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RULES OF THUMB FOR THE EXPANSION OF INDUSTRIES IN A PROCESS OF ECONOMIC GROWTH

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Consider a group of consumption goods industries in a process of economic growth. It is often assumed that the production of each sector will expand proportionately to the income elasticity of demand for its products. This simple rule may need some modification if capital can substitute for labour to different degrees in different sectors, and total capital stock grows at a rate which is different from the rate of growth of total labour input. Different rates of technical progress may also give rise to a need for modifying the above mentioned simple rule. Approximate formulas for these modifications are worked out in such a way that they can be applied for quantitative evaluations. Some rough numerical illustrations are offered in the concluding section of the paper.

1. INTRODUCTION

IN ECONOMIC long term planning or forecasting it is often assumed that sectors producing consumption goods should expand proportionately to the income elasticities of demand for their products. The purpose of this paper is to qualify this simple rule of thumb, taking into account the following facts: (a) If the total labour force and the total capital stock available for the consumption goods sectors grow at different rates, and if capital can be substituted for labour to different degrees in different sectors, then we should expect this to have some effect upon the pattern of expansion of consumption; and (b) if technical progress is not uniform for all sectors we should also expect this to influence the composition of consumption during the growth process.

Assuming a free market for consumption goods, the above effects must work through the influence of prices upon the direction of consumers' demand.

In our analysis we shall accept the following simplifying assumptions: (1) we consider a closed economy; (2) we assume the sectors to be vertically integrated; (3) we assume constant returns to scale in all sectors; (4) we assume independent utilities in each consumer's preference scale; (5) we assume technical progress in each sector to be neutral.

Under these assumptions we shall work out rather simple formulas for the above mentioned effects upon the pattern of expansion of industries and make an attempt at appraising their importance. This will also indicate

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how good an approximation we may obtain by applying the simple rule that sectors should expand proportionately to the income elasticities of demand for their products.

The basic model is presented in Section 2. The formulas for the effects mentioned above are worked out and interpreted in Sections 3 and 4. Some numerical illustrations are given in Section 5.

2. THE BASIC MODEL

We consider m sectors, $i = 1, \ldots, m$, producing consumption goods. For each sector we assume a production function of the Cobb-Douglas type with a neutral shift over time representing technical progress, i.e.,

$$X_i = A_i N_i^{\gamma_i} K_i^{\beta_i} e^{\varepsilon_i t} \tag{2.1}$$

where X_i is total production of good i, N_i is labour input, K_i is capital stock in sector i, t is time, and A_i, γ_i, β_i, and ε_i are constants, the last representing the rate of technical progress. We shall assume $\gamma_i + \beta_i = 1$ for all i.

We write the demand functions as

$$X_i = g_i\,(P_1, \ldots, P_m, Y) \tag{2.2}$$

where P_i is the price of good i and $Y = \Sigma P_i X_i$ is total consumption expenditure.[2] For convenience we shall assume prices to equal unity at a base point of time.

Let W be the wage rate and Q be the rate of return to (real) capital.[3] Assuming profit maximization we then obtain the familiar constancy of the income shares

$$WN_i = \gamma_i P_i X_i, \tag{2.3}$$

$$QK_i = \beta_i P_i X_i, \tag{2.4}$$

where, since $\gamma_i + \beta_i = 1$, $WN_i + QK_i = P_i X_i$.

We assume W to be constant, whereas Q may vary. The assumption that W is fixed is only a device for fixing the level of prices in the model and has no substantive content.

Finally we have

$$\Sigma N_i = N, \tag{2.5}$$

$$\Sigma K_i = K, \tag{2.6}$$

[2] Alternatively we might have introduced the total population as a factor before g_i, and let g_i be the average demand functions, assuming an exogenously given growth rate for the population. It would not be difficult to modify the following results accordingly. In fact, we may interpret X_i, N_i, K_i, and N and K (which are introduced below) as per capita concepts. In that case N would only change as a result of changed hours of work, changed age distribution, etc.

[3] If r is the rate of interest (including a risk premium), δ is the rate of depreciation and P_k the price of capital, then $Q = (r + \delta)P_k$.

where N and K are total labour force and total capital stock in the sectors under consideration.

Considering the growth of N and K as exogenously given, the above system will determine the growth of the X_i's. Our problem is to investigate to what extent the expansion of the X_i's fails to be proportional to the income elasticities of demand, or more correctly, the elasticities of demand with respect to total consumption expenditure.

For the following analysis it is convenient to introduce some matrix and vector notations.

