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SUBSTITUTION VERSUS FIXED PRODUCTION COEFFICIENTS IN THE THEORY OF ECONOMIC GROWTH: A SYNTHESIS

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Most growth models are based either on the assumption of fixed production coefficients for labour and capital or on the assumption of substitutability between factors. The present paper proposes a hypothesis which is a compromise between these extremes, viz., that any *increment* in production can be obtained by different combinations of *increments* in labour and capital inputs, whereas any piece of capital which is already installed will continue to be operated by a constant amount of labour throughout its life span. First, a "general model" is presented. Next, the model is solved in different special cases. In conclusion it is suggested that the proposed hypothesis would be particularly appropriate in studying the introduction of new techniques and the relationship between population growth, the rate of saving and "structural" unemployment.

1. INTRODUCTION

THE MODELS HITHERTO most widely applied in the theoretical analysis of problems of economic growth can be classified in the following three groups:

(a) Models with a given capital coefficient, where the labour input does not enter the analysis explicitly, but is treated rather vaguely in supplementary comments. The models of R. F. Harrod [**9**], Evsey D. Domar [**3**][2], Hans Brems [**2**], Robert Eisner [**5**] and Ingvar Svennilson [**22**] exemplify this class.

(b) Models with fixed production coefficients for labour input as well as for the capital stock, or some other kind of strict complementarity. As examples one might mention the analysis of D. Hamberg [**8**], the work on long-range projections at the Central Planning Bureau in the Netherlands, cf. e.g., P. J. Verdoorn [**25**], and furthermore the more disaggregated analysis by Wassily Leontief [**14**], Oskar Lange [**13**] and other authors in the field of input-output analysis.

(c) Models with explicitly expressed possibilities of substitution between total labour input and capital stock in a traditional production function. This type of model is exemplified by the publications of Jan Tinbergen [**23**], Trygve Haavelmo [**7**], Robert Solow [**19**] and Stefan Valavanis-Vail [**24**].

Models belonging to any of these groups may, of course, contain important

[1] I am indebted to professor Trygve Haavelmo and Mr. Hans Jacob Kreyberg at the University of Oslo, with whom I have discussed many of the problems analysed in this paper. Mr. Kreyberg has also read through the manuscript and given useful criticism.

[2] See in particular the Foreword and Essay III: "Capital Expansion, Rate of Growth, and Employment," (*Econometrica*, April 1946 pp. 137-147).

and realistic aspects and be well suited for certain objectives. I have, however, the feeling that many theorists, whether they apply models belonging to the group (a), (b) or (c), often have been working with a "guilty conscience"[3] regarding the realism of their assumptions. The purpose of this paper is to propose a kind of synthesis between the approaches in (b) and (c) above. The synthesis will be based on the following assumptions:

(1) Any gross[4] *increment* in the rate of production can be obtained by different combinations of *increments* in capital and labour input. We may perhaps express this in another way by saying that there are ex ante substitution possibilities between capital and labour, or that there are substitution possibilities *at the margin*.

(2) Once a piece of capital is produced and has been put into operation, it will continue to operate through all its life span in cooperation with a constant amount of labour input. We may perhaps express this by saying that there are no ex post substitution possibilities, or that there are no substitution possibilities between total labour input and existing capital stock.

Even if a more flexible framework may be imagined, I have the feeling that an analysis based on the assumptions (1) and (2) above will in most cases be more realistic than an analysis based on models belonging to any of the groups, (a), (b), or (c).

The idea may of course be applied at different levels of aggregation. We shall, however, as an illustration, apply it in a pure macro-analysis of growth problems.

In Section 2 the idea is worked out more precisely and included in a rather general model. In Sections 3, 4 and 5 the solution of the model is given and some special cases are discussed. Some concluding remarks are given in Section 6.

2. THE "GENERAL" MODEL

The model to be presented in this section is, of course, not general in any absolute sense. It is only general relative to the specializations discussed in Sections 3, 4 and 5.

The model will be characterized by the following properties:

(1) There are two factors of production, labour and capital, producing an output which may be used either for consumption or for accumulation.

(2) There are substitution possibilities ex ante, but not ex post, as explained in the introduction.

[3] Cf. Evsey D. Domar [**3**, p. 7].

[4] That means that we have not subtracted the decline in production caused by old capital being depreciated or scrapped.

(3) From the point of time when an amount of capital is produced, it will shrink according to a given function of its age. The labour input needed to operate the capital and the production achieved shrink proportionately. Even if other interpretations are possible, this is perhaps most easily accepted if we assume that each amount of capital consists of a certain number of identical pieces or units which are operated in the same way and retain their productive efficiency during their entire life time, and that there exists a "death rate table" for these capital units. As special cases, this assumption includes the case of capital of infinite duration and the case for which all capital units have the same finite life time.

(4) New production techniques can be introduced only by means of new capital equipment. This statement is not quite clearly expressed here, but will be clarified by the formulas below.[5]

(5) We assume either that net investment is a constant fraction of net income, or as an alternative, that gross investment is a constant fraction of gross income.[6]

(6) By computing the depreciation necessary to obtain the "net" concepts introduced by (5), a unit of capital is valued in proportion to its remaining life span.

(7) The total labour force is governed by an autonomous pattern of growth.

(8) There will always be full employment of labour and capital.

It is, of course, possible to analyse the effects of points (1) through (7) with some alternative instead of point (8). That will, however, not be done in this paper. At the end of this section, we shall comment on the interpretation of assumption (8).

