to the total initial amounts. This outcome implies a specific bundle of commodities for each agent. If there is at least one group of agents such that these agents could improve their situations by redistributing their own initial holdings instead of agreeing to the proposed outcome, then this outcome will not be realized. It will be "blocked" by the group or coalition mentioned, that is, they will refuse to accept it, because there is another arrangement which they can realize without requiring the cooperation of other agents, and which is better for each of them. It is then natural to ask: Is there a feasible outcome, or a set of outcomes, which will not be blocked by any coalition that can be formed by some agents of the economy (including degenerate coalitions consisting of single agents, and the grand coalition comprising all agents). If such an outcome, or set of outcomes exists, then this is the core.

An outcome belonging to the core is stable in a very important sense, different from the usual dynamic stability concept. It is stable against attempts by individuals and coalitions to find something better, because no possible coalition can do better by refusing to accept the outcome, and instead manage on the basis of the initial holdings of the members of the coalition.

We can now compare the allocation defined by the competitive equilibrium with core allocations. It is a simple matter to show, for exchange economies which we shall consider here, that the competitive equilibrium allocation belongs to the core. The result referred to above as the theory of the "shrinking of the core of an exchange economy to the competitive equilibrium (or set of equilibria) when the number of participants increases" is more striking and also more complicated to prove, and it is to this theme the present paper will be devoted. This connection between competitive equilibria and the core may, in my opinion, give rise to rather far-reaching speculations about economic systems and institutions.

The establishing of a core allocation by means of tentative formations of coalitions of all sizes and compositions, and comparisons of feasible outcomes for the various coalitions, will for an economy with more than a handful of agents represent a large effort in terms of communication and negotiations. In comparison the mechanism of competitive equilibrium is strikingly simple, requiring only individual decisions (when correct prices are given). If an economy has, so to speak, invented the price mechanism, and competitive equilibrium prices have been established, then an enormous organizational problem is solved in an easy manner, and the solution is stable in the sense described above, that is, any group which might contemplate breaking out of the market system will in the end find that it cannot improve its situation by doing so. Furthermore, if the economy consists of a number of agents "approaching infinity," then outcomes corresponding to the set of competitive equilibria are the *only* outcomes which satisfy this kind of stability requirement. I think these considerations go a long way towards explaining why competitive market mechanisms have appeared in almost all corners of the world and, under almost all conceivable circumstances, why they have proved to be so robust, why other arrangements tend to be less permanent, and why attempts to abolish the market mechanism have often failed in the sense that markets reappear unofficially parallel with the official nonmarket system. (These are, of course, sweeping statements which should not be taken too literally. They are meant only as suggestions of the perspectives opened up by a seemingly rather formal and esoteric theory.)

II. Strategy of Reasoning

As already suggested, this paper will treat only exchange economies, although extensions to production economies are possible. The main idea is to get as far as we can by means of simple calculus tools of analysis. We must then assume more of "smoothness" than necessary in more advanced expositions and proofs. In fact, we shall assume strictly convex preferences, representable by differentiable utility functions. The advantage gained by this is that we can exploit the possibility of approximating a utility function by its tangent in certain neighborhoods.

The strategy of the reasoning is first to limit considerations to Pareto optimal points since both competitive equilibria and core allocations belong to the set of Pareto optima. (The last part of this statement is true because the coalition of all agents would block, in the sense indicated above, any allocation which is not Pareto optimal. This, by the way, points to a limitation of the core theory in the form considered here. Whenever we consider, as we often do in welfare theory, situations which are not Pareto optimal, then we must implicitly assume some sorts of difficulties which prevent the formation of coalitions. It is, however, beyond the scope of this paper to pursue this idea.) Then we consider the various Pareto optimal allocations to see whether there are coalitions which would block them, and we shall find that, for any such allocation which does not belong to the set of competitive equilibria, we can construct such a coalition, that is, prove that the allocation does not belong to the core, provided that the number of agents is sufficiently large. (A certain regularity may be required concerning the way in which the number of agents is made large.)