Let x_i be the relative growth rate for X_i, i.e.,

$$x_i = \frac{1}{X_i} \frac{dX_i}{dt},$$

and similarly for n_i, k_i, p_i, q, and y. Then x denotes the column vector of $(x_1, \ldots, x_m)$ and similarly for n, k, and p. For N and K we introduce the special notation:

$$\bar{n} = \frac{1}{N} \frac{dN}{dt}, \quad \bar{k} = \frac{1}{K} \frac{dK}{dt}.$$

We introduce e_{ij} and E_i for the demand elasticities

$$e_{ij} = \frac{\partial g_i}{\partial P_j} \frac{P_j}{g_i}, \quad E_i = \frac{\partial g_i}{\partial Y} \frac{Y}{g_i},$$

and let e denote the matrix of the e_{ij}'s and E the column vector of the E_i's. For short we shall use the term "income elasticities" for the E_i's, although the correct expression would be "elasticities of demand with respect to total consumption expenditure."

γ, β and ε represent the column vectors of the γ_i's, the β_i's and the ε_i's, respectively.

ν and $\varkappa$ signify the vectors

$$\nu = \left(\frac{N_1}{N}, \ldots, \frac{N_m}{N}\right)', \quad \varkappa = \left(\frac{K_1}{K}, \ldots, \frac{K_m}{K}\right)'. \tag{2.7}$$

The symbol i signifies a column vector with m elements, each equal to unity. The symbol ^ is used to signify a diagonal matrix, i.e., $\hat{\gamma}$ is the diagonal matrix formed by $\gamma_1, \ldots, \gamma_m$, etc.

With the notations introduced above we can now derive from the model (2.1—7) the following equations involving the relative growth rates of the variables:

$$x = \hat{\gamma} n + \hat{\beta} k + \varepsilon, \tag{2.8}$$

$$x = ep + Ey, \tag{2.9}$$

$$n = p + x, \tag{2.10}$$

(2.11) $$n = iq + k,$$

(2.12) $$\nu' n = \bar{n},$$

(2.13) $$\varkappa' k = \bar{k}.$$

Under the assumptions of independent utilities the demand elasticities with respect to prices are related to the elasticities with respect to total expenditure by

(2.14) $$e = - E\alpha' - \mu\hat{E} + \mu EE'\hat{\alpha}$$

where α is the column vector of the budget percentages

$$\alpha_i = \frac{P_i X_i}{Y} \qquad \left(= \frac{X_i}{\sum X_j} \text{ at the base point of time}\right)$$

and the positive parameter μ is given by

$$\mu = - \frac{1}{\text{flexibility of the marginal utility of money}}.$$

This follows from formulas given by Ragnar Frisch.[4,5]

The coefficients entering the above model satisfy the following conditions:

(2.15) $$\gamma + \beta = i,$$

(2.16) $$\alpha' E = 1,$$

(2.17) $$ei = - E,$$

(2.18) $$\alpha' i = 1, \nu' i = 1, \varkappa' i = 1.$$

3. THE EFFECTS OF A DISPROPORTIONATE GROWTH OF TOTAL LABOUR FORCE AND TOTAL CAPITAL STOCK

We shall first work out the effects on the sectoral pattern of expansion of a disproportionate growth of the total labour force and the total capital stock, i.e., of a discrepancy between $\bar{n}$ and $\bar{k}$. A priori, it seems clear that the case $\bar{k} - \bar{n} > 0$ will favour the growth of sectors with high β_i's, and vice versa. Our intention is to work out formulas which may be used for a quantitative appraisal of the importance of this effect. We shall work out a first and a second approximation, the first being so simple that it still deserves to be labelled a rule-of-thumb, the second being not quite so simple.

In order to avoid repetition in the next section, in which we shall consider the effects of non-uniform technical progress, we retain the ε_i's within the formulas up to a certain stage in this section also.

[4] Cf. Ragnar Frisch [**1**, Excurs 18, **2**, or **3**, pp. 94–108]. Added in proof: Cf. also Ragnar Frisch's paper in *Econometrica*, April, 1959.

