SOVIET MATHEMATICAL ECONOMICS[1]

It is by now well known that mathematical methods have in recent years gained wide acceptance as an indispensable tool for economic analysis and planning in the Soviet Union. A decisive step forward in this direction was taken in 1959 with the publication of a big volume on mathematical methods in economics, edited by Academician V. S. Nemchinov. The volume under review contains translations into English of the main articles in this collection.

In order to prepare the ground for later advances in the use of mathematical methods, the volume should clearly serve three purposes:

(1) it should demonstrate the usefulness of mathematical methods, partly by general arguments and partly by presenting applications in special fields;

(2) it should prove the compatibility of mathematical methods in economics with basic Marxian theory;

(3) it should link up recent developments with earlier Soviet contributions to economic analysis.

This job was done mainly by the older generation of Soviet mathematical economists, headed by V. S. Nemchinov, V. V. Novozhilov and L. V. Kantorovich. In the very voluminous literature on mathematical economics which has followed since this first pioneering publication the emphasis has been more on special applications of mathematical methods, and the younger generation has been dominating to a far greater extent.

The first article in the volume is V. S. Nemchinov: " The Use of Mathematical Methods in Economics." This paper is very explicitly dedicated to the three purposes mentioned above, and the author brings home all the three points very forcefully. Nemchinov illustrates his points mainly by means of references to " chessboard balances " and input–output analysis, linear programming, and analysis of " expanded reproduction " in the Marxian sense.

The next paper in the volume is V. V. Novozhilov: " Cost–Benefit Comparisons in a Socialist Economy." This contribution makes up more than one-third of the volume. It is based on ideas which were partly published in the Soviet Union as early as in 1941 and 1946. In some sections it exploits the early linear-programming theory of L. V. Kantorovich (see below). The English title of Novozhilov's paper seems to me not quite appropriate, since there is not much in Novozhilov's paper about " benefits "

[1] Review of *The Use of Mathematics in Economics.* Edited by V. S. Nemchinov; English edition edited by A. Nove. (Edinburgh: Oliver and Boyd, 1964. Pp. xvii + 377. 105*s.*)

in the sense of Western cost–benefit analysis. The main problem of the paper is how to measure costs of production, for the purpose of taking rational decisions with respect to methods of production and selection of investment projects. Through most of the paper the output programme is taken as given from outside.

Some of the problems which are treated in the paper would be absent with a more rational price system. To this extent the arguments can easily be, and are in fact, turned into arguments about the system of prices in a planned economy. Novozhilov asks the question whether there must necessarily be "differences between enterprise profitability and national economic advantage," and argues that:

> "There is little doubt that it should be possible to construct prices and economic accounting indicators in such a way that economic accounting would become a reliable planning tool. We can imagine the basic lines of accounting and pricing methods to be so formulated that the production of the planned range of goods would be more profitable for an enterprise than violation of the plan, that savings in the accounting prime costs at enterprise level would be a reflection of savings in real cost from the standpoint of the economy as a whole, and that enterprises would have an interest in technical progress, in the best use of its productive capital" (p. 44).

In the first sections of his paper Novozhilov goes through several examples illustrating how to find the most efficient uses of investments in terms of saving of labour costs. Gradually it becomes clear that "the problem of finding the maximum effect of investments of the whole economy cannot be solved in isolation from the more general problem of finding the maximum effect of all limited means of production." He then sets himself the task of finding "standards of effectiveness" for all the limited means of production, these "standards" having the following meaning, which is easy to recognise: "From a mathematical point of view, the standards of effectiveness are auxiliary multipliers, which we can use to find the conditional extremum just as if the constraints were removed, as if we were finding the unconditional extremum" (p. 134). He poses the problem mathematically in the Lagrangian fashion and reaches a conclusion which is equivalent to a well-known statement by Paul Samuelson in his *Foundations*: that the Lagrangian multipliers give us "the standards of effectiveness" which are required. Furthermore, Novozhilov goes on to formulate the side conditions concerning the use of limited means of production as inequalities. The problem then appears as a programming problem. Thus, he ends up with "standards of effectiveness" which are equivalent to the shadow prices of linear programming.