III. Description of the Economy and Notation

I now introduce the notation necessary to describe the exchange economy to be considered. Let there be M perfectly divisible commodities indexed $i = 1, \dots, M$ and G "types" of individuals, indexed $j = 1, \dots, G$. All individuals of the same type have the same initial quantities of the various commodities and the same utility functions. The following notation is also introduced:

N_j = the number of individuals of type j $(j = 1, \dots, G)$.

$\bar{x}_{ij}$ = *initial quantity* of commodity i held by a person of type j $(i = 1, \dots, M;$ $j = 1, \dots, G)$.

x_{ij} = quantity of commodity i held by a person of type j after the exchange $(i = 1, \dots, M; j = 1, \dots, G)$. I call these *final quantities*.

$U_j = U_j(x_{1j}, \dots, x_{Mj})$ = utility function of an individual of type j $(j = 1, \dots, G)$. Assumptions about the utility functions have already been mentioned in Section II above.

$u_{ij} = \partial U_j / \partial x_{ij}$ = marginal utility of commodity i for an individual of type j $(i = 1, \dots, M; j = 1, \dots, G)$. I assume $u_{ij} > 0$ for all i, j.

The collection of all $\bar{x}_{ij}$ will be called the *initial allocation* or *initial point* and symbolized by $\bar{x}$. The collection of all x_{ij}, symbolized by x, will be called the *final allocation* or *final point*. I shall, furthermore, use x_{ij}^* and x^* to symbolize a Pareto optimal allocation.

An allocation which is feasible for the economy as a whole must satisfy

$$(1) \quad N_1 x_{i1} + \dots + N_G x_{iG} = N_1 \bar{x}_{i1} + \dots + N_G \bar{x}_{iG} \qquad (i = 1, \dots, M)$$

IV. Pareto Optimal Allocations and Competitive Equilibria

As already pointed out it follows from the definition of the core that a point which is not Pareto optimal cannot belong to the core. Hence, we need only consider Pareto optimal points as candidates for belonging to the core. Furthermore we shall consider as candidates only Pareto optimal points where individuals of the same type receive the same amounts of the various goods. This implies some loss in generality, but hardly serious for our purpose. Indeed, if $N_1, N_2, \dots, N_G$ have a greatest common divisor which is greater than one, then a very simple argument given by Jerry R. Green, which need not be repeated here, shows that core allocations have this "equal treatment property." (Convexity of preferences, as assumed above, is used in establishing this result.)

For our calculus approach it is assumed

that the optimizations defining Pareto optimal points yield interior solutions. Pareto optimal points with equal treatment as just described can then be characterized by the following conditions:

$$(2)\quad \frac{u_{1j}}{\lambda_1} = \ldots = \frac{u_{Mj}}{\lambda_M} = \mu_j \quad (j = 1, \ldots, G)$$

Pareto optimal allocations are allocations which satisfy these conditions in addition to the balances (1).

The symbols μ_j in (2), one for each type $1, \ldots, G$, are introduced for convenience as the common value of the proportions to the left. The factors of proportionality $\lambda_1, \ldots, \lambda_M$ in formula (2) can, of course, be interpreted as prices, but they are used here only as coefficients to characterize a Pareto optimal allocation, not to describe any particular institutional arrangement. Let x_{ij}^* denote the quantities corresponding to some Pareto optimal allocation, that is, an allocation satisfying (1) and (2).

I now introduce *imputed wealth*. For an individual of type j the imputed wealth in an arbitrary allocation x is defined by

$$(3)\quad y_j = \lambda_1 x_{1j} + \ldots + \lambda_M x_{Mj} \quad (j = 1, \ldots, G)$$

where the factors of proportionality are used from (2). For the initial allocation and for the Pareto optimal allocation we have, in particular, imputed wealth $\bar{y}_j$ and y_j^* respectively, defined by

$$(4)\quad \bar{y}_j = \lambda_1 \bar{x}_{1j} + \ldots + \lambda_M \bar{x}_{Mj} \quad (j = 1, \ldots, G)$$

$$(5)\quad y_j^* = \lambda_1 x_{1j}^* + \ldots + \lambda_M x_{Mj}^* \quad (j = 1, \ldots, G)$$

For the Pareto optimal allocation considered we do not necessarily have $y_j^* = \bar{y}_j$. If $y_j^* = \bar{y}_j$, then x^* represents a competitive equilibrium with prices $\lambda_1, \ldots, \lambda_M$ since the relations (2) then signify the adaptation of the various individuals to these prices and $y_j^* = \bar{y}_j$ represents the budget balance of an individual of type j. If $y_j^* \neq \bar{y}_j$ for some j, then we have a Pareto optimal point which is not a competitive equilibrium. It is well known that we may have more than one competitive equilibrium, that is, a set of equilibria. This does not matter for the following arguments.