[5] Although we do not actually need the inverse of e for the calculations presented in this paper, it may be useful for further explorations. I therefore give it here:

$$e^{-1} = -i\alpha' - 1/\mu\, \hat{E}^{-1} + 1/\mu\, i\alpha'\, \hat{E}^{-1}.$$

The easiest way of approaching the solution for x seems to be to look first for the solution for q and y. Eliminating x and p from (2.8—11) we obtain for n and k:

$$n = (I + e)\beta q + Ey - (I + e)\varepsilon, \tag{3.1}$$

$$k = [(I + e)\beta - i]q + Ey - (I + e)\varepsilon. \tag{3.2}$$

Inserting this in (2.12—13) we obtain the two following equations in the unknown scalars q and y:

$$\nu'(I + e)\beta q + \nu' Ey = \bar{n} + \nu'(I + e)\varepsilon, \tag{3.3}$$

$$[\varkappa'(I + e)\beta - 1]q + \varkappa' Ey = \bar{k} + \varkappa'(I + e)\varepsilon. \tag{3.4}$$

This system can be written as

$$d_1 q + (1 + d_2)y = \bar{n} + d_3, \tag{3.5}$$

$$(-1 + d_4)q + (1 + d_5)y = \bar{k} + d_6, \tag{3.6}$$

where the d's are defined by

$$\begin{aligned} d_1 &= \nu'(I + e)\beta, \quad d_2 = \nu' E - 1, \quad d_3 = \nu'(I + e)\varepsilon, \\ d_4 &= \varkappa'(I + e)\beta, \quad d_5 = \varkappa' E - 1, \quad d_6 = \varkappa'(I + e)\varepsilon. \end{aligned} \tag{3.7}$$

At this point we introduce the approximations. We also disregard the ε's, but with the intention of taking them up again in the next section.

The First Approximation

In this approximation we assume the d's to be negligible compared to unity. Let us justify this by taking a closer look at (3.7).

Using (2.14) we can write d_1 as

$$d_1 = \nu'\beta - \nu' E\alpha'\beta - \mu\nu'\hat{E}\beta + \mu\nu' EE'\hat{\alpha}\beta. \tag{3.8}$$

We now consider this formula term by term.

Because of the fact that $\nu' i = 1$, $\nu'\beta$ is an average of the β_i's.

The term $\nu' E\alpha'\beta$ is a product of an average of the E_i's and an average of the β_i's, the first with ν_i's and the second with α_i's as weights.

The term $\nu'\hat{E}\beta$ is an average of the elements $E_1\beta_1, E_2\beta_2, \ldots, E_m\beta_m$. Since the E's on the average are equal to 1 (cf. 2.16), we may expect $\nu'\hat{E}\beta$ to be approximately the same as an average of the β_i's.

Finally, the term $\nu' EE'\hat{\alpha}\beta$ is the product of an average of the E_i's with ν_i's as weights and an average of the β_i's with the $E_i\alpha_i$'s as weights (cf. 2.16).

Disregarding now the discrepancies between different averages of the E_i's and between different averages of the β_i's, it is seen that the terms in (3.8) offset each other. As a first approximation we therefore have $d_1 \approx 0$. It follows from similar considerations that we may as an approximation also assume d_2, d_4, and d_5 to vanish.

Since we are now disregarding the ε's, we are then left with the simple system

$$y = \bar{n}, \quad -q + y = \bar{k},$$

or

$$y = \bar{n}, \quad q = \bar{n} - \bar{k}. \tag{3.9}$$

The nature of this approximation can be illustrated by considering a one sector model corresponding to (2.1—6). In this case, (2.2) degenerates into $X = Y/P$, yielding $y = p + x$. By (2.10) we then see that $y = \bar{n}$. The result $q = \bar{n} - \bar{k}$ follows from (2.11). This means that (3.9) is the approximate solution we would have obtained for the growth of total consumption expenditure and the rate of return to capital if we had considered a pure macro model instead of the multisector model.

Seeking now the solution for x, we first insert from (3.1—2) for n and k in (2.8), obtaining

$$x = e\beta q + Ey. \tag{3.10}$$

Inserting for q and y from (3.9) and using (2.14—15) we obtain

$$\hat{E}^{-1}x = i(\bar{\gamma}\bar{n} + \bar{\beta}\bar{k}) + \mu(\beta - i\bar{\bar{\beta}})\,(\bar{k} - \bar{n}), \tag{3.11}$$

where we have introduced $\bar{\gamma}$ and $\bar{\beta}$ for the averages of the γ_i's and the β_i's with the budget percentages α_i as weights and $\bar{\bar{\beta}}$ for the average of the β_i's with the $\alpha_i E_i$'s as weights, i.e.,

$$\begin{aligned} \bar{\gamma} &= \alpha'\gamma, & \bar{\beta} &= (1 - \bar{\gamma}) = \alpha'\beta, \\ \bar{\bar{\gamma}} &= \alpha'\hat{E}\gamma, & \bar{\bar{\beta}} &= (1 - \bar{\bar{\gamma}}) = \alpha'\hat{E}\beta. \end{aligned} \tag{3.12}$$

For the interpretation of (3.11) it is convenient to spell out the formula in terms of scalars:

$$\frac{x_i}{E_i} = (\bar{\gamma}\bar{n} + \bar{\beta}\bar{k}) + \mu(\beta_i - \bar{\bar{\beta}})\,(\bar{k} - \bar{n}). \tag{3.13}$$

This allows a simple interpretation.