[5] This assumption probably corresponds to the idea expressed by Ingvar Svennilson [**21**, p. 208] in the following form: "The volume of investment, whether it constitutes a net addition to the stock of capital or not, can therefore be said to measure the rate at which capital is being modernized." Cf. also [**22**, p. 325]: "Technical progress will, however, mean that old capital goods are eliminated and new ones substituted." Compare further K. Maywald [**15**]: "It is assumed that only the best production process is used in every unit of equipment added in the course of each year to the total capacity of the industry or economy concerned. Each unit of equipment represents the technological stage of development reached in its year of origin, until the very end of its serviceable life." A similar hypothesis is also crucial, for instance, for important parts of Paul A. Baran's growth analysis [**1**,cf. e.g., p. 21 and pp. 78-79], and for S. G. Strumilin's analysis [**20**, cf. in particular p. 175].

[6] These are the savings hypotheses most widely applied in growth analysis. In his Essay VII in [**3**] ("Depreciation, Replacement, and Growth," *The Economic Journal*, 1953), Domar employs both hypotheses, maintaining that the gross concept is the more "applicable to a centrally directed economy, where a part of total output is set aside for investment, while the net concept is the more applicable to capitalist countries."

The unknowns that enter the model are:

$x(t)$, the rate of production at time t;
$N(t)$, the total labour force at time t;
$K(t)$, the total stock of capital at time t;
$k(t)$, the rate of gross investment at time t, i.e., $k(t)dt =$ the amount of capital produced and put into operation during the time interval $(t, t + dt)$;
$n(t)$, the rate of allocation of labour to newly constructed capital, i.e., $n(t)dt$ is the labour input allocated to the operation of the capital $k(t)dt$;
$y(t)$, the rate of gross increase in production at time t, i.e. the rate of increase in $x(t)$ caused by $k(t)$ and $n(t)$;
$V(t)$, the value of the capital stock at time t;
$D(t)$, the rate of depreciation at time t; and
$I(t)$, the rate of net investment at time $t = k(t) - D(t)$.

The main problem now is to provide a formal representation of a production process with the desired properties.

We first introduce the function φ describing the effects on production of the gross investment and the labour input used with this investment:

$$y(t) = \varphi(n(t), k(t), x(t), t). \tag{2.1}$$

If we now assume $\partial\varphi/\partial n > 0$ and $\partial\varphi/\partial k > 0$, $n(t)$ and $k(t)$ will be substitutable factors in the process which causes a certain gross rate of increase, $y(t)$, in production.

It is perhaps reasonable to assume φ to be homogeneous of degree one in n and k. We shall, however, not introduce this specialization in the "general" model.[7]

In (2.1) we have introduced $x(t)$ as an argument besides $n(t)$ and $k(t)$. The reason for this is the following: $n(t)$ and $k(t)$ in no way indicate the "pressure" on natural resources resulting from the rate of production. This pressure may, however, have important consequences. When the pressure is already high, a greater effort in the way of increases in labour

[7] There is no immediate connection between the question of homogeneity of φ in n and k and the question of homogeneity of an ordinary production function in N and K. The arguments raised in connection with the latter question are perhaps more relevant for the role played by the argument x in φ; cf. the following discussion of this point.

Under extremely simplifying conditions we may, however, relate the function φ to traditional microeconomic production functions in the following way. Suppose that any increase in total production is generated through establishment of new firms. Suppose further that all firms which are established simultaneously have identical production functions, the form of which is denoted by $\psi(\bar{n},\bar{k})$ where $\bar{n}$ and $\bar{k}$ stand for employment and capital per firm. In order to obtain a rate of gross increase y in and capital may thus be required to obtain a certain increase in production. As this pressure is mainly generated through the extraction from nature of raw materials which are required in rather fixed proportion to the amount of production, $x(t)$ may perhaps be a satisfactory indicator of this so-called pressure. Arguments might, however, be raised in favour of also introducing the total stock of capital $K(t)$ and the total labour input $N(t)$ in (2.1), indicating that the *way* in which production is carried out may possibly influence the degree of pressure on natural resources.

The arguments above imply $\partial\varphi/\partial x \leq 0$.

By contrast, one might perhaps also argue that $\partial\varphi/\partial x > 0$ on the basis of "external economies."

The symbol t is introduced as a separate argument in (2.1) to take care of the possible increase in productivity through improvements in "know-how," discoveries of new natural resources, etc.

Let us now study the shrinkage in capital over time. We introduce a function $f(\tau)$ with the following interpretation: *If an amount $k(t)dt$ of capital is produced in the time interval $(t - dt, t)$, then an amount $f(\tau)k(t)dt$ of this capital will still be active at time $t + \tau$ $(\tau \geqq 0)$.* It follows that $f(\tau)$ is monotonically non-increasing and that $f(0) = 1$.

As stated above (property 3) we assume that production shrinks proportionately with capital. This is equivalent to saying that if $k(\tau)d\tau$ (in cooperation with $n(\tau)d\tau$) caused an increase, $y(\tau)d\tau$, in the rate of production in the time interval $(\tau, \tau + d\tau)$, then the rate of production originating from this capital at time t equals $f(t - \tau)y(\tau)d\tau$. It is then obvious that the total rate of production at time t may be obtained by integrating the output from all layers of capital, with due account for the shrinkage:

$$x(t) = \int_{-\infty}^{t} f(t - \tau)y(\tau)d\tau. \tag{2.2}$$

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total production, it is then necessary to establish m new firms per unit of time, where $y = m\psi(\bar{n},\bar{k})$. We have further $n = m\bar{n}$ and $k = m\bar{k}$ which give $y = m\psi(n/m, k/m)$. This defines y as a function of n, k and m. Assume now that there exists for each expansion line in the $(\bar{n},\bar{k})$ space an optimal size of the firm (defined by the scale coefficient being equal to unity). Assume further that firms always attain this size. Then m will be a function $m(n,k)$ of n and k, and it is easily seen that $m(n,k)$ must be homogeneous of degree one in n and k. By these assumptions we get y as a function only of the variables n and k:

$$y = m(n,k)\psi\left(\frac{n}{m(n,k)}, \frac{k}{m(n,k)}\right),$$

and this function is homogeneous of degree one in n and k irrespective of the form and properties of the function ψ.