V. The Blocking of Pareto Optimal Allocations which are not Competitive Equilibria

I now raise the question as to whether a Pareto optimal point which is not necessarily a competitive equilibrium can be blocked by any coalition. Let a possible coalition consist of $n_1, n_2, \ldots, n_G$ individuals of the various types. This coalition can, on the basis of its own initial quantities, reach any final point x which satisfies the balance relations

$$(6)\quad n_1 x_{i1} + \ldots + n_G x_{iG} = n_1 \bar{x}_{i1} + \ldots + n_G \bar{x}_{iG} \quad (i = 1, \ldots, M)$$

The imputed wealth for individuals of type j in such a point is then given by (3).

The question now is whether there exists any feasible final point x for the coalition which is considered by all members to be better than the given Pareto optimal point x^*. It follows from what has been explained that, in order to show that x^* does not belong to the core, it is sufficient to construct *one* such coalition for which *one* such point exists. We then look for a simple way to do this, not for the most general characterization of the possibilities of blocking. If we tentatively limit attention to points x which are in the neighborhood of x^*, then imputed wealth can be used as a criterion to compare x and x^*. Since we have, approximately,

$$(7)\quad U_j(x_{1j}, \ldots, x_{Mj}) - U_j(x_{1j}^*, \ldots, x_{Mj}^*) \approx u_{1j} \cdot (x_{1j} - x_{1j}^*) + \ldots + u_{Mj} \cdot (x_{Mj} - x_{Mj}^*) = [\lambda_1(x_{1j} - x_{1j}^*) + \ldots + \lambda_M(x_{Mj} - x_{Mj}^*)]\mu_j = (y_j - y_j^*)\mu_j$$

and since $\mu_j > 0$, we have for x in the neighborhood of x^*:

$$(8)\quad y_j > y_j^* \Longrightarrow \text{an individual of type } j \text{ is better off in } x \text{ than in } x^* \quad (j = 1, \ldots, G)$$

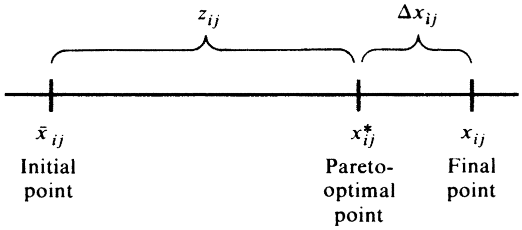

FIGURE 1

The question can now be posed as follows: Can we make $y_j > y_j^*$ hold for all j when the final point x is constrained by (6)? Introduce the following terms

(9) $$\Delta x_{ij} = x_{ij} - x^*_{ij}$$

(10) $$z_{ij} = x^*_{ij} - \bar{x}_{ij}$$

that is, Δx_{ij} is the deviation between the final point and the Pareto optimal point we are testing for possible blocking, and z_{ij} is the deviation between the Pareto optimal point and the initial point, as suggested in Figure 1.

We will now see if a change from x^* to x which, for each type, changes the quantities proportionately with z_{ij} will do for the purpose of blocking x^*, that is, for producing a final point which all members of the coalition find superior to x^*. We may think of this in the commodity space as drawing a straight line between the initial point $\bar{x}$ and the Pareto optimal point x^*, and then moving the final point for each group away from x^* along this ray, either towards $\bar{x}$ or further away from $\bar{x}$. Introduce the ratio

(11) $$s_j = \Delta x_{ij}/z_{ij} \qquad (i = 1, \ldots, M; \quad j = 1, \ldots, G)$$

If $s_j > 0$, then individuals of type j are moved further away than x^* from the initial point; if $s_j < 0$, then they are moved some distance back towards $\bar{x}$. (We may have $z_{ij} > 0$ or $z_{ij} < 0$. If, by coincidence, $z_{ij} = 0$ for some i, then also $\Delta x_{ij} = 0$, and s_j takes the value suitable for the changes in the quantities of the other commodities. If, for some j, we should happen to have $z_{ij} = 0$ for all i, then s_j is arbitrary. In the explanations which follow I shall, for brevity, neglect this special case.)