Consider first the term $(\bar{\gamma}\bar{n} + \bar{\beta}\bar{k})$. This is obviously a sort of average growth of production. If now *either* all goods were equally capital intensive (all β_i equal) *or* total labour force and total capital stock grew at the same rate ($\bar{k} = \bar{n}$), then the production of each sector would expand at a rate which would be proportional to the income elasticity of demand for its products, i.e., x_i/E_i would be the same for all sectors. This case corresponds to the most simple rule of thumb mentioned in the introduction.

If the capital intensity varies among the sectors and if total labour and capital grow at different rates, then this simple rule is invalidated by the second term in (3.13). From the interpretation of μ given in Section 2 we know that μ is positive. It is then evident that the second term in (3.13) favours the growth of the capital-intensive sectors (sectors for which

$\beta_i > \bar{\bar{\beta}}$) if total capital stock grows more rapidly than total labour force, and vice versa.

The importance of the second term depends essentially on the parameter μ, which means that it depends on the flexibility of the marginal utility of money. The smaller is the numerical value of this flexibility, the larger is μ, and the more important is the last term in (3.13). The reason why μ plays this crucial role is understood when one considers formula (2.14). In fact, a vanishing μ would rule out the substitution effect from the consumer's behaviour, and in our model the impact of the supply side upon the direction of the expansion of demand works by changing relative prices.

It is seen that formula (3.13) is quite simple and within reach of numerical evaluation.

The Second Approximation

In the first approximation we disregarded completely the d's in (3.5—6). In working out the second approximation we shall still treat them as quantities which are small when compared to unity, but we shall not disregard them. We therefore return now to (3.5—6) with the intention of working out a closer approximation to the solution than is given by (3.9).

Solving (3.5—6) for q and y and linearizing the expressions in the d's we obtain (disregarding d_3 and d_6 which relate to the ε's)

$$(3.14) \qquad \begin{aligned} y &= (1 - d_1 - d_2)\,\bar{n} + d_1\bar{k}, \\ q &= (1 - d_1 - d_2 + d_4 + d_5)\bar{n} - (1 - d_1 + d_4)\bar{k}. \end{aligned}$$

For the following derivations it is useful to introduce the covariance between the E_i's and the β_i's or γ_i's by

$$(3.15) \qquad \bar{m}_{E\gamma} = \sum a_i(E_i - 1)(\gamma_i - \bar{\gamma}),$$

and similarly for $\bar{m}_{E\beta}$. By means of (2.15) and the definitions in (3.12) it is easily seen that we have

$$(3.16) \qquad \bar{m}_{E\gamma} = \bar{\bar{\gamma}} - \bar{\gamma}, \quad \bar{m}_{E\beta} = \bar{\bar{\beta}} - \bar{\beta}, \quad \bar{m}_{E\gamma} + \bar{m}_{E\beta} = 0.$$

For the variance in the γ_i's or β_i's we introduce

$$(3.17) \qquad \begin{aligned} \bar{S} &= \sum a_i(\gamma_i - \bar{\gamma})^2 = \alpha'\hat{\gamma}\gamma - \bar{\gamma}^2, \\ \bar{\bar{S}} &= \sum a_i E_i(\gamma_i - \bar{\bar{\gamma}})^2 = \alpha'\hat{E}\hat{\gamma}\gamma - \bar{\bar{\gamma}}^2 \end{aligned}$$

where the bars distinguish the weight systems in the same way that they do in (3.12).

We also express ν and $\varkappa$ in terms of α, γ and β. At the base point of time, for which we carry out the differentiation, it follows from (2.3—7) that

$$(3.18) \qquad \nu = \bar{\gamma}^{-1}\hat{\alpha}\gamma, \qquad \varkappa = \bar{\beta}^{-1}\hat{\alpha}\beta.$$

Using (2.15) we obtain the following relation between ν and $\varkappa$:

$$\bar{\gamma}\nu + \bar{\beta}\varkappa = \alpha. \tag{3.19}$$

Thus equipped and using (2.14), (2.16), and (3.7) we can derive the following expressions for the d's involved in (3.14):

$$\begin{aligned} d_1 &= \bar{\gamma}^{-1}\{\bar{\beta}\bar{m}_{E\beta} - \bar{S} + \mu\bar{\bar{S}}\}, \quad d_2 = \bar{\gamma}^{-1}\bar{m}_{E\gamma}, \\ d_4 &= -\bar{\beta}^{-1}\{\bar{\beta}\bar{m}_{E\beta} - \bar{S} + \mu\bar{\bar{S}}\}, \quad d_5 = \bar{\beta}^{-1}\bar{m}_{E\beta}. \end{aligned} \tag{3.20}$$