Novozhilov starts his exposition with the statement that "all costs are labour costs exclusively." One might think, therefore, that he would not be able to treat effectively problems in which *time* enters in an essential way—

for instance, comparisons between investment alternatives with different timing of the outlays and savings. However, Novozhilov opens the possibilities for a rational treatment of problems involving time by observing that outlays at different points of time are not directly commensurable. In comparing current outlays with outlays on investment, the " standard effectiveness of investment " plays the same role as an interest rate, and Novozhilov even makes allowances for future changes in this standard effectiveness when selecting project variants. He also poses the problem of optimal accumulation, which he considers to be one of the most important problems in a socialist economy, but nevertheless a question which is " far from adequately understood in our economic literature." Novozhilov starts his analysis of this problem with the statement: " The most important and difficult aspect is the explanation of the conditions in which the interests of accumulation and consumption will, in general, coincide. This coincidence of interests is possible only in terms of long-term development, as they are opposed to one another during each separate short period of time" (p. 163). This is obviously a correct statement, and a good starting-point. However, the further analysis of this problem is somewhat disappointing, and does not result in any very clear conclusions, perhaps because it is not clear what is meant by his assertion that " the optimal relation between accumulation and consumption is that which secures the maximum continuous growth of labour productivity."

It would perhaps be tempting now to say that all the main conclusions of Novozhilov's work could be obtained more easily by going directly to the full programming formulation. This is not necessarily true. For the readers for whom Novozhilov's work is primarily written, I think that his exposition will give a more profound understanding of the role of efficiency prices than such a direct attack on the problem. Furthermore, as he goes along, Novozhilov has many interesting observations to offer also on problems which are not easily cast in terms of linear programming, and also on such broader questions as comparisons between Capitalism and Socialism and the transition into Communism. Even if one knows everything about linear programming and shadow prices, Novozhilov's paper is therefore still worth reading, and not only to learn about Soviet thinking in economics.

The next paper is Oskar Lange: " Some Observations on Input–Output Analysis." Starting with the Marxian schemes of simple and expanded reproduction, this paper gives an excellent treatment of physical and value aspects of input–output analysis, both static and dynamic. The contents of the paper will partly be known to Western readers from other writings of Oskar Lange. The most interesting thing is perhaps to see the simple and elegant way in which he deduces macro-relationships known from economic growth theory by aggregation in a dynamic input–output model. Oskar Lange is the only non-Soviet contributor to the volume.

Next follow two papers by L. V. Kantorovich, the first one reproducing with minor changes the now famous 1939 paper on " Mathematical Methods of Production, Planning and Organisation." It is now generally recognised that this paper establishes Kantorovich's priority in the field of linear programming. Nevertheless, it may not be out of place to indicate exactly what Kantorovich has done in this paper.

1. First of all he has described concretely and formulated mathematically several problems in the form of maximising or minimising a linear function in several variables subject to side conditions in the form of linear equations and inequalities. The first and most simple problem which he has treated is the following: Let there be n available machine tools to be used for production of articles, each of which consists of m components. Generally all the m components may be different. When machine tool i is used for producing component k it produces a_{ik} units per day. The problem is to distribute the production of the different components to the available machine tools so as to produce the maximum number of complete articles. Let h_{ik} be the fraction of the working day for which machine tool i is used to produce component k. These are then the unknown variables of the problem.

The number of component k produced per day amounts to $\sum_{i=1}^{n} a_{ik}h_{ik}$. Since each complete article consists of one of each of the m components, we require these sums to be equal for all $k = 1,2, \ldots, m$. Furthermore, we should maximise the common value of these sums. The problem can then be put in the following way, when z indicates the number of complete articles produced:

Maximise z subject to

(1) $$h_{ik} \geq 0 \quad (i = 1, \ldots n;\ k = 1, \ldots m)$$

(2) $$\sum_{k=1}^{m} h_{ik} = 1 \quad (i = 1, \ldots n)$$

(3) $$z = \sum_{i=1}^{n} \alpha_{ik}h_{ik} \quad (k = 1, \ldots m)$$

The problem (1)–(2)–(3), which is here slightly reformulated from Kantorovich so as to facilitate the comparison with the dual problem below, is obviously a rather special linear-programming problem. It is, however, generalised in several ways later in the same paper.

