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DISKUSSIONSINLÄGG

A MODEL OF ECONOMIC GROWTH WITH INCREASING EFFICIENCY OF CAPITAL

*By LEIF JOHANSEN**

The model under consideration in this note relates to the "basic model of growth" applied by Ingvar Svennilson in his study of capital accumulation and economic growth.[1] A main feature by which this model differs from the growth models of the Harrod–Domar type, is that the idea of increasing efficiency of capital is introduced. I suspect that professor Svennilson has failed to see all the implications of this feature of the model; the main object of this note is to throw some light on this point.

We divide the time in periods and adopt the following notations. The point of time t is the end point of period no. t. O_t = the gross national product in period no. t. I_t = the gross investment in period no. t. C_t = the amount of capital at the point of time t. All magnitudes are supposed to be measured in real terms.

Following professor Svennilson we assume that a constant fraction u of every age group of capital is eliminated each year. The total elimination of capital during period no. t will then be $u\,C_{t-1}$, and the amount of capital at the point of time t will be

$$(1) \qquad C_t = \sum_{\tau=0}^{\infty} I_{t-\tau}(1-u)^{\tau} = C_{t-1}(1-u) + I_t .$$

Further we follow professor Svennilson in assuming that the gross investment each year forms a constant quota s of the gross national product, i.e.:

$$(2) \qquad I_t = s\,O_t.$$

With regard to the productivity of capital, we assume that capital produced in different periods may have unequal efficiency, and further that the efficiency depends *only* on the period in which the capital is produced. Let α_t = the productivity of capital produced in period no. t. As the capital pro-

* I am indebted to Mr Hans Jacob Kreyberg for reading through the manuscript and giving useful hints.

[1] Cf. Ingvar Svennilson: A Note on Capital Accumulation and Economic Growth. Social Science Institute, University of Stockholm. Almqvist & Wiksell, Uppsala 1956. (A more extensive article is announced.)

duced in period no. $t-\tau$ and still existing in period no. t amounts to

$$I_{t-\tau}(1-u)^{\tau} \quad (\tau=0, 1, 2, \ldots),$$

we may write the production function

$$O_t = \sum_{\tau=0}^{\infty} \alpha_{t-\tau} I_{t-\tau}(1-u)^{\tau}. \tag{3}$$

We have here reckoned as if the capital existing at the *end* of period no. t takes part in the production during the period, i.e. as if all investment and depreciation is realized in the beginning of the period; cf. a note on this point further on.

The parameters u and s introduced above are both assumed to be positive and less than unity.

To solve the system (1)–(3), we combine the relations (2) and (3) to give

$$O_t = s \sum_{\tau=0}^{\infty} \alpha_{t-\tau} O_{t-\tau}(1-u)^{\tau}. \tag{4}$$

This equation may be written as

$$O_t = \frac{s}{1-s\alpha_t} \sum_{\tau=1}^{\infty} \alpha_{t-\tau} O_{t-\tau}(1-u)^{\tau}. \tag{5}$$

By writing out the equation (4) for O_{t-1} and arranging the terms in a suitable manner, we obtain

$$\sum_{\tau=1}^{\infty} \alpha_{t-\tau} O_{t-\tau}(1-u)^{\tau} = \frac{1-u}{s} O_{t-1} \tag{6}$$

which by insertion in (5) gives

$$O_t = \frac{1-u}{1-s\alpha_t} O_{t-1}. \tag{7}$$

Solving this difference equation, we get

$$O_t = \frac{(1-u)^t}{\prod_{\tau=1}^{t}(1-s\alpha_\tau)} O_0, \tag{8}$$

where O_0 is the gross product in period no. zero.

The model and the solution here will be meaningless if for any t $1-s\alpha_t \leqq 0$. We must therefore restrict our analysis to cases where this is not fulfilled.[1]

[1] The difficulty mentioned in the text stems from the fact that we have assumed the new capital produced during a period to take part in the production during the same period. If we drop this assumption, i.e. if we write

$$O_t = \sum_{\tau=1}^{\infty} \alpha_{t-\tau} I_{t-\tau}(1-u)^{\tau} \tag{3'}$$

forts.