These expressions are then inserted into (3.14) and the results for y and q inserted into (3.10). We then obtain a formula which can be written as

$$\frac{x_i}{E_i} = (3.13) - \frac{\mu}{\bar{\gamma}\bar{\beta}}(\beta_i - \bar{\beta})\,[(\bar{\gamma}\bar{n} + \bar{\beta}\bar{k})\bar{m}_{E\beta} - (\bar{S} - \mu\bar{\bar{S}})\,(\bar{k} - \bar{n})]. \tag{3.21}$$

Let us interpret this solution. We see that we have obtained a correction which is effective only if the capital intensities are not equal for all sectors. But it is interesting to note that it may operate even if $\bar{k} = \bar{n}$.

Suppose that $\bar{m}_{E\beta} > 0$, which means that the income elasticities of demand for the products of the sectors are positively correlated with the capital intensities. In this case the growth of income itself—independently of whether $\bar{n}$ or $\bar{k}$ is the greater—will tend to redirect demand in the direction of capital-intensive goods. This will make capital relatively more scarce and therefore cause (relatively) increasing prices for capital intensive goods, and thus, through substitution in the consumers' demand, to a certain extent counteract the original redirecting effect. It is this sort of "counteracting" effect which is expressed by the term involving $\bar{m}_{E\beta}$ in (3.21). Since $\bar{m}_{E\beta}$ can be expressed as the product of the correlation coefficient and the standard deviations, the effect increases both with the correlation between the E_i's and the β_i's and with the standard deviation in each of these series.[6]

The term involving $(\bar{S} - \mu\bar{\bar{S}})$ can work in either direction depending on whether μ is large or small. Let us consider the case for which μ is so large that $-(\bar{S} - \mu\bar{\bar{S}}) > 0$ and also for which $\bar{k} > \bar{n}$. Then the term under consideration will affect the capital-intensive sectors negatively. The reason for this may be sketched as follows. The fact that $\bar{k} > \bar{n}$ will cause capital-intensive goods to become cheaper compared to labour-intensive goods. When there is a strong substitution effect, however, (μ is large) this "first effect" (which is included in (3.13)) will be counteracted because it will induce "too strong" a redirection of demand in favour of capital-intensive goods, thus increasing the relative scarcity of capital and as a "second order

[6] Effects like those introduced in our model through $m_{E\beta}$ are discussed by Joan Robinson under the label "biased consumption," cf. [**8**, pp. 358–60].

effect" cause some increase again in the prices of capital-intensive goods. Since $\bar{S}$ and $\bar{\bar{S}}$ are not likely to deviate very much,[7] the critical value of μ for the direction of the effect now considered will be in the vicinity of unity, corresponding to a value of the flexibility of the marginal utility of money in the vicinity of -1.

The result that the critical value of μ should be in the vicinity of unity is intuitively plausible in view of the following observations. We noticed in connection with (3.9) that our first approximation for y and q corresponded to the solution of a pure macromodel in which the demand functions would degenerate into $X = Y/P$. This relationship implies a partial elasticity of X with respect to Y equal to unity and a partial elasticity of X with respect to P equal to -1. On the other hand, in (2.14) the values 1 and -1 for E_i and e_{ii}, respectively, imply $\mu = 1$.

Since the first approximation (3.13) was based on values for y and q which corresponded to the solution of a pure macromodel, whereas this is not the case for the second approximation (3.21), we may say that the new terms in (3.21) are corrections for an aggregation bias involved in the first approximation. The inexactitude which is still involved in (3.21) is the result of the linearization (in the d's) of the solution of (3.5—6).

4. THE EFFECTS OF TECHNICAL PROGRESS

In working out the effects of technical progress we shall proceed in two steps as we did in the preceding section.

The First Approximation

We return to equations (3.5—6) which determine y and q. The coefficients ε_i, which describe the technical progress, enter through d_3 and d_6. It is seen from (3.7) that d_3 and d_6 are formed in just the same way as d_1 and d_4. In Section 3 we argued that d_1 and d_4 could be disregarded as a first approximation. For exactly analogous reasons we may now, as a first approximation, disregard d_3 and d_6. We then end up with the solution (3.9) for y and q, even if the ε_i's are not now assumed to be equal to zero.