2. For such problems Kantorovich has proved the existence of " resolving multipliers " which are equivalent to what we usually call shadow prices, and gave the rules of correspondence between the structure of the system of shadow prices and the solution to the physical maximum problem given above.

Applying the theory of linear programming as presented, *e.g.*, in G. Dantzig: *Linear Programming and Extensions*, the dual of the problem (1)–(2)–(3) can be established as follows: Associate with each of the conditions in

(2) a variable t_i, and with each of the conditions in (3) a variable λ_k. The dual problem then appears as:[1]

Minimise $\sum_{i=1}^{n} t_i$ subject to

(4) $$\sum_{k=1}^{m} \lambda_k \geq 1$$

(5) $$\alpha_{ik}\lambda_k \leq t_i \quad (i = 1, \ldots n; \; k = 1, \ldots m)$$

It is clear from this that we must have

(6) $$t_i = \text{Max}[\alpha_{i1}\lambda_1, \alpha_{i2}\lambda_2, \ldots \alpha_{im}\lambda_m] \quad (i = 1, \ldots n)$$

Furthermore, the rules of correspondence between the dual and the primal problems imply that $h_{ik} = 0$ for those i,k which are such that (5) holds as a strict inequality. Comparing all this with Kantorovich's exposition (*e.g.*, on p. 251), we can easily verify that his " resolving multipliers " are equivalent to the variables of the dual problem as given above, and furthermore, that his explanation of how to find the solution of the problem (1)–(2)–(3) once the resolving multipliers are known, are equivalent to the rules of correspondence between the dual and the primal problem of linear programming. The same holds for the more general problems treated by Kantorovich, in which other types of conditions are present, and in which the coefficient matrix contains a relatively smaller number of zeros.

3. The observations given above mean that Kantorovich has given the true optimality criteria for his programming problems.

4. In addition to using the " resolving multipliers " as means of identifying the optimum solution, Kantorovich has interpreted them as shadow prices. In his words:

> " So far we have considered the resolving multipliers merely as a technical means for solving the problems A, B and C, and nothing more. It may appear, therefore, that this method of solving the problems A, B, C, offers no advantage as compared with other possible methods, except possibly, that it is simple and quick. This, however, is not so; resolving multipliers have a much wider significance. Not only do they produce the result of a problem, but they also provide a series of important characteristics of this result. Thus, a solution found by the method of resolving multipliers is much more valuable than a mere statement of numerical values h_{ik}. . . . Thus, if λ_k are known, questions connected with minor variations of the programme may be answered " (pp. 265–6).

5. As a computational method Kantorovich has proposed an interation procedure which works by successive adjustments of the " resolving multipliers," calculating for each adjustment the corresponding values of the variables of the primal problem and testing whether they satisfy the conditions. The deviations indicate the direction of the next adjustment of the " resolving multipliers," and so on until the true optimum is found (exactly

[1] When (2) and (3) are stated as equalities, the variables of the dual problem are unrestricted in sign. If the equality signs in (2) and (3) had been replaced by $\leq$ the variables of the dual problem should be restricted in sign.

or approximately). It is interesting to observe that while the first " Western " method—the simplex method—works directly on the primal problem, the first Soviet method worked more indirectly by successive adjustment of " multipliers " which could be interpreted as prices. Kantorovich is himself fully aware of the formal similarity between his method and a market mechanism; in his other paper in the volume he offers the following comment in connection with the prospects of applying his method to broader planning problems than the intra-firm problems treated in his first paper:

> " This method . . . assumes some approximate rating values, from which the most promising processes are defined. From the available resources we then attempt to develop a plan to fulfil the given task to the greatest extent. Thereby an excess or shortage of a certain product calls for a reduction or increase of its rating, respectively. The productive factors are treated similarly. In this manner we gradually approach the best plan. In a sense this process resembles the process of changing market prices with changes of supply and demand ratio (overproduction leads to price reduction, etc.). In our case, however, the competitive fight between different processes takes place merely within the framework of planning, without losses and without crises. Naturally, this is possible on a large scale only under the conditions of a planned socialist economy " (p. 293).