We have here for simplicity disregarded the arguments x and t in the production functions. The introduction of these arguments in ψ (for the reasons given in the text) does, however, in no way change the reasoning above.

The interpretation of $y(t)$ as the "gross" increase in $x(t)$ will now be clear. Suppose there is no shrinkage in capital, i.e., $f(\tau) \equiv 1$. Then $x(t) = \int_{-\infty}^{t} y(\tau)d\tau$ and consequently $\dot{x}(t) = y(t)$. We may therefore say that the increase $\dot{x}(t)$ consists of a gross increase $y(t)$ due to $n(t)$ and $k(t)$, while a deduction $y(t)—\dot{x}(t)$ is due to the shrinkage of the existing capital. If $f'(\tau)$ exists we have $\dot{x}(t) = y(t) + \int_{-\infty}^{t} f'(t—\tau)y(\tau)d\tau$ where $f' \leqq 0$.

A reasonable condition on the function φ is that $\varphi(0,0,x,t) \equiv 0$ identically in x and t. If $n(t) = k(t) = 0$ for $t > \theta$, then we shall have $x(t) = \int_{-\infty}^{\theta} f(t—\tau)y(\tau)d\tau$ for $t \geqq \theta$, and the only changes in $x(t)$ for $t > \theta$ will result from shrinkage in the existing capital. In this case, therefore, there will be no effect of increased "know - how" after the time θ. This illustrates the condition that the increased "know - how" in our model can be utilized only through the introduction of new capital equipment.

Let us now study the development of the labour input $n(t)$ available at any point of time to man the new capital equipment.

Our basic assumption is

(2.3) $N(t)$ is an exogenously given function of time.

This total labour force will be distributed over capital of different ages. Cooperating with the capital produced in the interval $(\tau, \tau + d\tau)$ will be the labour $n(\tau)d\tau$. At time t this will be reduced to $f(t — \tau)n(\tau)d\tau$. Accordingly, we have the following condition on the development of $n(t)$:

$$\int_{-\infty}^{t} f(t - \tau)n(\tau)d\tau = N(t). \tag{2.4}$$

By a similar integration we obtain an expression for the total amount of capital:

$$\int_{-\infty}^{t} f(t - \tau)k(\tau)d\tau = K(t). \tag{2.5}$$

In the traditional description of the production structure, x is related uniquely to N and K (and possibly also to t as a separate argument). In our approach it is, however, characteristic that it is in general *not* possible to derive any such unique relation which holds regardless of the development of $N(t)$ and $K(t)$. A necessary and sufficient condition for this possibility to exist is that φ be linear in n and k, and that x and t do not enter the function φ as separate arguments. (If the condition $\varphi(0,0,x,t) \equiv 0$ is abandoned, t may also enter φ. Then φ must be linear in n, k and any unique function of t).

The production model above recognizes fully the impossibility of changing at will the manning of capital equipment once constructed. There exists, however, another kind of rigidity which is not recognized above. To explain this rigidity, let us look at the capital equipment engaged in producing more capital equipment. This equipment is perhaps so constructed that it can produce only capital equipment designed to be manned in a definite way. For instance, a factory producing spinning-jennies may be equipped in such a way that it is only able to produce spinning-jennies which must be operated by a definite amount of labour. If this kind of rigidity is important, it will perhaps be difficult to realize the smooth adaption of capital equipment to the given $n(t)$-development which is implied by our model.[8] It would then perhaps be interesting to construct a model which would lie between the model presented here and one with no substitution possibilities.

Now for capital accumulation or savings. Different assumptions can be conceived of here. The possibility of *choice* open to society would make it desirable to investigate the consequences of various assumptions or to postulate some optimality criteria.[9] However, in order to conform to the most widely accepted models of growth on this point—where this paper does not attempt to make any contribution—I shall treat only the hypothesis of a fixed ratio of savings to income.

In order to define the "net" concepts, we need a rule for the valuation of capital. We then simply value a unit of capital proportionately to its remaining life span.

A newly produced unit of capital will on the average last

$$T(0) = \int_0^\infty f(\tau)d\tau \tag{2.6}$$

periods. A unit of capital already η periods of age will on the average have

$$T(\eta) = \frac{1}{f(\eta)} \int_\eta^\infty f(\tau)d\tau \tag{2.7}$$

periods left.[10]

We then say that a unit of capital η periods old is worth $T(\eta)/T(0)$ relative to a new one. By an integration similar to (2.5), we then get for the value of the total stock of capital

[8] Some rigidities of this kind must be implied by the analysis of Hans W. Singer [**18**], cf., e.g., p. 182: "The capital-intensive technology—which is the only now existing—. . ." and p. 183: "The absence of a technology which is at the same time modern (in the sense of incorporating the latest state of scientific knowledge) and yet in harmony with the factor endowment of under-developed countries must be classed as another major obstacle to economic development." Further, on p. 181: "In many respects, the technology of one hundred years ago would be preferable and would make their (the underdeveloped countries) economic development easier."

[9] Cf. on this point H. J. A. Kreyberg [**12**].