If such moves are feasible for the coalition considered, that is, satisfy (6), then we must have

$$n_1(x_{i1} - \bar{x}_{i1}) + \ldots + n_G(x_{iG} - \bar{x}_{iG}) = 0 \qquad (i = 1, \ldots, M)$$

which by use of (9)–(11) can be written as

(12) $$n_1(1 + s_1)z_{i1} + \ldots + n_G(1 + s_G)z_{iG} = 0 \qquad (i = 1, \ldots, M)$$

According to (8) individuals of type j are better off at x than at x^* if we have

$$y_j - y_j^* = \lambda_1(x_{1j} - x^*_{1j}) + \ldots + \lambda_M(x_{Mj} - x^*_{Mj}) > 0$$

or, in view of (9) and (11),

(13) $$y_j - y_j^* = (\lambda_1 z_{1j} + \ldots + \lambda_M z_{Mj})s_j > 0$$

Using (4), (5), and (10), this can also be written as

(14) $$y_j - y_j^* = (y_j^* - \bar{y}_j)s_j > 0$$

This requirement determines the sign of s_j for each type j. For members of the coalition belonging to each type j we must have

(15) $$s_j \gtrless 0 \quad \text{according as} \quad y_j^* \gtrless \bar{y}_j$$

This condition means that members of the coalition who have a larger imputed wealth at the Pareto optimal point considered than at the initial point should be moved further away from $\bar{x}$ through x^*, whereas members with higher imputed wealth in the initial situation than in the Pareto optimal point considered should be moved somewhat back from x^* towards $\bar{x}$.

I have not yet said anything about possible members for whom $y_j^* = \bar{y}_j$. This is a special case which will be disposed of later. For the moment it is assumed

(16) $$\bar{y}_j \neq y_j^* \text{ for } j = 1, \ldots, G$$

We have now considered feasibility and a criterion for positive gain by members of the coalition of the various types. The feasibility condition is dependent upon the number of members of the coalition belonging to each type, i.e., on $n_1, \ldots, n_G$. The crucial question now is whether it is possible to compose the coalition, that is, determine the numbers $n_1, \ldots, n_G$, in such a way that the feasibility condition (12) is fulfilled, while at the same time the condition (15) for a gain by all members in comparison with x^* is fulfilled.

For studying this it is convenient to introduce the proportions ν_j which members of each type form in the total coalition, i.e.,

$$(17) \quad \nu_j = \frac{n_j}{n_1 + \ldots + n_G} = \frac{n_j}{n} \quad (j = 1, \ldots, G)$$

In terms of these proportions the feasibility requirement (12) can be written as

$$(18) \quad \nu_1(1 + s_1)z_{i1} + \ldots + \nu_G(1 + s_G)z_{iG} = 0 \quad (i = 1, \ldots, M)$$

Observe that this condition is fulfilled for

$$(19) \quad \nu_1 = N_1/N, \ldots, \nu_G = N_G/N \qquad s_1 = \ldots = s_G = 0$$

where $N_1, \ldots, N_G$ are the total number of individuals of each type, and N is the total number of individuals, i.e., $N = N_1 + \ldots + N_G$. This follows from the fact that the Pareto optimal point x^* must be feasible for the exchange economy as a whole, that is, we must have

$$(20) \quad N_1 x_{i1}^* + \ldots + N_G x_{iG}^* = N_1 \bar{x}_{i1} + \ldots + N_G \bar{x}_{iG} \quad (i = 1, \ldots, M)$$

which is the feasibility condition (1) applied to the Pareto optimal point considered.