The method of calculation first developed by Kantorovich does not seem very efficient and easy to turn into a routine, since it requires quite a lot of inspection for each successive adjustment.

From the points given above it appears perfectly natural that the first Soviet method of linear programming theory should also give rise to ideas about decentralisation of economic decisions. There is not much about this in Kantorovich's first paper, which mainly treats intra-firm problems. But the decentralisation aspects are explicit in Novozhilov's paper, which, as already mentioned, uses Kantorovich's mathematical theory. The crucial point is the one already quoted above about the " standards of effectiveness " as multipliers that can be used " to find the conditional extremum just as if the constraints were removed." He furthermore uses the very apt expression that the standards " make the individual minima of outlays consistent." When discussing centralisation and decentralisation Novozhilov speaks about two forms of centralisation—direct and indirect:

> " The centralised management of the economy can be realised in two basic forms: direct and indirect. For example, the requirement in a scarce metal can be regulated either by limiting its outlay or by fixing a higher price.
>
> " Indirect centralisation consists in fixing standards for the calculation of outlay and results, with the help of which the ' provinces ' can themselves find the best variants of use of their efforts and means—the best from the point of view of the whole economy, corresponding to its optimal development plan.
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> " Planning includes both forms of centralisation, the experience of

socialist construction having shown both to be necessary. Direct centralisation is a basic form of planning; it is essential also to indirect centralisation, for a scientifically based system of standards for the calculation of outlay and results can be developed only from the plan as a whole. Thus, with indirect centralisation each question (even the smallest) is answered jointly by the centre and the ' provinces '. The centre works out the general standards for its solution, the ' province ' applies these standards to each particular case.

" Only by the combination of these two forms of centralisation can there be the greatest development of the planning principle and the widest democracy in economic construction " (p. 149).

The term " indirect centralisation " used in this way may be a convenient term in order to distinguish the system from a system in which also price-setting is decentralised.

The implications of programming theory for the question of decentralising economic decisions are also explicit in Kantorovich's second paper in the volume: " Further Development of Mathematical Methods and the Prospects of their Application in Economic Planning." In a way parallel to Novozhilov's he here argues that " apart from optimum planning, linear programming ought to be used to determine economic indicators, especially with regard to pricing." Kantorovich especially notes the following properties of the " resolving multipliers " or " ratings " which he also calls them, which are useful when they are to be used as prices:

" First, the ratings are *concrete* and *dynamic*, that is, they depend on the circumstances (definitions of the problem), and accordingly they change with any change of technological processes, resources, and assortment . . .

" Except in special cases, the extreme point will be inside one of the edges of the polyhedron [representing the set of feasible plans. L.J.], and not at its boundary. Consequently, a small change of the assortment (in the basic problem, of the resources) will result in a small displacement of the assortment line; the boundary (and with it the supporting plane) will remain as before, and the resolving multipliers will not change. Other small changes (those of processes) will affect the values of multipliers a little. Thus, resolving multipliers (ratings) possess, generally speaking, a certain *stability* despite changes in the targets " (p. 298).

And furthermore:

" Thanks to the relative stability of the ratings it is possible to neglect changes due to corrections of the plan, and to use previously established values. . . . Of course this statement refers to relatively small modifications which do not fundamentally change the situation (and ratings). Such is usually the case when questions of economics and planning have to be decided, when new jobs have to be incorporated in the plan, or when some tasks have to be replaced by others " (p. 300).

This second paper by Kantorovich presents results of research in the years after 1939, and generalises the theory presented in his first paper in several directions. His most general formulation is equivalent to a full linear activity analysis model, with an arbitrary number of technological processes, each of them involving an arbitrary number of inputs and outputs. His formulation is, however, in some respects slightly different from Western text-book formulations, and may for some applications prove more convenient.