[10] Cf. Gabriel A. Preinreich [**17**, p. 220].

$$V(t) = \frac{1}{T(0)} \int_{-\infty}^{t} f(t-\tau)T(t-\tau)k(\tau)d\tau \tag{2.8}$$

which gives

$$V(t) = \frac{1}{T(0)} \int_{\tau=-\infty}^{t} k(\tau) \int_{\xi=t-\tau}^{\infty} f(\xi)d\xi d\tau. \tag{2.9}$$

By differentiation this gives

$$I(t) = \dot{V}(t) = k(t) - \frac{1}{T(0)} K(t), \tag{2.10}$$

and for depreciation the familiar formula[11]

$$D(t) = \frac{1}{T(0)} K(t). \tag{2.11}$$

A constant fraction α of savings applied to the net concepts then gives $I(t) = \alpha(x(t) - 1/T(0)\ K(t))$, which can be written

$$k(t) = \alpha x(t) + \frac{1}{T(0)} (1-\alpha)K(t). \tag{2.12}$$

If we want to operate with a constant savings quota applied to the gross concepts, we need only neglect the last term in (2.12).

Considering our model as a whole now, we recognize that (2.1), (2.2), (2.3), (2.4), (2.5), and (2.12) where $T(0)$ is defined by (2.6) constitute 6 equations containing the six time functions $y(t)$, $n(t)$, $k(t)$, $N(t)$, $K(t)$, $x(t)$. This will be referred to in the following sections as our "general model."

One may now ask how it is that we have obtained a determinate model without any reference to the behaviour of the producers? In fact we have substitution possibilities "at the margin," and certain assumptions are therefore necessary to explain this behaviour.

The answer to this question is that a certain behaviour is tacitly implied by our assuming that $n(t)$ and $k(t)$ are always absorbed.

One explanation may be that our model applies to a centrally planned economy which at any time chooses to construct new equipment in such a way that the disposable labour is absorbed.

Another explanation may be that our model applies to an economy where production is governed by the profit motive. In that case a certain development of wages and the interest rate is implied by our model, namely, that development which makes entrepreneurs choose to absorb both the flow of savings and the flow of disposable labour at all times. These time functions for wages and the interest rate might be linked to our model. Many kinds of rigidities may, however, operate to make such smooth adaption impossible.[12]

[11] Cf. e.g., Essay VII in Domar [**3**].

[12] Cf. Robert M. Solow [**19**], D. Hamberg [**8**] and the discussion by Pilvin, Harrod, and Domar.

If the wage rate and the the rate of interest should move with rather different time shapes, a special problem would arise in connection with old capital. Capital which is constructed for instance at a time when wages are rather low and interest rates rather high may at a time of higher wages and lower interest rates be so unprofitable in use that it is scrapped prematurely or left idle for a while. Such a development may also be reflected in the valuation of the capital stock. These problems, however, are not taken care of formally in our model.[13]

In the case of a profit-motivated production process, it might be interesting to reverse the point of view described above and accepted in our model. Instead of assuming full employment of labour and capital, and implying tacitly the necessary development of wages and interest, we might assume certain developments for wages and interest, perhaps related to monetary aspects of the economy, and try to compute the time function for the possible unemployment which might occur. Such an attempt will however not be made in this paper.

The model above looks rather unmanageable in its general form. In the following sections we shall therefore work out the solutions for some special cases which are quite near to hand. The reason for working out the solution for different cases—that is, with different forms of the production function φ and the shrinkage function f—is partly that it is not obvious what functional forms are most realistic, and partly that I find it rather difficult to work out the solution if I try to combine that form of the production function (and the introduction of new techniques) which I personally find most interesting with that form of the shrinkage function which I would prefer if I had to choose.[14]

3. THE CASE WITH CAPITAL OF INFINITE DURATION

Rather important simplifications of the model are obtained if we consider the case with capital of infinite duration, i.e., the case in which

$$f(\tau) \equiv 1 \qquad \text{for } \tau \geqq 0. \tag{3.1}$$

[13] The technical changes which result from increasing "know-how," may, of course, also influence the valuation of capital. It is, however, not obvious how this ought to be introduced in the model, and I have therefore chosen to disregard it.

[14] In correspondence Robert Solow has made the following comment on an ambiguity which is not discussed above: "There is a little ambiguity involved in treating the 'physical' nature of the capital good as changing over time but at the same time assuming that the same commodity can be consumed without change. This of course is simply an aggregation difficulty; in a more complete model consumption goods will be separate from capital goods. But then in a more complete model the treatment of capital goods becomes more straightforward too. As time goes on and technical change occurs, some capital goods will be affected, others not. And one would naturally introduce different capital-labour substitution possibilities for different (including older and newer) machines, with rigidity appearing as a limiting case."

In that case (2.2) gives

$$x(t) = \int_{-\infty}^{t} y(\tau)d\tau. \tag{3.2}$$

Equation (2.4) gives

$$\int_{-\infty}^{t} n(\tau)d\tau = N(t), \tag{3.3}$$

and (2.12) gives simply

$$k(t) = ax(t) \tag{3.4}$$

as $T(0) = +\infty$.

In this case, there can, of course, exist costs of maintenance and repair. These costs must, however, be constant over time for every piece of capital. We can then define y and x net of these costs.

As a rather satisfactory form of φ we shall accept

$$\varphi(n,k,x,t) = An^a k^b x^{-c} e^{\varepsilon t} \tag{3.5}$$

with a, b, c and ε constant. Here $a + b = 1$ is possibly a realistic hypothesis, but in general we shall not make this assumption. The coefficient c is most probably $\geqq 0$, but we might, as mentioned in the discussion of the "general" model, have $c < 0$ as a consequence of "external economies." ε is the relative increase in productivity per period as a result of increased "know - how," etc. In general, we shall therefore have $\varepsilon \geqq 0$.

It is seen that both x and t have a neutral effect in φ in the sense that the marginal rate of substitution between n and k is not influenced by x and t.

Let us further assume an exponential growth of the labour force N, i.e.,

$$N(t) = N_0 e^{\nu t} \tag{3.6}$$

where ν is constant.