The statement just made simply means that a coalition with a composition proportional to the composition in the complete set of individuals can reach the Pareto optimal point under consideration on the basis of its own initial amounts. In order to construct a coalition which blocks the Pareto optimal point considered we shall try to find an allocation in the neighborhood of x^* which all members of the coalition prefer to x^*. We must then alter the composition of the coalition somewhat, but shall keep it *approximately* similar to the composition given by the first line of (19).

Now, in order that all members of the coalition gain by a move away from x^*, we must make $s_1, \ldots, s_G$ different from zero according to the sign pattern determined by (15). In order not to do violence to the local nature of the criterion that we use, we let $s_1, \ldots, s_G$ deviate only a little from zero. Let us for the moment treat $\nu_1, \ldots, \nu_G$ as free variables in the neighborhood of the values given by (19), restricted only by $\Sigma \nu_j = 1$. (This is a crucial point to which I shall return.) Then, for any given set of values for $s_1, \ldots, s_G$, some positive and some negative according to (15), we can clearly satisfy all equations in (18) by simply setting

$$(21) \quad \nu_1 = \frac{\alpha}{1 + s_1}\frac{N_1}{N}, \ldots, \nu_G = \frac{\alpha}{1 + s_G}\frac{N_G}{N}$$

since (18) by this insertion reduces to (20), which is known to be fulfilled. Here α is a parameter which is adjusted to that $\Sigma \nu_j = 1$.

By the procedure outlined above we have succeeded in constructing a coalition together with a feasible final point for the coalition which is superior to the Pareto optimal point x^* for all members of the coalition. By the definition of the core, we can accordingly conclude that x^* does not belong to the core. The argument is, however, not yet quite complete because of a couple of points which were temporarily put off in the development of the idea given above. Let us now return to these points.

VI. Some Special Points Needed to Complete the Argument

Let us first consider the assumption made by (16). If the Pareto optimal point considered should be such that for some type, $\bar{y}_j = y_j^*$, then individuals of this type cannot gain anything by being moved away from x^* in any direction according to the construction used above. However, it may be necessary to include a suitable number of individuals of this type in the coalition in order to give it the desired composition. For these members we set $s_j = 0$. We will then have $\nu_j = \alpha N_j/N$ according to (21). We now need these as members of the coalition, but they do not gain anything by it as compared with the Pareto optimal point x^*. However, since other members of the coalition, for whom $\bar{y}_j \neq y_j^*$, make a strictly positive gain, then a slight transfer so as to make these special members gain also could always be carried out if this is necessary for involving them in the coalition.

As already mentioned before the case in which $\bar{y}_j = y_j^*$ for *all* $j = 1, \ldots, G$ is the case in which x^* is the special Pareto optimal point representing the competitive equilibrium, or one of these if the competitive equilibrium is not unique. In this case the procedure outlined above will not succeed in constructing another feasible final point for a coalition which is superior to x^* for all members of the coalition. This is as it should be. It is well known that the competitive equilibrium belongs to the core so that no coalition can be constructed which can block such a point. (The fact that the competitive equilibrium belongs to the core is proved by elementary methods in many expositions and will not be taken up for further consideration here.)

In connection with the comparison between y_j and y_j^*, the following point may be observed. Consider equation (14). The difference $y_j^* - \bar{y}_j$ here decides the sign of s_j. If we multiply these differences by the number of individuals of each type and add over types we get

$$(22) \quad \sum_{j=1}^{G} N_j(y_j^* - \bar{y}_j) = \sum_{i=1}^{M} \lambda_i \Big(\sum_{j=1}^{G} N_j x_{ij}^* - \sum_{j=1}^{G} N_j \bar{x}_{ij}\Big) = 0$$

The last equality here follows from (1) which must hold for x^*. Since all $N_j > 0$ it follows from this that when some $y_j^* > \bar{y}_j$, then there must be at least one j for which the opposite inequality holds, and vice versa. Thus, if not all $y_j^* = \bar{y}_j$, then there will be at least one j for which $s_j > 0$, and at least one j for which $s_j < 0$.