Formula (3.10) must, however, be modified as follows:

$$x = e\beta q + Ey - e\varepsilon. \tag{4.1}$$

Inserting here from (3.9) and using (2.14) we obtain a result which may be written in the following way

$$\frac{x_i}{E_i} = (3.13) + \bar{\varepsilon} + \mu(\varepsilon_i - \bar{\bar{\varepsilon}}), \tag{4.2}$$

[7] They are equal when the E_i's are uncorrelated both with γ_i and γ_i^2 (using the α_i's as weights in computing the covariances).

where $\bar{\varepsilon}$ and $\bar{\bar{\varepsilon}}$ are averages of $\varepsilon_1, \ldots, \varepsilon_m$ and are defined analogously with (3.12):

$$\bar{\varepsilon} = \alpha'\varepsilon, \quad \bar{\bar{\varepsilon}} = \alpha'\hat{E}\varepsilon.$$

We see that we have obtained a correction to (3.13) consisting of two terms: first, all sectors are adjusted upwards corresponding to the average technical progress; next, they are pushed up or down according to whether the technical progress in the sector concerned is greater or smaller than the average. This latter effect is greater, the greater is μ; i.e., it is greater the more important is substitution in the consumers' behaviour.

Viewing (4.2) from another angle, we can say that technical progress in a particular sector affects *all* sectors through a term $(\bar{\varepsilon} - \mu\bar{\bar{\varepsilon}})$ and the sector itself through $\mu\varepsilon_i$. Since in "normal" cases $\bar{\varepsilon} \approx \bar{\bar{\varepsilon}}$,[8] we may conclude that technical progress in a sector affects the production of other sectors (with $E_i > 0$) positively or negatively according to whether the numerical value of the flexibility of the marginal utility of money is greater or smaller than a critical value which is in the vicinity of unity. With a low degree of substitution, factors will be transferred out of the sector which experiences the technical progress; with a high degree of substitution the cheapening of the goods in that sector causes factor transfers into it.

The Second Approximation

In the second approximation we do not neglect the *d's* in (3.5—6). Instead of (3.14), in which we neglected d_3 and d_6, we now obtain

$$\begin{aligned} y &= [(1 - d_1 - d_2)\bar{n} + d_1\bar{k}] + d_3, \\ q &= [(1 - d_1 - d_2 + d_4 + d_5)\bar{n} - (1 - d_1 + d_4)\bar{k}] + d_3 - d_6, \end{aligned} \tag{4.3}$$

where the terms inside the square brackets contain the solution (3.14).

The terms d_3 and d_6 can be worked out in a similar way as the other *d*'s in the preceding section. We introduce

$$\begin{aligned} \bar{m}_{\varepsilon\beta} &= \sum \alpha_i (\varepsilon_i - \bar{\varepsilon})(\beta_i - \bar{\beta}) = \alpha'\hat{\beta}\varepsilon - \bar{\beta}\bar{\varepsilon}, \\ \bar{\bar{m}}_{\varepsilon\beta} &= \sum \alpha_i E_i (\varepsilon_i - \bar{\bar{\varepsilon}})(\beta_i - \bar{\bar{\beta}}) = \alpha'\hat{E}\hat{\beta}\varepsilon - \bar{\bar{\beta}}\bar{\bar{\varepsilon}}. \end{aligned} \tag{4.4}$$

We then obtain

$$d_3 = \bar{\gamma}^{-1}[\bar{m}_{E\beta}\bar{\varepsilon} - (\bar{m}_{\varepsilon\beta} - \mu\bar{\bar{m}}_{\varepsilon\beta})], \quad d_6 = -\bar{\beta}^{-1}\bar{\gamma}d_3. \tag{4.5}$$

Inserting this in (4.3) and using (2.14) and (4.1) we obtain

$$\frac{x_i}{E_i} = (3.21) + \bar{\varepsilon} + \mu(\varepsilon_i - \bar{\bar{\varepsilon}}) - \frac{\mu}{\bar{\gamma}\bar{\beta}}(\beta_i - \bar{\bar{\beta}})[\bar{m}_{E\beta}\bar{\varepsilon} - (\bar{m}_{\varepsilon\beta} - \mu\bar{\bar{m}}_{\varepsilon\beta})]. \tag{4.6}$$

[8] Since we have $\bar{m}_{E\varepsilon} = \bar{\bar{\varepsilon}} - \bar{\varepsilon}$ analogously with (3.16), $\bar{\bar{\varepsilon}}$ and $\bar{\varepsilon}$ are equal when the technical progress coefficients are uncorrelated with the income elasticities of demand.