By differentiating (3.3) we then obtain

$$n(t) = \dot{N}(t) = n_0 e^{\nu t} \qquad \text{where } n_0 = \nu N_0. \tag{3.7}$$

By differentiating (3.2) we obtain $\dot{x}(t) = y(t)$. By means of (3.4), (3.5) and (3.7) we then get

$$\dot{x} = [An_0^a a^b] x^{b-c} e^{(a\nu+\varepsilon)t}. \tag{3.8}$$

This is a Bernoullian differential equation[15] which can be solved to give

$$x(t) = \left[\frac{An_0^a a^b (1-b+c)}{a\nu + \varepsilon} e^{(a\nu+\varepsilon)t} + C\right]^{\frac{1}{1-b+c}} \tag{3.9}$$

where C can be determined by means of $x(0)$. The solution is not valid for $a\nu + \varepsilon = 0$. We shall, however, disregard this special situation assuming $a\nu + \varepsilon > 0$.

[15] Cf. e.g., E. L. Ince [**10**, p. 22].

As t increases, it is here easily seen that the growth rate of $x(t)$ converges,

$$\frac{\dot{x}(t)}{x(t)} \rightarrow \frac{\alpha\nu + \varepsilon}{1 - b + c} \quad \text{as } t \rightarrow +\infty. \tag{3.10}$$

If we divide the solution for $x(t)$ by $N(t)$, we obtain for production per head,

$$\frac{x(t)}{N(t)} = \frac{1}{N_0}\left[\frac{An_0^a\alpha^b(1 - b + c)}{\alpha\nu + \varepsilon} e^{(\varepsilon - (1-a-b+c)\nu)t} + Ce^{-(1-b+c)\nu t}\right]^{\frac{1}{1-b+c}} \tag{3.11}$$

(where $n_0 = \nu N_0$ according to (3.7)).

Here, of course, different cases may be discussed. We shall, however, consider only the case for which

$$a > 0, b > 0, a + b \leqq 1, c \geqq 0, \varepsilon > 0, \nu > 0. \tag{3.12}$$

Then the last term within the bracket in (3.11) will vanish as t increases, and the solution will converge asymptotically,

$$\frac{x(t)}{N(t)} \rightarrow \frac{1}{N_0}\left[\frac{An_0^a\alpha^b(1-b+c)}{\alpha\nu + \varepsilon}\right]^{\frac{1}{1-b+c}} e^{\left[\frac{\alpha\nu+\varepsilon}{1-b+c} - \nu\right] t}. \tag{3.13}$$

For the growth rate in (3.13) to be positive, it is necessary and sufficient that

$$\nu < \frac{\varepsilon}{1-(a+b)+c}\left(= \frac{\varepsilon}{c} \text{ if } a + b = 1\right), \tag{3.14}$$

where the right hand side is always positive under our assumptions (3.12). (3.14) illustrates how an upper bound is set for the population growth by the requirement that average production shall not decline, and how this bound is influenced by technical change (the numerator of (3.14)) and by the scale properties of the production function (the denominator of (3.14)).

The most remarkable feature of the solution above is perhaps that the asymptotic growth rates given in (3.10) and (3.13) are independent of the propensity to save α.[16] Let us illustrate this by assuming two countries which are similar in all respects except for α. Asymptotically both countries will then obtain the same relative rate of growth in production per head, x/N, (and of course also in total production, x). It is, however, seen by the way in which α enters (3.13) that this asymptotic curve will lie on a higher level—and therefore the absolute rate of growth be greater—in the country with the higher propensity to save. This also implies that if both countries start from the same initial position, the country with the higher propensity to save will start out with the higher relative growth rate "before the asymptote is reached."

[16] This feature is not, however, dependent on our special way of introducing substitution in the model. A similar conclusion can be obtained by means of models with substitution possibilities of the traditional kind.

4. THE CASE WITH EXPONENTIAL SHRINKAGE OF CAPITAL

Let us now assume that the capital shrinkage function f defined in Section 2 has the exponential form

$$f(\tau) = e^{-\delta\tau}. \tag{4.1}$$

This would be the case if capital units are eliminated through "accidents," and for every unit of capital existing at a point of time t there exists a probability δdt that it will be destroyed by an accident in the time interval $(t, t + dt)$, where δ is a constant independent of the age of the capital unit.

For $x(t)$ we have then

$$x(t) = \int_{-\infty}^{t} e^{-\delta(t-\tau)}y(\tau)d\tau, \tag{4.2}$$

and similarly for $N(t)$ and $K(t)$,

$$\int_{-\infty}^{t} e^{-\delta(t-\tau)}n(\tau)d\tau = N(t) \tag{4.3}$$

and

$$\int_{-\infty}^{t} e^{-\delta(t-\tau)}k(\tau)d\tau = K(t). \tag{4.4}$$

Since

$$T(0) = \int_{0}^{\infty} e^{-\delta\tau}d\tau = \frac{1}{\delta}, \tag{4.5}$$

the savings equation (2.12) reduces to

$$k(t) = ax(t) + \delta(1 - a)K(t). \tag{4.6}$$

For the growth of the labour force we shall assume, as in the preceeding section, that $N(t) = N_0e^{\nu t}$.

In this case it turns out to be rather difficult to solve the system if we apply the function (3.5) with $c \neq 0$. Let us therefore study the special case in which

$$\varphi(n,k,x,t) = An^a k^b e^{\epsilon t}. \tag{4.7}$$

By differentiating (4.2) we now obtain

$$\dot{x}(t) = y(t) - \delta x(t), \tag{4.8}$$

and similarly for $N(t)$ and $K(t)$. For $n(t)$ this now gives

$$n(t) = n_0e^{\nu t} \qquad \text{where } n_0 = N_0(\nu + \delta). \tag{4.9}$$

The labour force available for new capital at any time consists therefore of the growth in the total labour force plus the workers who are set free from old capital which is eliminated.