How does the solution above differ from the solution of "traditional" models with constant productivity of capital? In these "traditional" models the solution gives a *constant* growth rate. By (7) and (8) the growth rate v of O_t is

$$(9) \qquad v = \frac{O_t - O_{t-1}}{O_{t-1}} = \frac{1-u}{1-s\,\alpha_t} - 1,$$

which according to our assumptions about u and s is an *increasing function of t* when α_t is increasing with t, i.e. when newer capital is always more efficient than older. On the other hand, for $\alpha_t = \alpha$ independent of t, we obtain[1]

$$(10) \qquad v = \frac{1-u}{1-s\,\alpha} - 1$$

which corresponds to the exponential solution

$$(11) \qquad O_t = \left(\frac{1-u}{1-s\,\alpha}\right)^t O_0.$$

I.e., we may say that *the increasing productivity of capital causes the rate of growth of the gross product to be an increasing function of time.*

For I_t we get the solution

$$(12) \qquad I_t = \frac{(1-u)^t}{\prod_{\tau=1}^{t}(1-s\,\alpha_\tau)} I_0$$

which for $\alpha_t = \alpha$ independent of t reduces to

$$(13) \qquad I_t = \left(\frac{1-u}{1-s\,\alpha}\right)^t I_0.$$

For C_t we obtain

$$(14) \qquad C_t = (1-u)^t \left\{ C_0 + I_0 \sum_{\Theta=1}^{t} \frac{1}{\prod_{\tau=1}^{\Theta}(1-s\,\alpha_\tau)} \right\}$$

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instead of (3), we obtain

$$(7') \qquad O_t = \{(1-u)\,(1+s\,\alpha_{t-1})\}\, O_{t-1}$$

instead of (7), with the explicit solution

$$(8') \qquad O_t = \left\{(1-u)^t \prod_{\tau=0}^{t-1}(1+s\,\alpha_\tau)\right\} O_0.$$

The setup here is perhaps more realistic than that used in the text, but the latter seems to correspond to professor Svennilson's model.

[1] Formula (10) is identical with professor Svennilson's formula for the growth rate, cf. formula (6) in the study cited above.

which for $\alpha_t = \alpha$ independent of t reduces to

$$C_t = (1-u)^t \left\{C_0 + I_0 \frac{1-(1-s\alpha)^t}{s\alpha(1-s\alpha)^t}\right\} = \left(\frac{1-u}{1-s\alpha}\right)^t C_0. \tag{15}$$

Professor Svennilson in his study assumes that the net effect of the shrinking of old capital expressed by the parameter u and the change in the efficiency of capital can be represented by a *constant* marginal return to capital. This is done "in order to study the interrelations in a process of even exponential growth". Consequently he discusses a solution which is formally equivalent to the solution above for $\alpha_t = \alpha$ independent of t.

The analysis above suggests that this may be self-contradictory. The consequence of introducing different efficiency of capital of different age is that the rate of growth of output will change with time, and therefore we will not get a process of even exponential growth (as long as the hypothesis expressed by equation (2) is retained). The reason for this is obviously that the changes in capital do not aggregate in such a way as to allow us to express the net effect of the different changes by means of a constant marginal return to capital.

If we want to retain the hypothesis of changing efficiency of new capital, and at the same time want to study a process of even exponential growth, we are forced to change the hypothesis of saving expressed by (2). It is easy to see that in order to obtain an even exponential growth of output, we must substitute for formula (2) the formula

$$I_t = \frac{s_0 \alpha_0}{\alpha_t} O_t, \tag{16}$$

where s_0 is a constant equal to the rate of saving in period no. zero. In this case we obtain the solution

$$O_t = \left(\frac{1-u}{1-s_0\alpha_0}\right)^t O_0 \tag{17}$$

with the growth rate

$$v = \frac{1-u}{1-s_0\alpha_0} - 1 \tag{18}$$

regardless of the development of the efficiency of capital.

In a special sense the hypothesis (16) may be said to represent a constant rate of saving, because the rate of capital accumulation decreases in just the same proportion as the efficiency of capital increases. We may therefore say that the rate of saving according to (16) is constant when we measure the new capital by its value in production.

COMMENT *by INGVAR SVENNILSON*

Leif Johansen's article forms an interesting complement to my earlier note. In my contribution to *25 Essays in Honour of Erik Lindahl*, I have taken into account his observations.