Kantorovich also considers problems covering several time periods. For such cases he points out how the " ratings " of products and factors can be defined separately for each period, and comments that

> " the ratings on one hand account for dynamic movement in time and on the other hand reduce all expenditure to one single moment of time (in a sense, a certain future expenditure is replaced by an equivalent smaller expenditure at the present moment) " (p. 315).

This shows how a rate of interest is implied by the optimal plan in the dynamic case.

Finally, Kantorovich offers a brief discussion on how his models could be applied in planning for the full national economy. His idea is close to a Tinbergenian " planning in stages ":

> "Evidently a realistic approach would be to construct a whole system of models. A model of the national economy should be made according to the most generalised main indicators for a relatively small number of products and factors. Alongside this main model auxiliary models should be developed of industrial branches and of locally grouped enterprises; the operative plan should be analysed and every possibility of variation studied. Probably the process of plan construction should comprise a series of stages of gradually increasing accuracy and co-ordination of original data, plans, budgets and indicators, with subsequent corrections as the plan is being put into effect " (pp. 320–1).

Mathematically, Kantorovich is, of course, very convincing. I therefore hesitate to say that I do not find his proof of Theorem 1 on p. 289 entirely satisfactory. The deficiency is probably only a slip, and there is no difficulty in remedying the proof.[1]

While all papers discussed above are of a very general interest, the last paper by A. L. Lur'e, is on a more special problem: " Methods of Establishing the Shortest Running Distances for Freights on Setting up Transportation Systems." The presentation of several graphical methods seems to me to require more explanation than is given by Lur'e. However, the analytical methods are very well presented. Although they are equivalent to methods which are now well known in Western countries, Lur'e's presentation is different and seems to offer some advantages over common Western text-book expositions. I cannot judge the originality of this special paper, but it is evident that original work on transportation programming has been done

[1] The point is this: In stating the problem on p. 284 Kantorovich specifies as one of the conditions that " expenditure of productive factors must not exceed the given values (available resources)," while the objective function does not involve the use of factors. In the proof on p. 289 it is assumed that if one plan is better than another, then the first plan does not use more of resources than the second plan. This is not justified.

at an early stage in the Soviet Union, mainly, perhaps, by Kantorovich, but also by others.

In addition to the papers discussed above, the volume contains a bibliography of linear programming and related problems, by A. A. Korbut, and a preface and a postscript by Nemchinov. There is a useful introduction to the English edition, written by A. Nove, explaining the general background for the volume, and aiding in understanding special terms and concepts used by the Soviet writers.

On all the subjects treated in the volume further advances have been made since 1959; the volume under review nevertheless carries exceptional importance because it more than any other work signifies the decisive—and irreversible—break-through for mathematical methods in Soviet economics.

Later mathematical–economic literature in the Soviet Union also covers subjects which are barely touched upon in the present volume, such as demand theory. It might be mentioned that Slutsky's fundamental paper in this field, which was first published in Italian in 1915, was translated into Russian and published in Moscow in 1963, with explanatory comments by Volkonsky and Konüs. This Konüs is identical with the author of the very important contribution on the theory of cost-of-living index numbers which was published in translation in *Econometrica* in 1939. In this field there is thus already some activity, and connections with the past are well established. Empirical research is also going on, and there are reasons to expect interesting contributions.

One field is still lacking, and that is welfare theory (beyond technical efficiency). My feeling is that one might also in the future expect this field to be cultivated and fruitfully applied, although on a somewhat different—*i.e.*, more social, less individualistic—basis than is conventional in welfare theory as known by now. Since the income distribution should be more directly socially determined and involve less antagonism than in a capitalist society, it may be argued that the chances for welfare theory to be fruitfully applied are better in a socialist than in a capitalist economy. Particularly, I think this could help very much to clarify the discussion on principles of pricing in the Soviet Union—in fact, I think it is the only way of settling the question.

For completeness I must end this review with a minor comment: there are too many misprints, such as wrong or misplaced subscripts, inequality signs in the wrong direction, etc., in the volume. There are also some examples of confusing expressions, particularly in connection with the mathematical parts of the texts, and some concepts are not used consistently throughout the volume. I have not checked how much of this is confined to the English edition.

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