By means of the equation for $K(t)$ and (4.6) we obtain

$$\dot{k}(t) = a\dot{x}(t) + \delta ax(t) - \delta ak(t). \tag{4.10}$$

By means of (4.7), (4.8) and (4.9) we then get the following differential equation for $k(t)$:

$$\dot{k} = [aAn_0^a]\, e^{(a\nu+\varepsilon)t}k^b - a\delta k. \tag{4.11}$$

This is a Bernoullian equation of a slightly more complicated form than (3.8). It is solved to give

$$k(t) = \left[\frac{An_0^a a(1-b)}{a\nu+\varepsilon+(1-b)a\delta}\, e^{(a\nu+\varepsilon)t} + Be^{-(1-b)a\delta t}\right]^{\frac{1}{1-b}}, \tag{4.12}$$

where B is determined by initial conditions.

If $\delta = 0$, and accordingly $k(t) = ax(t)$, it is easily seen that (4.12) corresponds to (3.9) for $c = 0$. In the case of $\delta \neq 0$, it is not so easy to obtain the solution for $x(t)$.

We see, however, that the last term in (4.12) tends to vanish for increasing t. Let us therefore consider only the asymptotic solution

$$\bar{k}(t) = \left[\frac{An_0^a a(1-b)}{a\nu+\varepsilon+(1-b)a\delta}\right]^{\frac{1}{1-b}} e^{\frac{a\nu+\varepsilon}{1-b}t} \tag{4.13}$$

which shows a constant relative growth rate.[17] The corresponding solution for $x(t)$, which we shall denote $\bar{x}(t)$, is then more easily obtained by means of (4.8) when we insert for y from (4.7):

$$\bar{x}(t) = Ge^{-\delta t} + He^{\frac{a\nu+\varepsilon}{1-b}t}, \tag{4.14}$$

where G is determined by initial conditions and H is given by

$$H = \frac{(1-b)\, An_0^a \left[\dfrac{An_0^a a(1-b)}{a\nu+\varepsilon+(1-b)a\delta}\right]^{\frac{b}{1-b}}}{a\nu+\varepsilon+\delta(1-b)}. \tag{4.15}$$

Since the first term in (4.14) tends to vanish, we observe that $x(t)$ in this case will tend to increase with the same relative rate of growth as in the case studied in the perceeding section (for $c = 0$) regardless of δ (cf. (3.10)).

It is seen that under the conditions (3.12) $H > 0$. Furthermore, H is larger, the larger is a. The role played by δ is more complicated. This seems reasonable since δ not only represents the shrinkage in capital but also influences the accumulation of capital through (4.6) and, under our assumptions regarding the production process, also influences the speed with which new techniques can be introduced.[18]

[17] A discussion of the admissibility of this kind of approximation can be found in Hans Brems [**2**]. Cf. also Evsey Domar [**3**], Essay IX ("A Soviet Model of Growth"), pp. 231-32.

[18] Cf. the discussion of the influences of the average life time of capital in Domar [**3**, Essay VII].

The development of the average income x/N can be studied in a way similar to that in the preceeding section.

In the case discussed in this section, important simplification is obtained if we substitute for the net savings equation, (4.6), a gross savings equation,

$$k(t) = \beta x(t). \tag{4.16}$$

It is then rather easy to solve the system even if we retain the production function (3.5) with c not necessarily equal to zero. The solution is

$$x(t) = \left[C'e^{-(1-b+c)\delta t} + \frac{(1-b+c)An_0^a\beta^b}{a\nu+\varepsilon+(1-b+c)\delta}e^{(a\nu+\varepsilon)t}\right]^{\frac{1}{1-b+c}} \tag{4.17}$$

where C' is determined by initial conditions. For $\delta = 0$, this solution clearly corresponds to (3.9). Since the first term in the bracket tends to vanish for increasing t when $\delta > 0$, a discussion of the long range development of x and x/N will follow lines similar to those of Section 3.

In this case δ also does not influence the asymptotic relative growth rate. But δ influences the *level* of the asymptotic development. It is possible for instance (remembering that $n_0 = N_0\ (\nu + \delta)$ and considering N_0 as given) to demonstrate that if $a + b = 1$ and $c = 0$, we shall have the following situation: If $\varepsilon = 0$ this level will be higher the smaller is δ. But for $\varepsilon > 0$, it is possible that the level is higher the higher is δ. There will then exist a (positive) optimal δ which is larger, the larger is ε. This clearly illustrates the interrelations between δ and the speed with which new techniques are introduced within our production theory framework.

Similar cases will, of course, exist also under assumptions less restrictive than that $a + b = 1$ and $c = 0$.

What is said here is that if we have a sufficiently fast technical development, there will exist an optimal positive δ (which implies an optimal average life time for capital units) *even if we disregard the different costs of producing capital equipment with higher and lower* δ. If these cost differences are taken into account, a positive optimal value of δ will, of course, exist *a fortiori*. It would be interesting to extend our analysis in this direction in order to obtain rules for a rational selection of δ in different cases.[19]

5. THE CASE WITH A FIXED LIFE TIME FOR EACH UNIT OF CAPITAL

In this section we shall assume $f(\tau)$ to have the following form:

$$f(\tau) = \begin{cases} 1 \text{ for } \tau \leqq \theta, \\ 0 \text{ for } \tau > \theta. \end{cases} \tag{5.1}$$

This means that every unit of capital has a finite life time θ, and that it retains its original productive capacity all through this life time.[20]

[19] Some elements for such an analysis might be found in S. G. Strumilin [**20**].

[20] This is the assumption adopted for instance by Domar [**3**, Essay VII] and by Hans Brems [**2**].