Here the terms $\bar{\varepsilon} + \mu(\varepsilon_i - \bar{\bar{\varepsilon}})$ are already discussed in connection with (4.1). The new terms discriminate among sectors according to the capital intensities.

Of the new terms, the one involving $\bar{m}_{E\beta\bar{\varepsilon}}$ is quite analogous to the term with $(\bar{\gamma}\bar{n} + \bar{\beta}\bar{k})\bar{m}_{E\beta}$ in (3.21) and needs no separate comment.

For the interpretation of the term containing $(\bar{m}_{\epsilon\beta} - \mu\bar{\bar{m}}_{\epsilon\beta})$, consider the case for which $\bar{m}_{\epsilon\beta} \approx \bar{\bar{m}}_{\epsilon\beta} > 0$, which means that technical progress is on the average more rapid in capital-intensive than in labour-intensive sectors. We then see that the term under consideration operates in the disfavour of the capital-intensive sectors if μ is large, i.e., if there is a high degree of substitution in consumers' behaviour. This is because the relative cheapening of the capital-intensive goods (for which technical progress is greater than for other goods) causes such a great redirection of demand towards these goods that capital becomes relatively more scarce than before, and we get a secondary effect counteracting the original cheapening of the capital-intensive goods. In the case of a small μ both effects work in the same direction. First, capital-intensive goods become cheaper because of reduced costs through increased productivity. Next, capital becomes relatively less scarce because consumers change their demands over to capital-intensive goods to a lesser extent than that corresponding to the possible increases in the production of these goods by constant factor inputs.

Again, the critical value of μ is in the vicinity of unity.

It may be doubted that the demand structure will remain unchanged under a process of technical progress. If technical progress takes the form of new or improved products, it may, to some extent, "create its own demand," and the benefits from technical progress will be taken in the form of increased production in those sectors which experience the technical progress to a greater extent than indicated by the above formulas.

5. NUMERICAL ILLUSTRATIONS

The purpose of this section is not to obtain results for specified industries, but rather to find out whether our formulas introduce excessive refinements which for practical purposes could be ignored. It is very difficult to say anything about that a priori; empirical evidence, however rough, is therefore desirable.

A crucial parameter in all the formulas is μ, which is defined as minus the inverse of the flexibility of the marginal utility of money. I have elsewhere estimated this flexibility to be approximately -2 for Norway in 1950 (average for all consumers).[9]

[9] The estimation is described (in Norwegian) in [**6**], and will be included in [**7**]. In [**3**, pp. 106–107], Ragnar Frisch has given some comments on the results. The estimation was based on formula (2.14) applied to independent evidence on income elasticities of demand and own-price elasticities for a few groups of goods. Added in proof: Cf. also Ragnar Frisch's paper in *Econometrica*, April, 1959, pp. 188–189.

This yields

$$\mu = 0.5.$$

The sector classification used contains 17 industries, obtained by a few aggregations on the basis of the sectors in the Norwegian National Accounts [**9**].

Rough estimates of $\bar{\beta}$ and $\bar{\bar{\beta}}$ are obtained on the basis of figures given in [**4**]. Since the present model assumes vertically integrated industries it seems reasonable to identify γ_i and β_i with the wage share and the capital share including the indirect contributions of labour and capital through the input-output structure.[10]

The budget percentages a_i and the income elasticities of demand E_i are the same as those used in [**6**] and [**7**]. The a_i's are originally based on the consumption figures in the National Accounts [**9**] with some corrections, and the E_i's are based on estimates in connection with the "Median Model" of the Oslo University Institute of Economics.[11] On this basis, the following values were obtained for $\bar{\beta}$ and $\bar{\bar{\beta}}$ defined by (3.12):

$$\bar{\beta} = 0.49, \quad \bar{\bar{\beta}} = 0.48.$$

The values of $\bar{\gamma}$ and $\bar{\bar{\gamma}}$ are correspondingly

$$\bar{\gamma} = 0.51, \quad \bar{\bar{\gamma}} = 0.52.$$

In the postwar expansion period in Norway total labour force grew about 1 per cent per year and total capital stock about 5 per cent per year. Assuming this to hold approximately for the totals of the consumption goods industries as well, we may put $\bar{n} = 0.01$ and $\bar{k} = 0.05$. With these data we can now illustrate formula (3.13). By insertions for $\bar{\gamma}$, $\bar{\beta}$, $\bar{n}$, $\bar{k}$, and μ we obtain

$$\frac{x_i}{E_i} = 0.030 + 0.020\,(\beta_i - \bar{\bar{\beta}}). \tag{5.1}$$

In the series of β_i's applied here, the minimal β_i occurs for Services; it equals 0.20, which yields $\beta_i - \bar{\bar{\beta}} = -0.28$. The maximal β_i occurs for