The second point which must be taken up refers to the assumption temporarily made that we could consider $\nu_1, \ldots, \nu_G$, i.e., the proportions of the representation of each type in the coalition considered, as free variables (restricted only by nonnegativity and $\Sigma\nu_j = 1$). When N is a finite integer and also $N_1, \ldots, N_G$ are integers, then we are in fact not entirely free in determining $\nu_1, \ldots, \nu_G$. These variables are defined by (17), and $n_1, \ldots, n_G$ must also be integers and restricted by $0 \leqq n_j \leqq N_j$. Suppose that we have tentatively determined $s_1, \ldots, s_G$ with correct signs and sufficiently small so as not to invalidate the application of our local criterion for gains. Then there may be no integers satisfying $0 \leqq n_j \leqq N_j$ which used in (17) produce the required $\nu_1, \ldots, \nu_G$ according to (21). The natural idea then is to make some small adjustments in $s_1, \ldots, s_G$ (without altering their signs) so as to make (21) hold good with values of $\nu_1, \ldots, \nu_G$ which can be produced by (17) with permissible integers for $n_1, \ldots, n_G$.

Now, this may be impossible if $N_1, \ldots, N_G$ are small integers. However, if $N_1, \ldots, N_G$ are large, then we are much more free in choosing $n_1, \ldots, n_G$, and it is easier to produce proportions $\nu_1, \ldots, \nu_G$ which satisfy the requirements needed for some sufficiently small $s_1, \ldots, s_G$ with correct signs. (According to what was said above in connection with (22), some of the types will be "overrepresented" in the coalition in the sense that $\nu_j > N_j/N$, and some types will be "underrepresented" in the sense that $\nu_j < N_j/N$. The factor α used to secure $\Sigma\nu_j = 1$ will be near to unity when $s_1, \ldots, s_G$ are small.)

If we increase $N_1, \ldots, N_G$ beyond all limits, then we approach a situation in which the restriction that $n_1, \ldots, n_G$ have to be integers is no longer an effective restriction on the possibilities for choosing $\nu_1, \ldots, \nu_G$. Then the construction of the coalitions as given above can be carried out for any Pareto optimal point x^* which is not a competitive equilibrium, that is, for any x^* for which at least one type (and then necessarily at least two) have $y_j^* \neq \bar{y}_j$. This shows that when the number of individuals of all types increases beyond all limits, then only competitive equilibrium solutions remain in the core. (In order to make the comparison between smaller and larger economies meaningful, it is easiest to think of the larger economy as one in which the number of individuals of each type has been blown up proportionately. Then we may speak about "the same point" x^* in the smaller and the larger economy, the only difference being in the absolute numbers of individuals enjoying the various commodity bundles.)

VII. A Final Remark

The construction presented above can also be used to say something more intuitive about the size of the core when the number of individuals is finite, and in general other (Pareto optimal) points besides the competitive equilibrium belong to the core. For instance, if the indifference surfaces corresponding to the utility functions of individuals of the various types are very strongly curved, then there will be less freedom in the choice of $s_1, \ldots, s_G$, while smaller curvature makes for wider ranges of permissible choices of $s_1, \ldots, s_G$. The less free we are in choosing $s_1, \ldots, s_G$, the more difficult will it be to find permissible $\nu_1, \ldots, \nu_G$ when we have a limited number of individuals of each type to select $n_1, \ldots, n_G$ from. Thus, for an economy with a given number of individuals, the blocking procedure used here seems to be more powerful in excluding points from the core when there is a moderate curvature than when there is strong curvature in the indifference surfaces in the neighborhood of the point tested. By similar reasoning one may also get the impression that it will normally be easier to exclude points which are far away from the competitive equilibrium than points in its neighborhood. However, these suggestions are only hints about directions in which the arguments can be developed. A complete analysis of the question as to which points belong to the core and which ones do not, for a given number of individuals of each type, is a much more difficult task than the one tackled above. In order to show that a point does *not* belong to the core, it is sufficient to construct *one* particular coalition which is able to block the point in *one* particular way as we have done above. In order to show that a point belongs to the core, one must show that all possible coalitions with all their feasible reallocations fail to produce a point which is superior to the point considered for all members. Except for competitive equilibrium points, this is usually a complicated matter.

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