This further gives

$$T(0) = \theta. \tag{5.2}$$

In this case it seems rather difficult to solve the system with production functions such as those applied in the preceeding sections. We shall therefore now have to be content with a linear and homogeneous function

$$\varphi(n,k,x,t) = an + bk, \tag{5.3}$$

where a and b are constants.

It would perhaps be possible to add a term depending on t on the right hand side of (5.3). This would, however, not satisfy the restriction $\varphi(0,0,x,t) \equiv 0$ and would therefore not be altogether meaningful within the framework of our approach. At the end of this section we shall, however, briefly discuss the case in which a is not constant, but depends on t.

From (2.2), (2.4) and (2.5) we now get

$$x(t) = \int_{t-\theta}^{t} y(\tau)d\tau = aN(t) + bK(t), \tag{5.4}$$

which means that there now exists a production function relating x uniquely to N and K. Formally our model in this case does not, therefore, differ from the more common growth models on this point. The concrete meaning underlying the relation is, however, still different from that of the more common production functions, as explained in Sections 1 and 2.

For the savings function we have

$$k(t) = \alpha x(t) + \frac{1}{\theta}(1 - \alpha)K(t). \tag{5.5}$$

Assuming as before that $N(t) = N_0 e^{\nu t}$, we obtain for $n(t)$:

$$n(t) = n_0 e^{\nu t} \quad \text{where } n_0 = \frac{\nu N_0}{1 - e^{-\nu\theta}}. \tag{5.6}$$

Instead of the usual differential equations as were obtained in the preceeding sections, we now get a mixed difference-differential equation to solve:

$$\dot{k}(t) = \gamma[k(t) - k(t - \theta)] + \alpha a \nu N_0 e^{\nu t} \tag{5.7}$$

where

$$\gamma = \frac{1}{\theta}[1 + \alpha(\theta b - 1)]. \tag{5.8}$$

Such an equation may show many curious solutions, among them discontinuous ones. The discontinuous solutions are, however, less interesting, at any rate in growth analysis. We shall therefore restrict our discussion to continuous solutions.

The results of James and Belz [**11**] then imply that the solution of the homogeneous equation corresponding to (5.7) can be expressed as a sum of exponential expressions (with real and complex exponents).

Let us insert an expression $Ce^{\varrho t}$ for $k(t)$ in the equation obtained by disregarding the last term in (5.7). We then get the characteristic equation[21]

$$\varrho = \gamma(1 - e^{-\varrho\theta}). \tag{5.9}$$

Let us first consider only the real solutions for ϱ, which are the most interesting from the point of view of growth analysis. At the end of this section we shall briefly comment on the complex solutions.

We observe first that $\varrho = 0$ is a solution of (5.9). The equation has however one more real solution if $\gamma\theta = [1 + a(\theta b - 1)] \neq 1$.[22] It is also easy to demonstrate that this solution will be a value $\varrho > 0$ if $\gamma\theta > 1$.

The latter case is, however, the more interesting from an economic point of view. The condition $\theta b > 1$, which assures $\gamma\theta > 1$, may in fact be obviously interpreted as the condition for the profitability of round-about production through the employment of capital, b representing the productivity of capital and θ its productive life time. We can therefore conclude that (5.9) has two real solutions, one equal to zero and one positive, so that we can write the solution for the homogeneous part of (5.7) as

$$k^*(t) = C_1 + C_2 e^{\varrho t} \tag{5.10}$$

where ϱ now designates the positive root of (5.9) and C_1 and C_2 are arbitrary constants.

By adding the particular solution resulting from the last term in (5.7), we obtain the following solution for $k(t)$ (where we are still disregarding the complex solutions of (5.9)):

$$k(t) = C_1 + C_2 e^{\varrho t} + \frac{a\alpha\nu N_0}{\nu - \gamma(1 - e^{-\nu\theta})} e^{\nu t}. \tag{5.11}$$

For $x(t)$ we obtain by integrating $k(t)$ to give $K(t)$ and by inserting in (5.4):

$$x(t) = C' + C'' e^{\rho t} + aN_0\left[1 + \frac{ab(1 - e^{-\nu\theta})}{\nu - \gamma(1 - e^{-\nu\theta})}\right] e^{\nu t} \tag{5.12}$$

where C' and C'' are constants which may be determined by means of initial conditions.[23]

For the average income we obtain

$$\frac{x(t)}{N(t)} = \frac{1}{N_0} C' e^{-\nu t} + \frac{1}{N_0} C'' e^{(\rho - \nu)t} + a\left[1 + \frac{ab(1 - e^{-\nu\theta})}{\nu - \gamma(1 - e^{-\nu\theta})}\right]. \tag{5.13}$$

If now $\nu > \varrho$, the first two terms of (5.13) tend to vanish, and the last term remains as a constant asymptote. In this case it is easy to demonstrate that $\nu > \gamma(1 - e^{-\nu\theta})$ so that the asymptotic value for the average income is greater than a, which is obviously reasonable.

[21] Our equation (5.9) is equivalent to equation (5.3) in Domar [**3**, Essay VII].

[22] If $\gamma\theta$ should be equal to one, $\varrho = 0$ would be a double root.

[23] If we require $K(t) \to 0$ for $t \to -\infty$, then $C' = C_1 = 0$.

If $\nu < \varrho$, the second term on the right hand side of (5.13) will dominate in the long run. It is easy to prove that C'' must in this case be positive, provided that $K(t) \to 0$ for $t \to -\infty$, and x/N will therefore *increase* in the long run.

The special case $\varrho = \nu$ shall not be discussed here. (5.13) does not give the general (real, continuous) solution in this case.

The values of C' and C'' might be discussed further with reference to initial conditions. This discussion is, however, tedious and will be omitted here.