[10] The table used was (7c.5) in [**4**]. This table includes also an indirect tax share and an import share. I added, however, the wage and capital shares and considered each of them as a fraction of the total thus obtained. This is justified if imported inputs and indirect taxes are proportional to total output. The capital share includes depreciation. This is as it should be if depreciation depends only upon the capital stock and not upon the amount of labour combined with the capital. Furthermore, I made some rough corrections in the shares by imputing wages to employers and own-account workers in industries where this represents a considerable part of total labour input, using Table 39 in the Norwegian National Accounts [**9**]. Finally I had to perform some aggregation in order to make the sector classification correspond to that for which I had the income elasticities of demand.

[11] See Hans Heli [**5**].

Dwellings; this β_i is 0.95, which yields $\beta_i - \bar{\bar{\beta}} = 0.47$.[12] For these two sectors we then obtain

$$x_i = 0.024\, E_i \quad \text{for Services,}$$
$$x_i = 0.039\, E_i \quad \text{for Dwellings,}$$

whereas the relationship for an "average industry" ($\beta_i = \bar{\bar{\beta}}$) would be

$$x_i = 0.030\, E_i \quad \text{for an ``average industry.''}$$

We see that for extreme sectors like those above the term $\mu(\beta_i - \bar{\bar{\beta}})(\bar{k} - \bar{n})$ introduces considerable correction to the most simple rule of thumb which says that industries should expand proportionately to the income elasticities of demand for their products.

Let us then turn to the second approximation given in formula (3.21). By formula (3.16) we have

$$\bar{m}_{E\beta} = 0.48 - 0.49 = -0.01,$$

which means that income elasticities of demand and capital intensities are slightly negatively correlated.[13] The term involving $\bar{m}_{E\beta}$ in (3.21) then becomes

$$(5.2) \qquad -\frac{\mu}{\bar{\gamma}\bar{\beta}}(\bar{\gamma}\bar{n} + \bar{\beta}\bar{k})\, \bar{m}_{E\beta}\, (\beta_i - \bar{\bar{\beta}}) = 0.0006\, (\beta_i - \bar{\bar{\beta}}).$$

It favours capital-intensive industries, but it does not correct significantly what is already contained in (5.1).

The variances defined by (3.17) are computed to be

$$\bar{S} = 0.031, \quad \bar{\bar{S}} = 0.029.$$

The difference is seen to be negligible. The term in (3.21) involving these variances becomes

$$(5.3) \qquad \frac{\mu}{\bar{\gamma}\bar{\beta}}(\bar{S} - \mu\bar{\bar{S}})\, (\bar{k} - \bar{n})\, (\beta_i - \bar{\bar{\beta}}) = 0.001\, (\beta_i - \bar{\bar{\beta}}).$$

Like (5.2) it favours capital-intensive industries, but it is negligible compared with what is already included in (5.1).

In conclusion, therefore, we seem to be justified in the double statement:

(1) The effect treated in Section 3 above may entail a considerable correction to the simple rule that sectors should expand proportionately to the income elasticities of demand for their products.

(2) The effect seems to be taken care of with sufficient accuracy by "the first approximation" given in formula (3.13).

[12] In this sector the output is "dwelling services," whereas the dwelling itself enters as capital input.

[13] With one decimal place more, $\bar{m}_{E\beta}$ is -0.014.

By using the values obtained above in formulas (3.20) it is easily seen that all the d's are so small that the linearization implied in (3.14) should not cause any inaccuracy worth noticing.

We have not much to say about the formulas for the effects of non-uniform technical progress, since suitable measurements are hardly available. We have, however, already quoted an estimate for μ, and it is then easy to appraise the importance of the terms in (4.2) in hypothetical cases. With $\mu = 0.5$, (4.2) reads

$$\frac{x_i}{E_i} = (3.13) + \bar{\varepsilon} + 0.5\,(\varepsilon_i - \bar{\bar{\varepsilon}}). \tag{5.4}$$

Since technical progress may be very different in different sectors, the final term here may be of a magnitude worth consideration.

The additional terms in (4.6) are not likely to be very important. With the value of $\bar{m}_{E\beta}$ obtained above, the term involving $\bar{\varepsilon}$ will be negligible. Also the term involving $(\bar{m}_{\varepsilon\beta} - \mu\bar{\bar{m}}_{\varepsilon\beta})$ seems incapable of producing considerable correction, even when the ε_i's are strongly correlated with the β_i's.

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