As regards the dependence of the growth rate ϱ on the parameters of the model discussed in this section, I shall only mention that ϱ now depends essentially on the propensity to save, α, and is higher the higher is α. For different values of α we have then the following possibilities: For low values of α, $\varrho < \nu$ and average income tends to a stationary value. This level is higher, the higher is α. When α is above a certain critical value, average income will, however, tend to increase in the long run, and increase faster, the higher is α.

In addition to the real solutions discussed above, the characteristic equation (5.9) contains an infinity of complex solutions. By means of the results of Frisch and Holme [**6**] it is easily seen[24] that in our case (with $\theta b > 1$) these complex solutions will correspond to one cycle with a period in each of the intervals (the bounds included)

$$(5.14) \qquad \left(\frac{\theta}{j+\frac{1}{2}}, \frac{\theta}{j}\right) \qquad (j = 1,2,3,\ldots),$$

i.e., in the intervals $\left(\frac{2}{3}\,\theta,\, \theta\right)$, $\left(\frac{2}{5}\,\theta,\, \frac{\theta}{2}\right)$, $\left(\frac{2\theta}{7}, \frac{\theta}{3}\right)$, etc.

As regards the dampening or explosion of the fluctuations, I have not reached any really general result. It is, however, rather easy to demonstrate[25]

[24] Cf. Table 2, case $C > 1$. In our case, Frisch and Holme's parameters a and c are both equal to our γ, and their C equals $\gamma\theta - \text{Log}_e\, \gamma\theta$, which is greater than 1 because we have assumed $\gamma\theta > 1$.

[25] I shall briefly give the proof by help of the results of Frisch and Holme [**6**]. Write $\varrho = \beta + i\alpha$ (where α and β are of course not identical with the propensities to save used in the text). We further write $u = \alpha\theta$ and $v = \beta\theta$. A necessary and sufficient condition for dampening is then $v < 0$. By Frisch and Holme's equation (25) we have

$$\text{(i)} \qquad v = \text{Log}_e\left(\gamma\theta\,\frac{\sin u}{u}\right)$$

where we have introduced our γ instead of Frisch and Holme's c. Now Frisch and Holme further state (p. 232) that the u's corresponding to the cyclic components in our case [$C = \gamma\theta - \text{Log}_e\gamma\theta > 1$ since $\gamma\theta > 1$] will lie in the intervals

$$\text{(ii)} \qquad 2j\pi \leqq u \leqq (2j+1)\,\pi \qquad (j = 1,2,\ldots..),$$

that means that 2π is a lower bound for any u. Since $\sin u \leqq 1$, $\gamma\theta < 2\pi$ then implies $\gamma\theta \sin u/u < 1$, which by (i) is sufficient for $v < 0$ and thus for dampening.

These results are in conformity with some results by Hans Neisser [**16**] and by Domar [**4**] in similar cases. Their analyses are however carried out in terms of pure difference equations.

that all the cycles are damped for acceptable values of γ and θ, a sufficient condition for dampening being that

$$\gamma\theta < 2\pi, \quad \text{i.e., } \theta b < 1 + \frac{2\pi - 1}{a}. \tag{5.15}$$

With the interpretation of θb given above, it is very unlikely that this condition should not be fulfilled.

The possible relevance of the results in this section for business cycle analysis and for the question of interaction between growth and cycles shall not be discussed here.

In the case discussed in this section, it seems to be rather difficult to introduce increasing technical efficiency in a satisfactory way. We might, however, introduce a changing marginal productivity of labour by letting a be a given exponential function of time:

$$\varphi(n,k,x,t) = a_0 e^{\lambda t} n + bk \tag{5.16}$$

where a_0 is the marginal productivity of labour at $t = 0$ and λ is the (constant) relative rate of increase in this productivity. This satisfies $\varphi(0,0,x,t) \equiv 0$. However it obviously represents a quite special non-neutral change in productivity.

The solution for $x(t)$ with the assumption (5.16) takes the form (5.12) where $\nu + \lambda$ is substituted for ν and $a_0\,(1 - e^{-(\nu+\lambda)\theta})/(1 - e^{-\nu\theta})$ is substituted for a.

The solution for $x(t)/N(t)$ then takes the form

$$\frac{x(t)}{N(t)} = Pe^{-\nu t} + Qe^{(\rho-\nu)t} + Re^{\lambda t} \tag{5.17}$$

where P, Q and R are constants. (This solution is not valid in the special case, $\varrho = \nu + \lambda$.) When $\lambda > \varrho - \nu$, so that the last term in (5.17) will dominate in the long run, it is easily seen that R (which is independent of the initial situation except for a_0) is positive.

6. CONCLUDING REMARKS

As stated in the introductory remarks and in Section 2 of this paper, I find the hypothesis of "ex ante" substitution possibilities, but no such possibilities "ex post," more realistic than the hypotheses about the production process most widely accepted in *theoretical* growth analysis. In fact, I have the feeling that the hypothesis applied in this paper is closer to the experience of many students of economic growth who approach these questions from a "practical" point of view, and it may possibly be helpful in removing some of the "guilty conscience" of some theorists in the field who rely either on fixed coefficients or on full substitutability in a "classical" sense.

In particular, I find the hypothesis outlined in this paper important when technical progress is to enter the analysis as a main factor.

Sections 3, 4 and 5 show that our hypothesis does not make our models unmanageable in cases where on points other than the substitution possibilities we rely on hypotheses usually applied in theoretical growth analysis.

It seems that the conclusions are in some respects more sensitive to shifts in the *form* of the production function than they are to the shift from the assumption of substitutability in the usual sense to our assumption of substitutability only "at the margin." If the study had been directed more specially towards such subjects as, say, the importance of the rate of investment for the possibilities of adopting new techniques, the importance of obsolescence in the process of growth, the relation between population growth and "structural" unemployment, etc., then the conclusions would depend more specifically on the choice of what kind of substitutability one assumes.

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